Algebra 2*, wiosna 2012, seria VI
Zadania na 19 marca. Tym razem po angielsku, ponieważ większość zadań jest wziętych z pierwszej serii zadań do mini-kursu *Invariants of algebraic groups actions, an introduction*.

For hints and more problems on symmetric polynomials look at chapter 7 of Cox, Little, O'Shea book. For hints on quotients of the plane and more problems look at notes of Miles Reid. The last two problems are harder than the rest.

1. Symmetric polynomials. Recall that a polynomial $f \in \mathbb{K}[x_1, \ldots, x_n]$ is called symmetric if $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for every permutation $\pi \in S_n$ of the set $[n] = \{1, \ldots, n\}$. Recall also that, for $k \leq n$ an elementary symmetric polynomial $\sigma_k \in \mathbb{K}[x_1, \ldots, x_n]$ is defined as

$$\sigma_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

   (a) Prove that

$$(y - x_1)(y - x_2) \cdots (y - x_n) = y^n - \sigma_1 y^{n-1} + \cdots + (-1)^{n-1} \sigma_{n-1} y + (-1)^n \sigma_n$$

   (b) Prove that, if σ_k^n denotes the k-th elementary symmetric polynomial in n variables, then $\sigma_k^n = \sigma_k^{n-1} + x_n \sigma_{k-1}^{n-1}$

   (c) Find a formula (e.g. a generating function) for the dimension of the $\mathbb{K}$-linear space of symmetric polynomials in n variables of total degree d.

2. Write the following symmetric functions as polynomials in elementary symmetric functions:

   (a) $\sum_{i \neq j} x_i^2 x_j$, $\sum_{i \neq j} x_i^2 x_j^2$,

   (b) $\prod_{i \neq j} (x_i - x_j)$ for $n = 3$ (discriminant)

   Hint: Use Gauss algorithm or the fact that homogeneous symmetric polynomials of degree d are linear combinations of monomials in elementary symmetric functions of the appropriate degree.

3. Let $h_i(x_1, \ldots, x_n)$ denote the sum of all monomials of degree i. For every $k \geq 1$ prove the identity

$$\sum_{i=0}^{k} (-1)^i h_{k-i}(x_1, \ldots, x_n) \sigma_i(x_1, \ldots, x_n) = 0$$

where $\sigma_0 = 1$ and $\sigma_i = 0$ for $i > n$. *Hint: Cox, Little, O'Shea, p.325.*

4. More symmetric polynomials. Let us define $s_k = \sum_{i=1}^{n} x_i^k$. Prove the following identities

   (a) if s_k^n denotes the respective function in n variables then $s_k^n = s_k^{n-1} + x_n^k$

   (b) $s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \cdots + (-1)^{k-1} \sigma_{k-1} s_1 + (-1)^k n \sigma_k = 0$, for $1 \leq k \leq n$

   (c) $s_k - \sigma_1 s_{k-1} + \sigma_2 s_{k-2} - \cdots \pm \sigma_n s_{k-n} = 0$, for $k > n$

   Conclude that, if the characteristic of $\mathbb{K}$ is zero or bigger than n, then functions s_k generate the ring of invariants $\mathbb{K}[x_1, \ldots, x_n]^{S_n}$.

5. Useful identity. Let $\mathbb{K}$ be a field of characteristic zero. Prove the following identity in $\mathbb{K}[x_1, \ldots, x_n]$:

$$n! \cdot x_1 \cdots x_n = \sum_{I \subseteq [n]} (-1)^{|I|} (\sum_{i \in I} x_i)^n$$

   where I runs over all subsets of $[n] = \{1, \ldots, n\}$.

6. Cyclic group action. Let $G_m \subset \mathbb{C}^*$ be the multiplicative group of m-th roots of unity generated by $\epsilon = \epsilon_m = \exp(2\pi i/m)$. Let G_m act faithfully on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ with weights $(a_1, \ldots, a_n)$, where $0 < a_i < m$. This means that $\epsilon(x_i) = \epsilon^{a_i} \cdot x_i$.

   (a) Without using Hilbert-Noether theorem, prove that the algebra of invariants $\mathbb{C}[x_1, \ldots, x_n]^{G_m}$ is generated by a finite number of monomials. Bound the number of generators in terms of n and m.

   (b) Prove that $x_1^{b_1} \cdots x_n^{b_n} \in \mathbb{C}[x_1, \ldots, x_n]^{G_m}$ if and only if $m | \sum a_i b_i$.

7. Cyclic 2-dimensional quotients. Notation as above, now assume $n = 2$

(a) Find generators of $\mathbb{C}[x_1, x_2]^{G_m}$ for $(a_1, a_2) = (a, m - a)$, with $0 < a < m$. Find relations between these generators.

(b) Find generators of $\mathbb{C}[x_1, x_2]^{G_m}$ for $(a_1, a_2) = (1, 1)$. Find relations between these generators.

8. Let $G \subset GL(n, \mathbb{C})$ be a finite group.

   (a) Prove that every element of G is diagonalizable, that is its matrix is similar (or conjugate) to a diagonal matrix.

   (b) An element of G is called quasi-reflection if it its identity eigenspace is of dimension $n-1$. Is the product of quasi reflections a quasi-reflection again?

   (c) Prove that the subgroup generated by all quasireflections in G is normal.

9. Let us consider the dihedral group $D_{2m} \subset GL(2, \mathbb{C})$ generated by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix}$$

with $\epsilon = exp(2\pi i/m)$ the m-th root of unity. The group D_{2m} acts linearly on $\mathbb{C}[x_1, x_2]$. Find the generators of the ring of invariants and relations between them.

10. † Let us consider the binary dihedral group $BD_{4m} \subset SL(2, \mathbb{C})$ generated by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \epsilon \end{pmatrix}$$

with $\epsilon = exp(2\pi i/2m)$ the $2m$-th root of unity. Find the rank of the group and the relation between the generators. The group BD_{4m} acts linearly on $\mathbb{C}[x_1, x_2]$. Find the generators of the ring of invariants and relations between them. Do the case $m = 2$ for the start.

11. † For $i = \sqrt{-1}$ let us consider the binary tetrahedral group BT generated by the following matrices

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \tfrac{1}{2} \begin{pmatrix} 1+i & -1+i \\ 1+i & 1-i \end{pmatrix}$$

Find the rank of the group and the relation between the generators. The group BT acts linearly on $\mathbb{C}[x_1, x_2]$. Find the generators of the ring of invariants and relations between them.