Assignment 1

Assume that formulas of a system $\mathcal{L}$ have the form

$$\varphi, \psi ::= \forall x.\sigma \mid \exists x.\varphi$$
$$\sigma, \tau ::= P(x) \mid \sigma \vee \tau \mid \sigma \wedge \tau \mid \neg\sigma$$

where P ranges over an infinite set $\mathcal{P}$ of predicate variables of arity 1. The semantics of this logic is the semantics of the first-order classical logic.

Prove the following proposition.

> *For each sentence φ of $\mathcal{L}$ if there is a model $\mathcal{M}$ of φ then there is a model $\mathcal{M}'$ of φ such that the size of the carrier of $\mathcal{M}'$ is less than or equal to $|\varphi|$ (i.e. the size of the sentence φ).*

Solution The sentence φ has the general form: $\exists x_1 \ldots \exists x_n.\forall y.\varphi_0$ where φ_0 does not contain quantifiers. If there is a model $\mathcal{M}$ such that $\mathcal{M} \models \varphi$ then there is a valuation v such that $\mathcal{M}, v \models \forall y.\varphi_0$. Consider a model $\mathcal{M}|_A$ the carrier of which is $A = \{v(x_1), \ldots, v(x_n)\}$ and predicates are interpreted as in $\mathcal{M}$. It is obvious that $\mathcal{M}|_A, v \models \forall y.\varphi_0$. The carrier size of $\mathcal{M}|_A$ is less than or equal to $n \leq |\varphi|$.

Assignment 2

Show that the provability problem for the logic in Assignment 1 is decidable.

Solution We have the following statement

$$\vdash \varphi \quad \text{iff} \quad \not\models \neg\varphi.$$

If $\varphi = \exists x_1 \ldots \exists x_n.\forall y.\varphi_0$, where φ_0 is a monadic formula without quantifiers, then $\neg\varphi$ is equivalent to $\psi = \forall x_1 \ldots \forall x_n.\exists y.\varphi_0$. This formula is a formula of the fragment $[all, (\omega), (\omega)]$, which is decidable. So if ψ is satisfiable, we announce that φ is unprovable. Otherwise φ is provable.

Assignment 3

Prove that the satisfiability problem for the logic in Assignment 1 is NP-complete.

Solution The satisfiability problem is in NP, because it is enough to guess a model of size not greater than the size n of the formula $\varphi = \exists x_1 \ldots \exists x_n.\forall y.\varphi_0$ that is verified for satisfiability and then guess the assignment of the monadic predicates that occur in φ_0 (see the solution of assignment 1). In this way we obtained a model $\mathcal{M}$. Guess an assignment v of $x_1, \ldots, x_n$ to elements of the model. After that for each of the elements a of the model check the value of

$\mathcal{M}, v[y := a] \models \varphi_0$. The latter can be computed in $O(n^2)$ time and must be repeated n times. Thus the whole procedure is polynomial.

To prove that the problem is NP-hard, we reduce the SAT problem. Consider a formula σ of the propositional logic. For each propositional variable α in σ we take a monadic predicate P_α and translate σ to $\overline{\sigma}$ by induction as follows

- $\overline{\alpha} = P_\alpha(x)$
- $\overline{\sigma_1 \vee \sigma_2} = \overline{\sigma_1} \vee \overline{\sigma_2}$
- $\overline{\sigma_1 \wedge \sigma_2} = \overline{\sigma_1} \wedge \overline{\sigma_2}$
- $\overline{\neg\tau} = \neg\overline{\tau}$

Finally, we take the formula $\exists x.\overline{\sigma}$. Suppose that σ is satisfied by a boolean valuation v_b. Consider a model $\mathcal{M}$ with carrier $\{0\}$ and predicates defined so that $P_\alpha^{\mathcal{M}}(0)$ iff $v_b(\alpha) = 1$. By a straightforward induction over σ we show that $\mathcal{M} \models \exists x.\overline{\sigma}$.

If there is a model $\mathcal{N}$ of $\exists x.\overline{\sigma}$ then there is a valuation v such that $\mathcal{N}, v \models \exists x.\overline{\sigma}$. We take a boolean valuation $v_b(\alpha) = 1$ iff $P_\alpha^{\mathcal{N}}(v(x))$. Again, by a straightforward induction over σ we show that $v_b(\sigma) = 1$.