Growth alternative for Hecke-Kiselman monoids

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Hecke-Kiselman monoid

Definition (Ganyushkin, Mazorchuk, 2011)

For any simple digraph Θ of n vertices the corresponding monoid HK_Θ generated by idempotents e_i, $i \in \{1, \ldots, n\}$ is defined by the following relations, for any $i \neq j$:

1. $e_i e_j = e_j e_i$, when there is no edge/arrow between i, j in Θ,
2. $e_i e_j e_i = e_j e_i e_j$, when we have an edge $i - j$ in Θ,
3. $e_i e_j e_i = e_j e_i e_j = e_i e_j$, when we have an arrow $i \to j$ in Θ.

Motivations (1): finite $\mathcal{J}$-trivial monoids

Question (Ganyushkin, Mazorchuk, 2011)

For which graphs Θ is the monoid HK_Θ finite? When is it $\mathcal{J}$-trival, namely, do we have

$$\mathrm{HK}_\Theta\, a\, \mathrm{HK}_\Theta = \mathrm{HK}_\Theta\, b\, \mathrm{HK}_\Theta \implies a = b$$

for all $a, b \in \mathrm{HK}_\Theta$?

Two extreme cases:

- when Θ is unoriented, then HK_Θ is the 0-Hecke monoid of a Coxeter group W with graph Θ and $|\mathrm{HK}_\Theta| = |W|$.
- when Θ is oriented, then HK_Θ is finite if and only if Θ is acyclic.

In general: an open question.

Motivations (2): f.g. algebras of alternative growth

Some relevant examples of classes of finitely generated algebras $A \simeq K\langle X\rangle/I$ with alternative growth:

- finitely presented monomial algebras,
- the Gröbner basis of I is finite,
- automaton algebras (the language of normal forms of words is recognized by a finite automaton).

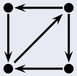
A motivating example of automaton algebras: algebras of Coxeter groups and the 0-Hecke algebras.

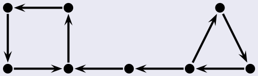
The main result – for oriented Θ

Theorem

Assume that Θ is a finite oriented simple graph. The following conditions are equivalent.

(1) Θ does not contain two different cycles connected by an oriented path of length ≥ 0, for instance





(2) the monoid algebra $K[HK_\Theta]$ satisfies a polynomial identity,

(3) $\mathrm{GKdim}(K[HK_\Theta]) < \infty$,

(4) the monoid HK_Θ does not contain a free submonoid of rank 2.

Two cycles and the oriented path

Let Θ be an oriented graph. Then:

- if Θ contains a graph with two oriented cycles joined by an oriented path, then HK_{Θ} contains a free monoid $\langle x, y \rangle$.

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- if $\Theta = \Theta' \cup \{v\}$, where v is a source or sink vertex then:

$$\mathrm{GKdim}(K[\mathrm{HK}_\Theta]) < \infty \iff \mathrm{GKdim}(K[\mathrm{HK}_{\Theta'}]) < \infty,$$

$$K[\mathrm{HK}_\Theta] \text{ is PI} \iff K[\mathrm{HK}_{\Theta'}] \text{ is PI.}$$

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- if we „keep removing” sources and sinks from Θ (along with the adjacent arrows), which does not contain two oriented cycles joined by an oriented path, we can only arrive at the following connected components:
  - an acyclic graph Θ', and $\mathrm{GKdim}(K[\mathrm{HK}_{\Theta'}]) = 0$,
  - an oriented cycle Θ'', and $\mathrm{GKdim}(K[\mathrm{HK}_{\Theta''}]) = 1$.

The mixed graph case – remarks

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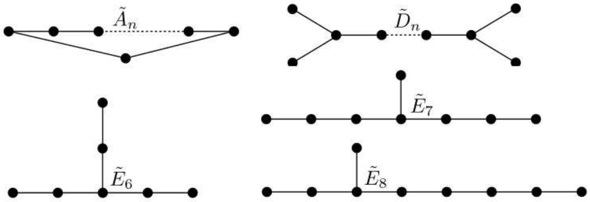
- (trivial) if you replace an oriented arrow by an unoriented edge, the Gelfand-Kirillov dimension of $K[HK_\Theta]$ will not decrease (replace $aba = bab = ab$ with $aba = bab$).
- (Tsaranov, de la Harpe) if Θ is unoriented then HK_Θ is the 0-Hecke monoid of the Coxeter monoid of Θ and $GKdim(K[HK_\Theta]) < \infty$ if, and only if Θ is a disjoint union of extended Dynkin diagrams:



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