A SHARP L^p ESTIMATE FOR THE TOTAL VARIATION PROCESS

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Abstract. For a given $p \geq 2$, let X be an L^p bounded martingale and let Y be a martingale of bounded mean oscillation. The paper contains the proof of the estimate

$$\left\| \int_0^\infty |\mathrm{d}\langle X, Y\rangle_t| \right\|_p \leq p\|X\|_p \|Y\|_{bmo}.$$

The inequality is sharp for each p and the range $p \geq 2$ cannot be expanded without additional assumptions on X and Y. The proof rests on the existence of a certain special function, enjoying appropriate size and concavity requirements.

1. Introduction

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space equipped with a discrete-time filtration $(\mathcal{F}_n)_{n\geq 0}$. Let $\mathcal{H}$ be a given separable Hilbert space, with the norm $|\cdot|$ and the scalar product denoted by the dot $\cdot$; with no loss of generality, we may and will assume that $\mathcal{H} = \ell^2$. Suppose further that $f = (f_n)_{n\geq 0}$, $g = (g_n)_{n\geq 0}$ are two adapted martingales taking values in $\mathcal{H}$. The associated difference sequence $df = (df_n)_{n\geq 0}$, $dg = (dg_n)_{n\geq 0}$ are defined by $df_0 = f_0$ and $df_n = f_n - f_{n-1}$, and similarly for dg. The square function of f is given by

$$S(f) = \left(\sum_{n=0}^{\infty} |df_n|^2 \right)^{1/2}$$

and the p-th norm of f is defined by $\|f\|_p = \sup_{n\geq 0} \|f_n\|_p$, $0 < p < \infty$. The martingale g is said to belong to BMO, the class of martingales of bounded mean oscillation, if it is uniformly integrable and we have the estimate

$$\|g\|_{BMO} = \operatorname*{esssup}_{n\geq 0} \left\| \mathbb{E}\big(|g_\infty - g_{n-1}|^2 \,|\, \mathcal{F}_n \big) \right\|_\infty^{1/2} < \infty$$

(with the convention $g_{-1} = 0$). The space BMO, introduced by John and Nirenberg [4] in the analytic setup, plays a fundamental role in probability and harmonic analysis. For example, it forms a natural substitute for the space L^∞: many classical operators (e.g. singular integral) are not bounded on L^∞, but map L^∞ to BMO. This weaker boundedness is strong enough to apply appropriate interpolation and deduce the boundedness of operators on other function spaces. Another important property was established by Fefferman in the seventies: BMO is a dual space to the Hardy space H^1. The martingale counterparts of these statements were proved by Getoor and Sharpe [1]. In particular, we have the following quantitative version of Fefferman's result: for any martingale $f \in H^1$

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and any $g \in BMO$ with $g_0 = 0$, we have

$$|\mathbb{E}\langle f_\infty, g_\infty\rangle| \leq \sqrt{2}\|f\|_{H^1}\|g\|_{BMO} \tag{1.1}$$

and the constant is the best possible (cf. [5]). Here

$$\langle f_\infty, g_\infty\rangle = \sum_{n=1}^{\infty} \mathbb{E}(df_n \cdot dg_n|\mathcal{F}_{n-1})$$

is the discrete-time version of the skew bracket of f and g, and the H^p norm of f equals

$$\|f\|_{H^p} = \left(\mathbb{E}\left(\sum_{n=0}^{\infty}|df_n|^2\right)^{p/2}\right)^{1/p}, \qquad 0 < p < \infty.$$

In our considerations below, we will work with a slightly different class of martingales, which originates in the works of Herz [2, 3]. We say that g lies in bmo, if it is uniformly integrable and

$$\|g\|_{bmo} = \sup_{n\geq 0}\left\|\mathbb{E}\big(|g_\infty - g_n|^2\,|\,\mathcal{F}_n\big)\right\|_\infty^{1/2} < \infty.$$

We easily check that bmo is bigger than BMO: this follows directly from the estimate

$$\mathbb{E}\big(|g_\infty - g_{n-1}|^2\,|\,\mathcal{F}_n\big) = \mathbb{E}\big(|g_\infty - g_n|^2\,|\,\mathcal{F}_n\big) + |dg_n|^2 \geq \mathbb{E}\big(|g_\infty - g_n|^2\,|\,\mathcal{F}_n\big).$$

Thus, there is a natural question about versions of the above Fefferman's inequality under the assumption $g \in bmo$. It is not difficult to construct examples showing that (1.1) fails to hold, with any finite constant, if we replace $\|g\|_{BMO}$ with $\|g\|_{bmo}$. This leads to the question about the L^p-extensions of the above estimate. We will prove the following.

Theorem 1.1. *Let $p \geq 2$ be a fixed exponent. Then for any L^p-bounded martingale f and any $g \in bmo$ we have the inequality*

$$\|\langle f_\infty, g_\infty\rangle\|_p \leq p\|f\|_p\|g\|_{bmo} \tag{1.2}$$

and

$$\|\langle f_\infty, g_\infty\rangle\|_p \leq p\|f\|_{H^p}\|g\|_{bmo}. \tag{1.3}$$

The constant p is the best possible in both estimates. Furthermore, both inequalities do not hold for $p < 2$ with any finite constant.

To the best of our knowledge, this is a new result. Actually, we will be able to prove the following stronger form of (1.2):

$$\left\|\sum_{n=1}^{\infty}\mathbb{E}(|df_n|\cdot|dg_n|\,|\mathcal{F}_{n-1})\right\|_p \leq p\|f\|_p\|g\|_{bmo}. \tag{1.4}$$

The following simple argument links the above estimates with classical Hardy inequalities on the positive halfline. Suppose that the probability space is the unit interval $[0,1]$ with its Borel subsets and Lebesgue's measure. For a given $\delta \in (0,1)$, consider the filtration $(\mathcal{F}_n)_{n\geq 0}$, where $\mathcal{F}_n$ is generated by $[0,(1-\delta)^n]$ and all Borel subsets of $((1-\delta)^n, 1]$. Then the random variable $g(\omega) = -\ln\omega$ satisfies $\|g\|_{bmo} = 1$. Indeed, we have

$$g_n(\omega) = \begin{cases} -\ln(1-\delta)^n + 1 & \text{if } \omega \leq (1-\delta)^n, \\ -\ln\omega & \text{if } \omega > (1-\delta)^n \end{cases}$$

and hence for each n,

$$\left\|\mathbb{E}[(g-g_n)^2|\mathcal{F}_n]\right\|_\infty = \frac{1}{(1-\delta)^n}\int_0^{(1-\delta)^n} (\ln(\omega(1-\delta)^{-n})+1)^2 \mathrm{d}\omega = \int_0^1 (\ln u+1)^2 \mathrm{d}u = 1.$$

Now, fix arbitrary real-valued variables $f \in L^p$ and $h \in L^{p'}$ (where $p' = p/(p-1)$) and let $(f_n)_{n\geq 0}$, $(h_n)_{n\geq 0}$ stand for the associated martingales. Then we have

$$\mathbb{E}\left(\sum_{n=1}^\infty \mathbb{E}(h|\mathcal{F}_{n-1})dg_n\right) f = \mathbb{E}\sum_{n=1}^\infty \mathbb{E}(h|\mathcal{F}_{n-1})dg_n df_n = \mathbb{E}\langle f_\infty, g_\infty\rangle h \leq p\|f\|_p\|h\|_{p'},$$

by Hölder's inequality and (1.2). Taking the supremum over all f with $\|f\|_p \leq 1$, we get

$$\left\|\sum_{n=1}^\infty \mathbb{E}(h|\mathcal{F}_{n-1})dg_n\right\|_{p'} \leq p\|h\|_{p'}.$$

It remains to note that as $\delta \to 0$, the expression $\sum_{n=1}^\infty \mathbb{E}(h|\mathcal{F}_{n-1})dg_n$ converges almost surely to the random variable $Th(\omega) = \omega^{-1}\int_0^\omega h$. Thus by Fatou's lemma and some standard dilation (which allows to expand $[0,1]$ to the whole $\mathbb{R}_+$), we get Hardy's inequality $\|Th\|_{L^{p'}(\mathbb{R}_+)} \leq p\|h\|_{L^{p'}(\mathbb{R}_+)}$, with the best constant (but for $p' \in (1,2]$ only).

By a straightforward approximation, Theorem 1.1 immediately leads to a sharp estimate for continuous-time martingales. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ be adapted martingales taking values in a Hilbert space $\mathcal{H}$. We impose standard regularity requirements on the trajectories of these processes: the paths are assumed to be right-continuous and to have limits from the left. Next, we say that Y belongs to the space bmo, if it is uniformly integrable and

$$\|Y\|_{bmo} = \sup_{t\geq 0}\left\|\mathbb{E}\left[|Y_\infty - Y_t|^2 \,\middle|\, \mathcal{F}_t\right]^{1/2}\right\|_\infty < \infty.$$

Furthermore, the symbol $\int_0^\infty |\mathrm{d}\langle X,Y\rangle|$ will denote the total variation of the skew bracket $\langle X,Y\rangle = \sum_{j=1}^\infty \langle X^j, Y^j\rangle$, where X^j, Y^j denote the j-th coordinates of X and Y, respectively. Here is the continuous-time version of Theorem 1.1 and (1.4).

Theorem 1.2. *Let $p \geq 2$ be a fixed exponent. Then for any L^p-bounded martingale X and any $Y \in bmo$ we have the sharp estimate*

$$\left\|\int_0^\infty |d\langle X,Y\rangle_t|\right\|_p \leq p\|X\|_p\|Y\|_{bmo} \tag{1.5}$$

and

$$\left\|\int_0^\infty |d\langle X,Y\rangle_t|\right\|_p \leq p\|[X,X]^{1/2}\|_p\|Y\|_{bmo}. \tag{1.6}$$

The inequalities do not hold, with any finite constant, in the range $p < 2$.

As we mentioned above, it is enough to focus on Theorem 1.1. The proof of (1.2) will rest on the existence of a certain special function of four variables, enjoying appropriate size and concavity requirements. This type of approach, called Burkholder's method or Bellman function method, has been exploited intensively in the last forty years and yielded numerous significant results in probability theory and harmonic analysis.

The remaining part of the paper is divided into two sections. Section 2 is devoted to the analysis of the special function and the proof of (1.2). The last part of the paper contains examples showing that the constant p, as well as the range $p \geq 2$, are optimal.

2. Proof of (1.2) and (1.3)

Let $p \geq 2$ be a fixed parameter and let $\mathcal{D} = \mathcal{H} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$. The proof of the L^p estimate rests on the special function $U = U_p : \mathcal{D} \to \mathbb{R}$, given by

$$U(x, s, y, z) = s^p(1 + z - y^2) - p^2|x|^2 s^{p-2}.$$

We will prove the following property of this object.

Lemma 2.1. *Suppose that a point $(x, s, y, z) \in \mathcal{D}$ satisfies $z \leq y^2 + 1$. Let d, h, k be centered random variables, with d taking values in $\mathcal{H}$ and h, k taking values in $\mathbb{R}$. If $\mathbb{E}|d|^p < \infty$ and $\mathbb{E}h^2 < \infty$, then*

$$\mathbb{E}U(x + d, s + \mathbb{E}|dh|, y + h, z + k) \leq U(x, s, y, z). \tag{2.1}$$

Proof. Since the variables d, h and k are centered, we compute that

$$\begin{aligned}
&\mathbb{E}U(x + d, s + \mathbb{E}|dh|, y + h, z + k)\\
&= \mathbb{E}\Big[(s + \mathbb{E}|dh|)^p(1 + z + k - (y + h)^2) - p^2(s + \mathbb{E}|dh|)^{p-2}(x + d)^2\Big]\\
&= \mathbb{E}\Big[(s + \mathbb{E}|dh|)^p(1 + z - y^2) - p^2(s + \mathbb{E}|dh|)^{p-2}x^2\Big]\\
&\quad - \mathbb{E}(s + \mathbb{E}|dh|)^{p-2}\Big[(s + \mathbb{E}|dh|)^2h^2 + p^2d^2\Big].
\end{aligned}$$

But $(s + \mathbb{E}|dh|)^2h^2 + p^2d^2 \geq 2p(s + \mathbb{E}|dh|)|dh|$ almost surely, so we obtain

$$\begin{aligned}
&\mathbb{E}U(x + d, s + \mathbb{E}|dh|, y + h, z + k)\\
&\leq (s + \mathbb{E}|dh|)^p(1 + z - y^2) - p^2(s + \mathbb{E}|dh|)^{p-2}x^2 - 2p(s + \mathbb{E}|dh|)^{p-1}\mathbb{E}|dh|.
\end{aligned}$$

Now we substitute $t = \mathbb{E}|dh|$ and maximize the right-hand side with respect to t. To this end, observe that the function

$$t \mapsto (s + t)^p(1 + z - y^2) - p^2(s + t)^{p-2}x^2 - 2p(s + t)^{p-1}t$$

is decreasing on $\mathbb{R}_+$; indeed, its derivative is equal to

$$p(s + t)^{p-2}\big[(s + t)(z - y^2 - 1) - 2(p - 1)t\big] - p^2(p - 2)(s + t)^{p-3}x^2 \leq 0,$$

where the latter bound comes from the assumed estimates $p \geq 2$ and $z \leq y^2 + 1$. Thus,

$$\mathbb{E}U(x + d, s + \mathbb{E}|dh|, y + h, z + k) \leq s^p(1 + z - y^2) - ps^{p-2} = U(x, s, y, z),$$

which is precisely the claim. □

Proof of (1.2). Fix martingales $f \in L^p$ and $g \in bmo$ as in the statement. Introduce the auxiliary martingale $h = (h_n)_{n\geq 0}$ and the increasing process $s = (s_n)_{n\geq 0}$ given by $h_n = \mathbb{E}(g^2|\mathcal{F}_n)$ and $s_n = \sum_{k=1}^n \mathbb{E}(|df_k| \cdot |dg_k|\,|\mathcal{F}_{k-1})$ for $n = 0, 1, 2, \ldots$ (we set $s_0 = 0$). By homogeneity, we may and do assume that $\|g\|_{bmo} \leq 1$; then the two-dimensional process (g, h) takes values in the parabolic domain $\{(y, z) : y^2 \leq z \leq y^2 + 1\}$. The key ingredient of the proof is the observation that the process

$$\left(U\left(f_n, s_n, g_n, h_n\right)\right)_{n\geq 1}$$

is a supermartingale. To show this, fix $n \geq 1$ and apply the estimate (2.1), or rather its conditional version with respect to $\mathcal{F}_{n-1}$, with $x = f_{n-1}$, $s = \sum_{k=1}^{n-1} |df_k||dg_k|$, $y = g_{n-1}$, $z = h_{n-1}$ and $d = df_n$, $h = dg_n$ and $k = dh_n$ (if $n = 1$, then $s = 0$). Then d, h and k are centered (relative to $\mathcal{F}_{n-1}$) and we have the estimate $z \leq y^2 + 1$, directly from the condition $\|g\|_{bmo} \leq 1$. Note that the conditional version of (2.1) is precisely the aforementioned supermartingale property and hence we may write

$$\mathbb{E}U\left(f_n, s_n, g_n, h_n\right) \leq \mathbb{E}U(f_0, 0, g_0, h_0) = 0.$$

However, we have

$$s_n^p(1 + h_n - g_n^2) - p^2 s_n^{p-2} f_n^2 \geq s_n^p - p^2 s_n^{p-2} f_n^2 \geq \frac{2}{p}\left(s_n^p - p^p|f_n|^p\right),$$

where the first estimate follows from Schwarz' inequality, while the second is due to Young's inequality. Thus we obtain

$$\mathbb{E}(s_n^p - p^p|f_n|^p) \leq 0,$$

and letting $n \to \infty$ completes the proof. □

Proof of (1.3). Consider the larger Hilbert space $\mathcal{K} = \ell^2(\mathcal{H})$. Consider the $\mathcal{K}$-valued martingales

$$F_n = (df_0, df_1, df_2, \ldots, df_n, 0, 0, \ldots) \quad \text{and} \quad G_n = (g_n, 0, 0, \ldots).$$

Since $g \in bmo$, we have $G \in bmo$ as well and $\|G\|_{bmo} = \|g\|_{bmo}$. Thus, by (1.4), we get

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} \mathbb{E}(|df_n||dg_n|\,|\mathcal{F}_{n-1})\right\|_p &= \left\|\sum_{n=1}^{\infty} \mathbb{E}(|dF_n||dG_n|\,|\mathcal{F}_{n-1})\right\|_p \\ &\leq p\|F\|_p\|G\|_{bmo} = p\|F\|_p\|g\|_{bmo}.\end{aligned}$$

It remains to observe that $\|F\|_p = \|f\|_{H^p}$ and hence the claim follows. □

3. Sharpness and the optimality of the range of p

We start with showing that the estimates (1.2), (1.3), (1.5) and (1.6) fail to hold for $p < 2$, no matter what the multiplicative constant is. By a straightforward embedding argument, it is enough to focus on the discrete-time case. Fix $\delta > 0$ and let $f = g$ be real-valued martingales starting from zero, satisfying

$$\mathbb{P}(df_1 = -\delta) = (1 + \delta^2)^{-1} = 1 - \mathbb{P}(df_1 = \delta^{-1})$$

and $df_2 = df_3 = \ldots = 0$. Let $(\mathcal{F}_n)_{n \geq 0}$ be the natural filtration. Then

$$\|g\|_{bmo} = \|f\|_{bmo} = \left\|\mathbb{E}(df_1^2|\mathcal{F}_0)\right\|_\infty^{1/2} = \left(\mathbb{E}df_1^2\right)^{1/2} = 1$$

and $\|f\|_p^p = \|f\|_{H^p}^p = \delta^p(1 + \delta^2)^{-1} + \delta^{2-p}(1 + \delta)^{-1} \to 0$ as $\delta \to 0$. Since $\langle f_\infty, g_\infty\rangle = \mathbb{E}df_1^2 = 1$ almost surely, the ratio

$$\frac{\|\langle f_\infty, g_\infty\rangle\|_p}{\|f\|_p\|g\|_{bmo}} = \frac{\|\langle f_\infty, g_\infty\rangle\|_p}{\|f\|_{H^p}\|g\|_{bmo}}$$

can be made arbitrarily big by picking δ sufficiently small, and hence indeed the range $p \in [2, \infty)$ cannot be expanded.

Next, we turn our attention to examples showing that the constant p in the estimates (1.2) and (1.3) is optimal. Fix $p \geq 2$ and let $\alpha < p^{-1}$, $\delta \in (0, 1)$ be auxiliary parameters.

Introduce the sequence $p_n = (1-2\delta)^n$, $n = 0, 1, 2, \ldots$, and consider the probability space $((0,1], \mathcal{B}(0,1), |\cdot|)$, where $|\cdot|$ stands for the Lebesgue's measure. We endow the space with the filtration $(\mathcal{F}_n)_{n\geq 0}$, where

$$\mathcal{F}_n = \sigma\left((0,p_n], \left(p_k, \frac{p_k+p_{k-1}}{2}\right], \left(\frac{p_k+p_{k-1}}{2}, p_{k-1}\right] \,:\, k = 1, 2, \ldots, n\right).$$

Next, consider the real-valued martingale f with the differences given by $df_0 = 0$ and

$$df_n = \begin{cases} \alpha(1+2\alpha\delta)^n & \text{on } ((p_n+p_{n-1})/2, p_{n-1}]\,, \\ -\alpha(1+2\alpha\delta)^n & \text{on } (p_n, (p_n+p_{n-1})/2]\,, \\ 0 & \text{elsewhere} \end{cases}$$

for $n = 1, 2, \ldots$. Note that the supports of the differences are pairwise disjoint, and hence in particular we have

$$S(f) = |f| = \alpha(1+2\alpha\delta)^n \quad \text{ on } (p_n, p_{n-1}] \tag{3.1}$$

almost surely. Finally, let g be the martingale generated by the sign of f: then is, we let $g_n = \mathbb{E}(\operatorname{sgn} f|\mathcal{F}_n)$, $n = 0, 1, 2, \ldots$. Obviously, g is a martingale bounded by 1, in particular, we have the estimate $\|g\|_{bmo} \leq 1$. We easily compute that $dg_0 = 0$ and

$$dg_n = \begin{cases} 1 & \text{on } ((p_n+p_{n-1})/2, p_{n-1}]\,, \\ -1 & \text{on } (p_n, (p_n+p_{n-1})/2]\,, \\ 0 & \text{elsewhere} \end{cases}$$

for $n = 1, 2, \ldots$. Directly from the above formulas for df and dg, we compute that

$$\begin{aligned} \mathbb{E}(|df_n||dg_n||\mathcal{F}_{n-1}) &= \begin{cases} \frac{1}{p_{n-1}} \cdot \alpha(1+2\alpha\delta)^n(p_{n-1}-p_n) & \text{on } (0, p_{n-1}], \\ 0 & \text{elsewhere}, \end{cases} \\ &= \begin{cases} 2\alpha\delta(1+2\alpha\delta)^n & \text{on } (0,p_{n-1}], \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$

and hence we have

$$\sum_{k=1}^{\infty} \mathbb{E}(|df_n||dg_n||\mathcal{F}_{n-1}) = (1+2\alpha\delta)^n - 1 = \alpha^{-1}|f| - 1 \quad \text{ on } (p_n, p_{n-1}].$$

Here in the last passage we have used the formula (3.1). We are ready to compare the L^p norms of $\sum_{k=1}^{\infty} \mathbb{E}(|df_n||dg_n||\mathcal{F}_{n-1})$ and f. Directly from (3.1), we check that

$$\|f\|_p^p = \sum_{n=1}^{\infty} \alpha^p(1+2\alpha\delta)^{np}(p_{n-1}-p_n) = \frac{2\alpha^p\delta}{1-2\delta}\sum_{n=1}^{\infty}\left[(1+2\alpha\delta)^p(1-2\delta)\right]^n.$$

Recall that $\alpha < p^{-1}$; consequently, if δ is sufficiently close to 0, then the above series is convergent. On the other hand, if α is taken close enough to p^{-1}, then the value of $\|f\|_p$ can be made arbitrarily large. Denoting by C_p the optimal constant in (1.5), we obtain

$$C_p \geq \frac{\|\sum_{k=1}^{\infty}\mathbb{E}(|df_n||dg_n||\mathcal{F}_{n-1})\|_p}{\|f\|_p} \geq \frac{\alpha^{-1}\|f\|_p - 1}{\|f\|_p}.$$

By the above discussion, the right-hand side can be made arbitrarily close to p. This gives the desired sharpness of (1.5). Since $S(f) = |f|$ almost surely, the constant p in (1.3) is also the best possible.

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