University of Warsaw

Faculty of Mathematics, Informatics and Mechanics

Łukasz Treszczotko

Functional limit theorems related to particle systems

PhD dissertation

Supervisor

dr hab. Anna Talarczyk-Noble
Institute of Mathematics
University of Warsaw

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Author's declaration:

I hereby declare that this dissertation is my own work.

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Łukasz Treszczotko

Supervisor's declaration:

The dissertation is ready to be reviewed.

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....
dr hab. Anna Talarczyk-Noble

Abstract

This thesis investigates functional limits of functionals related to systems of independently moving particles with weights/charges. A very large class of self-similar processes with stationary increments can be obtained as a scaling limit (spatial or temporal) of such systems, including fractional Brownian motion, the Rosenblatt process and an increasing number of processes with heavy-tailed finite-dimensional distributions.

The first part of the thesis establishes a particle picture interpretation for the asymmetric Rosenblatt processes and Hermite process of any order by considering functionals of intersection local times of the moving particles.

The second part is concerned with studying the behaviour of such systems when the weights attached to the particles have heavy-tailed distributions. The processes obtained as scaling limits correspond to a new class of stable H-sssi processes introduced recently by Samorodnitsky, et al.

The last part considers a discrete framework of random walks in random scenery. Therein, by introducing additional randomness, we are able to provide a natural way in which a certain class of stable H-sssi processes can be obtained, This is especially important, given that the aforementioned class was, in general, previously representable only in an abstract way.

Keywords: local times; Lévy processes; stable self-similar processes; random walks in random scenery; Hermite processes; Rosenblatt process; particle systems

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Streszczenie

Praca bada funkcjonalne twierdzenia graniczne związane z układami niezależnie poruszających się cząstek z dołączonymi wagami. Wiele procesów samopodobnych ze stacjonarnymi przyrostami można otrzymać jako obiekty graniczne przeskalowanych (czasowo lub przestrzennie) funkcjonałów tychże układów cząstek. Dotyczy to przykładowo ułamkowego ruchu Browna, procesu Rosenblatta i sporej liczby procesów o ciężko-ogonowych rozkładach brzegowych.

Pierwsza część pracy skupia się na stworzeniu cząsteczkowej interpretacji asymetrycznego procesu Rosenblatta i procesów Hermite'a dowolnego rzędu. Jako narzędzia użyte są tu czasy lokalne samoprzecięć trajektorii cząstek.

Druga część pracy dotyczy badania sytuacji, w której wagi dołączone do cząstek mają rozkłady o ciężkich ogonach. Otrzymane procesy graniczne odpowiadają nowym klasom stabilnych procesów samopodobnych wprowadzonych niedawno przez Samorodnitsky'ego i in.

Ostatnia część pracy poświęcona jest dykretnym modelom błądzenia losowego w losowym środowisku. W części tej, dodając dodatkowe źródło losowości, jesteśmy w stanie otrzymać naturalnie wyglądający model w którym procesy graniczne należą do klasy stabilnych procesów samopodobnych, która dotąd (w ogólności) została wprowadzona tylko w abstrakcyjny sposób.

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Contents

Part I

Introduction and background

Chapter 1

Introduction

The main theme of the thesis may be succinctly described as ”investigating scaling functional limit theorems for systems of moving particles with view to non-Gaussian and usually stable limits”.

1.1 Functional limit theorems

1.1.1 Sums of stationary sequences

Classical functional limit theorems investigate convergence in law, as $n \to \infty$, of the processes of the form

$$X_n(t) = \frac{1}{F_n} \sum_{k=1}^{\lfloor nt \rfloor} \xi_k, \quad t \geq 0, \tag{1.1.1}$$

where $(\xi_k))_{k\in\mathbb{Z}}$ is a stationary sequence of random variables and F_n is a suitable norming. More precisely, often one considers a continuous interpolation of X_n, that is the processes

$$\widetilde{X}_n(t) = X(\lfloor nt \rfloor) + (nt - \lfloor nt \rfloor)\big(X(\lceil nt \rceil) - X(\lfloor nt \rfloor)\big), \quad t \geq 0. \tag{1.1.2}$$

We will briefly recall the results in this case, even though in the thesis we are dealing with somewhat more complicated models related to particle systems. This is because we shall see similar phenomena arising in both cases. The classical Donsker theorem (see [5, Section 8]) states that if ξ_i are i.i.d. random variables with finite variance σ^2, then the suitable normalization is $F_n = \sigma\sqrt{n}$ and the processes $\widetilde{X}_n$ converge in $\mathcal{C}[0,\infty)$ to the standard Brownian motion. If ξ_k are i.i.d. random variables with infinite variance and are in the domain of attraction of α-stable laws, then $F_n = n^{\frac{1}{\alpha}}L(n)$, with L slowly varying at infinity, then $\widetilde{X}_n$ converge in law in $\mathcal{D}[0,\infty)$ to an α-stable Lévy process.

The picture becomes more complicated when the random variables ξ_i are dependent. There is a variety of different limits possible. These limit processes are usually self-similar, i.e., for some $H > 0$ and any $a > 0$ the process $(X(at))_{t\geq 0}$ has the same law as $(a^H X(t))_{t\geq 0}$. They also have stationary increments if (ξ_k) is stationary. We will use the abbreviation "H-sssi" to indicate that the process is self-similar with stationary increments.

Self-similar processes with stationary increments (see Section 2.1) are (by Lamperti's theorem) the only possible limits of normalized partial sums of stationary sequences. Whenever $(\xi_k)_{k\in\mathbb{Z}}$ is a stationary sequence of random variables the processes given by (1.1.1) converge in the sense of finite dimensional distributions for some normalizing sequence F_n which increases to $+\infty$, then the limit process must be an H-sssi process (see Theorem 8.1.5 in [42]).

The most famous of these *non-central functional-limit theorems* is perhaps the one in which *fractional Brownian motion* arises as a limit. Recall that fractional Brownian motion, with Hurst coefficient $H \in (0,1)$, is a centred Gaussian process B_H with covariance

$$\mathbb{E}(B_H(t)B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

Let $(\xi_n)_{n\in\mathbb{Z}}$ be a centred stationary Gaussian sequence with variance equal to 1 such that

$$r(n) := \mathbb{E}(\xi_n\xi_0) = n^{2H-2}L(n), \tag{1.1.3}$$

with $H \in (\frac{1}{2}, 1)$ and L - a function slowly varying at infinity. Then the processes (1.1.1) converge in law to a fractional Brownian motion with Hurst coefficient H (see [45])

Another important example is when one considers a stationary sequence $(\rho_n)_{n\in\mathbb{Z}}$ with

$$r(n) := \mathbb{E}(\rho_n\rho_0) = n^{\frac{2H-2}{k}}L(n) \tag{1.1.4}$$

and in (1.1.1) considers the sequence $(\xi_n)_{n\in\mathbb{Z}}$, with $\xi_n = (\rho_n^2 - 1)$, $n \in \mathbb{Z}$. The result which is originally due to Taqqu ([45]) states that, with $F_n = n^H$, the corresponding limit process is the so called *Rosenblatt process* which is non-Gaussian but has the same covariance function as the fractional Brownian motion. This is a self-similar process with stationary increments living in the so called *second*

Wiener chaos. See Section 1.2.2 for details. These types of processes are still actively investigated (see for instance [3] or [4]).

Another important research direction deals with the case when ξ's are heavy tailed. The possible limits form a very wide family of self-similar processes. In particular, so called *linear fractional stable motions* arise as limits of moving average processes with noise sequence given by a stationary sequence (ξ_k) with heavy-tailed marginal distributions (see [42, Proposition 9.5.7]).

1.1.2 Functional limit theorems related to particle systems

A different class of problems in which functional limit theorems were investigated relates to particle systems. Similar types of limits appear in that context, although the picture is much less complete. The present thesis answers some of the questions in that framework.

Deuschel and Wang in [14] studied the following model: suppose that at time 0 we have a system of particles in $\mathbb{R}^d$ whose positions are determined by a Poisson random measure with Lebesgue intensity measure, then each of the particles moves independently according to a Brownian motion. The positions of the particles at time t are described by the empirical process N_t, such that for $A \in \mathcal{B}(\mathbb{R}^d)$ $N_t(A)$ is the number of particles in A at time t. Deuschel and Wang investigated the limit of rescaled occupation time, i.e.,

$$X_T(t) = \frac{1}{F_T} \int_0^{Tt} (N_s - \mathbb{E}N_s) ds, \tag{1.1.5}$$

where $T \to \infty$ and F_T is an appropriate normalization. They showed that for $d = 1$ and any "sufficiently nice" test function ϕ the functional X_T evaluated at ϕ (denoted by $\langle X_T, \phi \rangle$), converges in law in $\mathcal{C}[0, \infty)$ to a fractional Brownian motion with Hurst parameter $H = \frac{3}{4}$. This result has been later extended to the case where the particles were moving independently according to symmetric α stable Lévy processes. In [6] and [10] the processes of the form (1.1.5) were considered in the space of tempered distributions. In particular, it was shown that if $d < \alpha$, then for any $\phi \in \mathcal{S}(\mathbb{R}^d)$-Schwartz space of smooth functions decreasing rapidly at infinity, the processes $(\langle X_T(t), \phi \rangle)_{t \geq 0}$, converge to a fractional Brownian motion with Hurst paramter $H = 1 - \frac{1}{2\alpha}$, while if $d \geq \alpha$, then the limit is a Brownian motion (up to a multiplicative constant). Bojdecki, Gorostiza and Talarczyk in their series of papers [7], [9], and others concentraded principally on branching systems (the moving particles could die or give birth to new particles). It was shown that, depending on the parameters of the system in question, the limiting object of the process $(\langle X_T(t), \phi \rangle)$ can be either a Brownian motion or a stable Lévy process or a self-similar process with dependent (and usually non-stationary) increments. The latter could be a Gaussian or a stable process. In their model stable limit appeared due to branching when the distribution of offspring had infinite variance. Analogous models, considering both branching particle systems as well as superprocesses were studied by other authors e.g. [26], [32], [39] and [40].

Another result of Bojdecki Gorostiza and Talarczyk [8] which will be of importance to us gives a particle picture representation of the Rosenblatt process. Therein the Rosenblatt process arises as a limit of a functional involving *intersection local times* of a system of particles evolving according to α-stable Lévy processes. This result can be considered as particle system analog of the result of Taqqu regarding the Rosenblatt process in [45] described in the previous section.

Recently, other functonal limit theorems regarding scaling limits of dynamical systems (see [35], [20]) and particle systems (see [12]) were proved resulting in some new interesting stable self-similar processes.

1.1.3 Motivation

Our research presented in this dissertation concerns limit theorems related to particle systems in the spirit of the results described in Section 1.1.2. The motivation was two-fold: on the one hand the aim was to get a better understanding of some particle systems, e.g. a natural extension of the simple Poisson system particles moving according to α-stable lévy processes, from [30], [14] and [6], where additionally each of the particles has a certain weight, which can have a heavy-tailed distribution. On the other hand, in view of the fact that the processes such as Brownian motion, fractional Brownian motion, Lévy stable processes and the Rosenblatt process turned out to arise as limits of certain functionals of particle systems, we wanted to see what other interesting classes of processes can be obtained from particle systems in a reasonably natural way.

The problems that we study in this thesis split into three groups:

1. ”particle picture” interpretations of self-similar processes with stationary increments which live in Wiener chaoses, such as non-symmetric Rosenblatt process and Hermite processes (Chapter 3),
2. particle picture interpretations of stable self-similar processes described in [15], [35] and [20] obtained as limits of systems of weighted moving particles (Chapter 4),
3. a scaling limit in terms of systems of random walks in random scenery leading to a stabel self-similar process with stationary increments first described in [20] in an abstract way, thus giving an interpretation to that process (Chapter 5).

Presently we give a short description of our results

1.2 Overview of the results

In this section we provide a brief overview of the results of our research. In order to make it transparent and concise we do not introduce the full terminology and background, which we provide in Chapter 2. Here we concentrate on the most important results we have obtained.

1.2.1 Particle systems

In chapter 3 and 4 we will be using a particular version of a system with initial positions of particles given by a Poisson random measure and their movement governed by Lévy processes. The system can be intuitively described as follows. We have a countable number of particles in $\mathbb{R}$ whose initial positions are given by points of a Poisson random measure with Lebesgue intensity. As time goes by they move in space independently according to some predefined Lévy process. To each of the particles we independently assign random variables, which may be interpreted as weights or charges. This makes the system much more heterogeneous and allows for more complexity. As the particles evolve we observe their behaviour indirectly through some observable or statistic. For instance we may look at the number of particles in a specific set $A \subset \mathbb{R}$ at any given time. We may also choose some function $\phi : \mathbb{R} \to \mathbb{R}$ and consider the functional of the form

$$X(\phi) := \sum_j z_j \int_0^t \phi(x_j + \eta_s^j) ds, \quad t \geq 0,$$

where z_j are the weights/charges, x_j are the starting points and η_j are the processes determining the movement of the particles. This corresponds to the occupation time process evaluated on a test function ϕ, i.e. $\int_0^t \langle N_s, \phi \rangle ds$, in the setting of [6] in the case without branching and with $z_j \equiv 1$,

Thus, we track the movement of the system as time goes by. Considering the possible limits of the functionals of the above form turns out to be very fruitful and leads to appealing representations of a number of important stochastic processes.

The particle system described here is a variation of a model (without branching) considered by Bojdecki et al. in [6], [12] and [8], except that in the papers mentioned above, we have either $z_j \equiv 1$ (and the corresponding processes have to be centred) as in [12] and [6] or z_j are i.i.d. Bernoulli random variables with $\mathbb{P}(z_j = 1) = \mathbb{P}(z_j = -1) = \frac{1}{2}$ as in [8].

In Chapters 3 and 4 of the thesis we will consider the following system. Let (x^j) be a Poisson system with Lebesgue intensity measure on $\mathbb{R}$ and let $(\eta^j)_{j=1}^{\infty}$ be independent symmetric α-stable Levy processes with the index of stability $\alpha \in (0,2)$. Assume that these processes are independent of the points (x^j). Finally, assume that $(z_j)_{j=1}^{\infty}$ are i.i.d random variables independent of everything else, where z_j represents the signed weight of j-th particle. The particle system is given by $(x^j + \eta_t^j)_{t\geq 0, j=1,2,\ldots}$ and $(z_j)_{j=1,2,\ldots}$. Thus the initial position of the particles is given by the points (x^j) and they evolve independently according to the symmetric α-stable processes. In some cases we allow the particle motions to be governed by a more general Lévy process.

1.2.2 Representation of the non-symmetric Rosenblatt process and Hermite processes

For $0 < H < 1$ there is only one (up to a multiplicative factor) H-self-similar Gaussian process with stationary increments, namely the fractional Brownian motion B_H. Due to this fact, in many models it appears in a natural way as a limit process in some limiting procedures. It is a member of a larger family of *Hermite processes*. For $k \geq 1$ a k-Hermite process is an H-sssi process which "lives" in the k-th Wiener chaos and can be represented with the help of multiple Wiener-Itô integrals:

$$Z_H^k(t) := \int_{\mathbb{R}^k}' \int_0^t \prod_{j=1}^k (s - x_j)_+^{d-1} ds W(dx_1) \ldots W(dx_k), \tag{1.2.1}$$

where W is a two-sided Brownian motion, $\frac{1}{2}(1 - \frac{1}{k}) < d < \frac{1}{2}$ is the number satisfying $H = kd - k/2 + 1$ so that $1/2 < H < 1$. Other possible representations are given in Section 3.1.1.

k-Hermite processes arise as functional limits of the processes of the form (1.1.1) once we choose $\xi_j = H_k(\rho_j)$, where H_k is the k-th Hermite polynomial and (η_j) is a stationary Gaussian sequence satisfying

$$r(n) := \mathbb{E}(\rho_n \rho_0) = n^{\frac{2H-2}{k}} L(n), \tag{1.2.2}$$

with $H \in (\frac{1}{2}, 1)$, $k \geq 1$ and L - a function slowly varying at infinity.

Recently in [8] it was shown that the Rosenblatt process can be obtained from a Poisson system of particles as in Section 1.2.1, where z_j are random signs (denoted by σ_j in [8]): (σ_j) is an i.i.d. sequence such that σ_j takes values 1 and -1 with equal probability and the processes η^j are α-stable Lévy processes with the index of stability $\alpha \in (\frac{1}{2}, 1)$. It was shown that the process given by

$$\xi_t^T = \frac{1}{T} \sum_{j \neq k} \sigma_j \sigma_k \langle \Lambda(x^j + \eta^j, x^k + \eta^k; T), \mathbf{1}_{[0,t]} \rangle, \quad t \geq 0, \tag{1.2.3}$$

where Λ is the *2-intersection local time* of two independent α-stable Lévy processes (see section 2.3 in [8], [1]), converges, as $T \to \infty$, (up to a constant) for $\alpha \in (1/2, 1)$, in $\mathcal{C}[0, \tau]$ for $\tau \in (0, \infty)$, to the Rosenblatt process with the Hurst coefficient $H = \alpha$. Informally k-intersection local time of cadlag processes $\rho^1, \ldots, \rho^k$ at time $T \geq 0$ (denoted here by $\Lambda^{(k)}$) can be defined by

$$\left\langle \Lambda^{(k)}(\rho^1, \ldots, \rho^k; T), \phi \right\rangle = \\ = \int_{[0,T]^k} \phi(\rho^1_{s_1}) \delta_0(\rho^2_{s_2} - \rho^1_{s_1}) \ldots \delta_0(\rho^k_{s_k} - \rho^1_{s_1}) ds_1 \ldots ds_k, \tag{1.2.4}$$

where δ_0 is the Dirac distribution at 0 and $\phi \in \mathcal{S}$ (the Schwartz space of rapidly decreasing function). One gives a meaning to (1.2.4) by approximating δ_0 by smooth functions, see e.g. [44].

We extend this result to all k-Hermite processes by showing (see Theorem 3.2.3) that if for $k \geq 2$, $\alpha \in (1 - \frac{1}{k}, 1)$ and $T > 0$ we denote

$$\rho_t^T := \frac{1}{T^{k/2}} \sum_{j_1 \neq \ldots \neq j_k} \sigma_{j_1} \ldots \sigma_{j_k} \langle \Lambda^{(k)}(x^{j_1} + \xi^{j_1}, \ldots, x^{j_k} + \xi^{j_k}; T), \mathbf{1}_{[0,t]} \rangle, t \geq 0, \tag{1.2.5}$$

then, the process ρ^T has a continuous modification and as $T \to \infty$, it converges in law in $\mathcal{C}[0, \tau]$ for every $\tau > 0$, to the k-Hermite process Z_H^k with the Hurst coefficient equal to $H = 1 - (1 - \alpha)k/2$. In the proof we rely on the toolbox of spectral multiple Wiener-Itô integrals (see Section 2.5.5), which is a different path from the one chosen by the authors in [8], who used the method of moments.

It was not at first obvious whether or not Hermite processes were the only self-similar processes with stationary increments in their respective Wiener chaoses. The only H-sssi process in the first Wiener chaos is the fractional Brownian motion (see Theorem 1.3.3 in [16]). This is not true for Wiener chaoses of order $k \geq 2$ and the first example of this fact was the *non-symmetric Rosenblatt process* (see [27] and [49] for an introduction to this process), which is obtained if we replace the kernel $\prod_{j=1}^k (s - x_j)_+^{d-1}$ in equation (1.2.1) (for $k = 2$) by

$$g(x, y) = x^{-1+\alpha/2} y^{-1+\beta/2} \mathbf{1}_{\{x>0, y>0\}}, \tag{1.2.6}$$

with $\alpha, \beta \in (0, 1)$ and $\alpha + \beta > 1$. Even in this relatively simple case with $k = 2$, $\alpha, \alpha', \beta, \beta' \in (0, 1)$ and $\alpha + \beta = \alpha' + \beta' > 1$ the corresponding non-symmetric processes have different laws for different choices of α, α', β and β' (see Proposition 3.10 in [49]). More generally, the initial kernel can be replaced by even more general functions to obtain the so called *generalized Hermite processes* introduced and investigated in [3].

In Theorem 3.2.2 we prove that if we consider a different functional of the same particle system, i.e.,

$$\zeta_t^T = \frac{1}{T}\sum_{j\neq k}\sigma_j\sigma_k\int_0^T\int_0^T \mathbf{1}_{[0,t]}(x^j+\eta_r^j)\frac{1}{|x^k+\eta_s^k-x^j-\eta_r^j|^{1-\frac{\beta-\alpha}{2}}}drds \quad (1.2.7)$$

for $T>0, t\geq 0$ and α and β such that $1>\beta>\alpha>0$, $\alpha+\beta>1$, then, as $T\to\infty$ the processes η^T converge, in $\mathcal{C}[0,\tau]$ for every $\tau>0$, to the non-symmetric Rosenblatt process with parameters (α,β), up to a multiplicative constant.

1.2.3 Infinite variance H-sssi processes as limits of particle systems and random walks in doubly random scenery

In recent years a number of stable non-Gaussian H-sssi processes were introduced which further enrich the already vast number of such processes investigated by researchers. They often arise quite naturally as scaling limits of (spatial or temporal) of dynamical systems (see [35] and [20] for example) or particle systems (see [7]). We will briefly describe the main results of these papers in Section 4.3. One of our research interests was evidently in that direction. It resulted in two publications: [47] and [46] which are the content of chapters 5 and 4, respectively. In our approach we obtain in the limit the processes (1.2.10) and (1.2.11), thus obtaining their particle picture interpretation, as well as obtain some new self-similar stable processes.

A wide range of stable self-similar processes has the integral representation of the form

$$X_t = \int_S f_t(x)M(dx), \quad t\in T, \quad (1.2.8)$$

where T is some index set, $(S,\mathcal{S},\mu)$ is a measure space, $f_t: S\to\mathbb{R}$ are deterministic functions and M is a stable random measure as described in Section 2.3.2. Generally speaking the random measure M is responsible for how heavy the tails of the marginal distributions of the process X are and the family of functions $(f_t)_{t\in T}$ determines its dependence structure.

In chapter 4 we consider a system of particles as in 1.2.1 where the particles (x^j) are drawn according to a Poisson random measure on $\mathbb{R}$ with Lebesgue intensity, the movements of the particles are given by i.i.d. Lévy processes (η^j). The quantities that we study are given by the functional

$$G_t^T = \frac{1}{F_T}\sum_j z_j\int_0^{Tt}\phi(x^j+\eta_u^j)du, \quad T>0, \quad (1.2.9)$$

where F_T is some appropriate norming, z_j are i.i.d. *weighted charges* attached to the particles and ϕ is some integrable test function. We were interested in functional limit theorems for processes of the form (1.2.9) in the regime in which the weights z_j are *heavy tailed*. The results that we have obtained are fairly general and we had to overcome a number of technical difficulties to make them so (see Section 4.6 for details). However, in this introductory note we will provide a simplified version which, nonetheless, provides a good overview of the type of limits obtained.

Suppose at this moment that the particle motions are given by symmetric β-stable Lévy processes with $1 < \beta < 2$ and that the common distribution of (z_j) has, up to a multiplicative constant, density of the form $|z|^{-1-\alpha}\mathbf{1}_{\{|z|>1\}}dz$, for $1 < \alpha < 2$. The limit processes depend mainly on two properties of ϕ:

- whether $\int_{\mathbb{R}} \phi(y)dy = 0$ or not;
- how does $\phi(y)$ behave as $y \to \pm\infty$.

First order limit theorem (see Theorem 4.6.1)

If $\int_{\mathbb{R}} \phi(y)dy \neq 0$, then under normalization $F_T = T^{1-\frac{1}{\beta}+\frac{1}{\alpha\beta}}$

$$G^T \xrightarrow{\mathcal{C}[0,\infty)} K \int_{\mathbb{R}} \phi(y)dy \times X,$$

where K is a positive constant and X is the process given by

$$X = \left(\int_{\mathbb{R}\times\Omega'} L_t(x,\omega')M_\alpha(dx,d\omega') \right)_{t\geq 0}, \tag{1.2.10}$$

where $(L_t(x,\omega'))_{t\geq 0, x\in\mathbb{R}}$ is a jointly continuous version of the local time of the β-stable Lévy process, with $\beta \in (1,2)$ (defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$) and M_α is a symmetric α-stable random measure on $\mathbb{R}\times\Omega'$ with control measure $\lambda_1 \otimes \mathbb{P}'$, which is itself defined on some other probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Second order limit theorems

If $\int_{\mathbb{R}} \phi(y)dy = 0$, to obtain a non-trivial limit of G^T one has to take a larger normalization. Also, the situation is more complicated, as the behaviour of G^T depends on the rate of decay of ϕ at $+\infty$ and $-\infty$.

Case 1 (see Theorem 4.6.2)

If ϕ is sufficiently quickly vanishing at infinity, then, with the normalization $F_T = T^{\frac{\beta-1}{2\beta}+\frac{1}{\alpha\beta}}$, we have

$$G^T \overset{f.d.d.}{\Longrightarrow} K_1(\phi)Y,$$

where Y is given by

$$(Y(t))_{t\geq 0} = \left(\int_{\Omega'\times\mathbb{R}} W(L_t(x,\omega'),\omega')M_\alpha(d\omega',dx) \right)_{t\geq 0}, \tag{1.2.11}$$

where everything is as in the previous case with additional randomness provided by a Brownian motion W defined on $(\Omega', \mathcal{F}', \mathbb{P}')$ independent of the local time $(L_t)_{t\geq 0}$. The precise condition on ϕ are given in Section 4.4 but it suffices e.g. if

$$\int_{\mathbb{R}} |\phi(y))||y|^\kappa dy < \infty$$

for some $\kappa > \frac{\beta-1}{2}$.

Case 2 (see Theorem 4.6.4)

If ϕ vanishes relatively slowly at infinity, then the limit of (1.2.9) depends on finer behaviour of ϕ at $+\infty$ and $-\infty$. Suppose that $\phi(x) \sim \frac{c}{|x|^\gamma}$, where $1 < \gamma < 1 + \frac{\beta-1}{2}$. Then, for $F_T = T^{1+\frac{1}{\alpha\beta}-\frac{\gamma}{\beta}}$

$$G^T \overset{f.d.d.}{\Longrightarrow} K_2(\phi)V,$$

where $K_2(\phi)$ is a positive constant and V is given by

$$V_t = \int_{\mathbb{R}\times\Omega'} Z_t(x,\omega')M_\alpha(dx,d\omega'), \tag{1.2.12}$$

where M_α as before and

$$Z_t(x,\omega') = \int_0^\infty |y|^{-\gamma} \left(L_t(x+y,\omega') - L_t(x-y,\omega')\right) dy, \quad x \in \mathbb{R}, \tag{1.2.13}$$

with $(L_t)_{t\geq 0}$ also as in the previous two cases. There are also other cases in which on of the tails of ϕ dominates the other, which lead to similar results with a modified prozess Z. Details can be seen in Section 4.6.2. The processes (Z_t) is a fractional derivative of stable local time and its properties were extensively studied in [17]. The process V appears to be new.

1.2.4 Random walks in doubly random scenery

Description and known results

The third part of the thesis is closely related to the second one, inasmuch as the limit processes obtained belong to the same class. The methodology and modelling approach are, however, discrete, and therefore quite different both in their use and interpretation.

The models that we consider in this part are in the vein of *random walks in random scenery* models going back to the paper [22] by Kesten and Spitzer. These models consider a system of random walks (often called *users*) moving on $\mathbb{Z}^d$, $d \in \mathbb{N}$ and collecting *rewards* (ξ_k) that are attached to the points of the lattice $\mathbb{Z}^d$. The user earns random *rewards* (governed by the random scenery) associated to the points in the network that they visit. Thus, models of this form should have a lot in common with the particle systems with weights that we have described in Section 1.2.1.

The quantity of interest is then the total amount of rewards collected. If we denote the random walk on $\mathbb{Z}$ by $(S_n)_{n\geq 0}$, and the scenery by $(\xi_k)_{k\in\mathbb{Z}}$, then the total sum of rewards collected by time $n \in \mathbb{N}$ is given by

$$Z_n := \sum_{k=1}^{n} \xi_{S_k}. \tag{1.2.14}$$

In [22] the authors assume that $S_n = X_1 + \ldots + X_n$ is such that the X_i are i.i.d. and belong to the normal domain of attraction of a β-stable law with $\beta \in (1, 2]$. In particular the linearly interpolated version of S_n (as in (1.1.2)) converges in law in $\mathcal{D}[0, \infty)$ to a β-stable Lévy process Y.

The ξ_i are i.i.d., independent of the X_i and in the normal domain of attraction of an α-stable law and as in the previous paragraph the linearly interpolated version of their partial sums converges, in law in $\mathcal{D}[0, \infty)$, after rescaling to an α-stable Lévy process W.

Then they prove that the appropriately rescaled and linearly interpolated process $(Z_{[nt]})_{t\geq 0}$ converges in $\mathcal{C}[0, \infty)$, as $n \to \infty$ to the process $(\Delta_t)_{t\geq 0}$, which has the integral representation given by

$$\Delta_t = \int_{\mathbb{R}} L_t(x) W(dx), \quad t \geq 0,$$

where L_t is the jointly continuous version of the local time of the process Y. Note that the process Δ is not a stable process.

Following the above result, [15] the authors considered a scaling limit of a large number of independent users collecting rewards from independent sceneries. The scaling limit in the corresponding functional limit theorem (see Theorem 1.2 in [15]) leads to the process that we have already described, namely (1.2.10).

Motivation and results

In [20] the authors have introduced a new class of stable H-sssi processes which can be represented as

$$(Y_{\alpha,\beta,\gamma}(t))_{t\geq 0} = \left(\int_{\Omega'\times\mathbb{R}} S_\gamma(L_t(x,\omega'),\omega')dZ_\alpha(\omega',x) \right)_{t\geq 0}, \tag{1.2.15}$$

where S_γ is a symmetric γ-stable Lévy process, $(L_t(x))_{t\geq 0}$ is the local time of a symmetric β-stable Lévy process independent of the process S_γ (both defined on $(\Omega', \mathcal{F}', \mathbb{P}'))$, and Z_α is a symmetric α-stable random measure on $(\Omega', \mathbb{R})$ with control measure $\mathbb{P}' \otimes \lambda_1$. The parameters α, β and γ satisfy $0 < \alpha < \gamma \leq 2, \beta \in (1, 2]$. The authors have obtained only $Y_{\alpha,\beta,2}$ as a limit of their model. Given that the process (1.2.10) can be obtained as a limit of a fairly intuitive aggregation of discrete structures, we have conjectured that the same may be true of the more complicated objects of the form (1.2.15). This turns out to be true and we have managed to provide a relatively uncomplicated scenario in which it happens.

We briefly describe the setting of our result and its formulation. The model is similar to the one introduced in [15] with the caveat that we introduce an additional source of randomness. We consider only the simple one-dimensional integer lattice. In our model, each user generates a sequence $Y_1, Y_2, \ldots$ of i.i.d. random variables which are independent of the ξ_x's and his movement. Now, any time the walker visits a point $x \in \mathbb{Z}$ he/she gets a reward (or receives punishment) given by $Y_k \times \xi_x$, where k is number of times that the walker has already stayed at x (including the current visit). Thus, the amount by which a potential reward is being multiplied depends only on the number of the visits. The total reward/punishment at time n in this scheme is given by

$$\sum_{x\in\mathbb{Z}} \Big(\sum_{k=1}^{N_n(x)} Y_k \Big) \xi_x, \tag{1.2.16}$$

where

$$N_n(x) := \sum_{k=1}^{n} \mathbf{1}_{\{S_k = x\}} \tag{1.2.17}$$

denotes the number of visits to the point $x \in \mathbb{Z}$ up to time $n \in \mathbb{N}$.

We are interested in the scaling limit in which we consider the aggregate behaviour of a large number of independent walkers with independent strategies and having independent environments from which they collect the rewards. We consider an i.i.d. sequence of processes $\big((D_n^{(i)}(t))_{t\geq 0}\big)_{i=1}^{\infty}$, $n \geq 1$ and define for $t \geq 0$

$$G_n(t) := \frac{1}{c_n^{1/\alpha}} \sum_{i=1}^{c_n} D_n^{(i)}(t), \quad n \geq 1, \tag{1.2.18}$$

where c_n is any sequence of positive integers. We prove (see Theorem 5.2.2) that under certain assumptions (see Section 5.2) for any $0 < \alpha < \gamma \leq 2$ the process $(G_n(t))_{t \geq 0}$ defined by (1.2.18) converges (up to a multiplicative constant) as $n \to \infty$, in the sense of finite-dimensional distributions, to the process given by (1.2.15).

1.3 Organization of the thesis

The thesis is split into two parts. The first one provides an introduction to the subject matter of the thesis, delineates main research areas and objectives, gives a brief overview of the most important results. Moreover, it includes a relatively self-contained overview of the main mathematical tools and concepts used. The second part contains the precise formulation of all results and their proofs.

In Chapter 1 we provide a brief overview of the main results of our research with some introductory remarks needed to understand our motivations and interests.

In Chapter 2 we delineate the most important technical tools that we use throughout the thesis so that the whole exposition is as self-contained as it reasonably can be. We first fix the notation, then give a brief introduction to self-similar processes, stable random variables, Lévy processes, local times, infinitely divisible random measures and integrals with respect to these measures. Finally we provide some technical results relating to tightness criteria for cadlag processes and some basic tools connected to regular variation.

Chapter 3 is the content of our first major research direction which was aimed at providing a representation for the non-symmetric Rosenblatt process and Hermite processes of any order. In the introduction to this section we provide additional material regarding, among other things, Wick polynomial generalized random fields, random Gaussian spectral measures and integrals with respect to these measures. We then go on to state the main results and prove them rigorously.

Chapter 4 we provide a particle picture representation of a number of new classes of stable self-similar processes with stationary increments introduced by Samorodnitsky et al. in [35] and [20]. It starts with a description of the frameworks in which the above-mentioned classes arose, then describes our own with a number of theorems providing alternative and intuitively appealing models in which they may arise. Some additional results concerning occupation times of stable Lévy processes are also presented and later proven.

Chapter 5 is intimately realted to Chapter 4 as the limit processes obtained belong to the same class of processes. However, it uses discrete structures to arrive at those limits. It presents a random walk in random scenery limit representation of a class of stable processes introduced in [20]. It manages to provide a natural interpretation this class which was previously, in general, only representable in an abstract way.

Some of the four main sections described above are followed by appendices in which some additional technical results are proved.

Chapter 2

Notation and background

In this section we fix basic notation and recall some of the concepts and theorems later used in the main part of the thesis, in particular random measures and integrals with respect to them, multiple Wiener-Itô integrals, random spectral measures, weak convergence of processes and regular variation.

Nomenclature

$\mathcal{B}(S)$	Borel σ-field on a metric space S
$\mathcal{S}$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}$
$\mathcal{S}(\mathbb{R}^d)$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$
$\mathcal{S}^c$	the Schwartz space of complex-valued rapidly decreasing functions
$\overset{\mathcal{C}[0,\tau]}{\Rightarrow}$	convergence in law in the space of continuous on $[0,\tau]$ functions with supremum norm
$\overset{f.d.d.}{=}$	equality in the sense of finite-dimensional distributions
$\overset{f.d.d.}{\Rightarrow}$	convergence of finite-dimensional distributions
$\widehat{\phi}$	the Fourier transform of a function ϕ, $\widehat{\phi}(x) := \int_{\mathbb{R}^d} e^{i\langle x,y\rangle}\phi(y)dy$
$i.i.d$	independent and identically distributed

2.1 Stationarity, self-similarity and infinite divisibility

2.1.1 Self-similar processes and stationarity

Definition 2.1.1. *A stochastic process* $(X_t)_{t\in\mathbb{R}}$ *is* stationary *if for any* $h \in \mathbb{R}$

$$(X_{t+h})_{t\in\mathbb{R}} \overset{f.d.d.}{=} (X_t)_{t\in\mathbb{R}}. \tag{2.1.1}$$

Definition 2.1.2. *A stochastic process* $(X_t)_{t\geq 0}$ *has* stationary increments *if for any* $h > 0$

$$(X_{t+h} - X_h)_{t\geq 0} \overset{f.d.d.}{=} (X_t - X_0)_{t\geq 0}. \tag{2.1.2}$$

Definition 2.1.3. *A stochastic process* $(X_t)_{t\geq 0}$ *is* self-similar *if there exists an* $H > 0$ *such that for all* $c > 0$,

$$(X_{ct})_{t\geq 0} \stackrel{f.d.d.}{=} c^H (X_t)_{t\geq 0}. \tag{2.1.3}$$

The parameter H *is called the* Hurst exponent. *We also say that* X *is* H-self-similar *(or* H-ss *for short). If, additionally, the process* X *has stationary increments, we then say that it is* H-sssi.

The intuitive understanding of self-similarity can be briefly sketched as follows. Whenever a stochastic process X is H-self-similar, rescaling time by a factor of $c > 0$ and, at the same time, rescaling space by a factor c^H will result in a process which looks statistically the same. The classic example is that of Brownian motion for which the Hurst exponent is equal to $1/2$.

Let us recall some basic properties of H-*sssi* processes (see [37, Proposition 2.5.6]).

Proposition 2.1.4. *Let* $X = (X_t)_{t\geq 0}$ *be an* H*-sssi process. Then the following properties hold:*

(i) $X_0 = 0$ *a.s.,*

(ii) If $H \neq 0$, $H \neq 1$ *and* $\mathbb{E}|X_1| < \infty$, *then* $\mathbb{E}X_t = 0$, *for all* $t \geq 0$,

(iii) If $H = 1$ *and* $\mathbb{E}|X_1| < \infty$, *then* $X_t = t$ *a.s. for all* $t \geq 0$,

(iv) If $\mathbb{E}|X_1|^2 < \infty$, *then* $\mathbb{E}|X_t|^2 = t^{2H}\mathbb{E}|X_1|^2$, *for all* $t \geq 0$,

(v) If $\mathbb{E}|X_1|^2 < \infty$, *then, for* $0 \leq s, t < \infty$,

$$cov\,(X_t X_s) = \frac{\mathbf{Var}(X_1)}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \tag{2.1.4}$$

(in particular this means that in this case we must have $H \leq 1$ *since otherwise the right-hand side of* (2.1.4) *is not non-negative definite).*

If X is a centred Gaussian H-sssi process, then X must be a fractional Brownian motion since the covariance structure uniquely determines the finite-dimensional distributions for a centred Gaussian process. If we do not make the gaussianity assumption, then there is large number of H-sssi processes, from Hermite processes to stable H-sssi processes, which will be described in a more detailed fashion in 3.1.1 and 2.3.2, respectively.

2.1.2 Infinite divisibility

Now, let us introduce the notion of *infinite divisibility*, which needs to be mentioned to make our presentation complete.

Definition 2.1.5. *A random variable X is infinitely divisible if for every $n \in \mathbb{N}$ there exist i.i.d. random variables $Y_1^n, \dots, Y_n^n$ such that*

$$X \stackrel{d}{=} Y_1^n + \ldots + Y_n^n.$$

If we denote the distribution of X by μ_X and its characteristic function by $\phi_X(\cdot)$, then the following are equivalent (see [2, Proposition 1.2.6]):

(i) X is infinitely divisible;

(ii) For every $n \in \mathbb{N}$ there exists a probability measure μ_n such that

$$\mu_X = \mu_n^{*^n},$$

where $*$ denotes convolution of measures;

(iii) For every $n \in \mathbb{N}$ ϕ_X has an n-th root, i.e., there exists a function $\phi_{X,n}(\cdot)$, with $\phi_X(\cdot) = (\phi_{X,n}(\cdot))^n$, such that $\phi_{X,n}$ is itself a characteristic function.

Usually the easiest way to determine whether a random variable is infinitely divisible is to check the condition (iii) above. Examples of infinitely divisible random variables include Gaussian and Poisson random variables. Stable random variables discussed in Section 2.2.2 are also infinitely divisible.

Full characterization of infinitely-divisible random variables is provided by the famous Lévy-Khinchine theorem which we formulate for completeness.

Theorem 2.1.6 (Lévy-Khinchine)**.** *A random variable X with values in $\mathbb{R}^d, d \geq 1$ is infinitely divisible if and only if there exists $a \in \mathbb{R}^d$, a positive-definite symmetric matrix Σ and a measure ν on $\mathbb{R}^d \setminus \{0\}$, satisfying*

$$\int_{\mathbb{R}^d} \left(1 \wedge |z|^2\right) \nu(dz) < \infty, \tag{2.1.5}$$

such that for for all $\theta \in \mathbb{R}^d$ we have

$$\mathbb{E} \exp\left(i\langle \theta, X\rangle\right) = \exp\left(-\psi(\theta)\right), \tag{2.1.6}$$

where

$$\psi(\theta) = -i\langle a, \theta\rangle + \frac{1}{2}\langle \theta, \Sigma\theta\rangle - \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\langle \theta, z\rangle} - 1 - i\mathbf{1}_{\{|z|\leq 1\}}\langle \theta, z\rangle\right) \nu(dz). \tag{2.1.7}$$

The measure ν is usually called the Lévy measure *of X.*

For reference see [2, Theorem 1.2.14]. The triple (a, Σ, ν) is called the *characteristic triple* of X. The function $z \mapsto \mathbf{1}_{\{|z|\leq 1\}}\langle \theta, z\rangle$ may be replaced by other bounded functions which near 0 behave as $\langle \theta, z\rangle$. This changes only the value of a.

Definition 2.1.7. *A stochastic process $(X_t)_{t\in T}$, where T is some index set, is said to be* infinitely divisible *if for every integer $n \geq 0$, there exists a stochastic process $(Y_t)_{t\in T}$ such that*

$$(X_t)_{t\in T} \stackrel{d}{=} \left(\sum_{k=1}^{n} Y_t^{(k)}\right)_{t\in T},$$

where $((Y_t^{(k)})_{t\in T})_{k\in\mathbb{N}}$ are i.i.d. copies of Y.

Similarly as in the case of infinitely divisible random variables we have a Lévy-Khinchine representation of the characteristic function of an infinitely divisible process X which says that to every infinitely divisible stochastic process there corresponds a unique characteristic triple (a, Σ, ν) as in Theorem 2.1.6, such that for $t \geq 0$

$$\mathbb{E}\exp(i\theta X_t) = \exp(-t\psi(\theta), \quad \theta \in \mathbb{R}, \tag{2.1.8}$$

where ψ is as in 2.1.7 (see Corollary 2.4.20 in [2]).

2.2 Lévy processes stable random variables and stable processes

2.2.1 Lévy processes

The brief introduction to Lévy processes presented in this section follows [2].

Definition 2.2.1. *Let $X = (X_t)_{t\geq 0}$ be a stochastic process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that X is a* Lévy process *if the following three conditions are satisfied:*

(1) $X_0 = 0$ almost surely;

(2) X has independent and stationary increments;

(3) X is continuous in probability, i.e., for every $\epsilon > 0$ and $s, t \geq 0$

$$\lim_{t\to s} \mathbb{P}(|X_t - X_s| > \epsilon) = 0.$$

It can be shown that such an X always has a cadlag version and from now on we always take such a version. The marginals of Lévy processes are in a one-to-one correspondence with infinitely divisible distributions. More precisely, if $X = (X_t)_{t\geq 0}$ is a Lévy process, then X_t is infinitely divisible for every $t \geq 0$. On the other hand, for any infinitely divisible random variable Z there exists a Lévy process X such that $X_1 \stackrel{d}{=} Z$ (see [2, Chapter 1.4]). The equivalent of (2.1.6) for a Lévy processes X is given by

$$\mathbb{E}e^{i\theta X_t} = \mathbb{E}e^{-t\psi(\theta)}. \tag{2.2.1}$$

Here ψ is as in (2.1.7) and corresponds to an infinitely divisible random variable X_1. Examples of Lévy processes include Brownian motion, Poisson process, compound Poisson process and stable Lévy motions, i.e., Lévy processes which have stable finite-dimensional distributions.

2.2.2 Stable random variables and processes

Definition 2.2.2. *Let X, X' be independent copies of the same random variable in $\mathbb{R}^d$. Then, they are stable iff there exists an $\alpha \in (0,2]$ such that for all $A, B \in \mathbb{R}$*

$$AX + BX' \stackrel{d}{=} (A^\alpha + B^\alpha)^{1/\alpha} X + D$$

for some $D \in \mathbb{R}^d$. If $D = 0$, then X is strictly stable.

The characteristic function of a stable random variable (for $d = 1$ as we will only consider this case) is given in the following proposition (see for instance [43]).

Proposition 2.2.3. *An α-stable random variable X with $\alpha \in (0,2]$, has the following characteristic function:*

$$\mathbb{E}\big(\exp(i\theta X)\big) = \begin{cases} \exp\Big(-\sigma^\alpha |\theta|^\alpha \big(1 - i\beta \operatorname{sign}(\theta) \tan(\pi\alpha/2)\big) + \mu\theta\Big), & \text{if } \alpha \neq 1, \\ \exp\Big(-\sigma|\theta| \big(1 + i\beta \tfrac{2}{\pi} \operatorname{sign}(\theta) \log(|\theta|)\big) + \mu\theta\Big), & \text{if } \alpha \neq 1, \end{cases} \tag{2.2.2}$$

where $\sigma > 0$ is the scale parameter, $\beta \in [-1,1]$ is the skewness parameter and $\mu \in \mathbb{R}$ is the shift parameter.

In the thesis we will only consider symmetric α-stable ($S\alpha S$ for short), i.e., with β and μ in (2.2.2) equal to 0. Gaussian random variables correspond to $\alpha = 2$ and are usually treated independently. Following [37], we recall that stable random variables are the only possible weak limits of normalized partial sums

$$\frac{\sum_{k=1}^n X_k - a_n}{b_n}, \tag{2.2.3}$$

where the X_k are i.i.d., a_n is chosen so that (2.2.3) are centered and b_n are positive such that $b_n \to \infty$. If the sums in (2.2.3) converge weakly to a stable random variable, we then say that X_1 is in the *domain of attraction* of this variable.

Stable random variables are infinitely-divisible, with corresponding Lévy measures of the form

$$\nu(dz) = \mathbf{1}\{z < 0\}\frac{a}{|z|^{\alpha+1}} + \mathbf{1}\{z > 0\}\frac{b}{|z|^{\alpha+1}},$$

for some $a, b > 0$. This choice corresponds to $\beta = \frac{b-a}{a+b}$.

Stable processes are stochastic processes with stable finite-dimensional distributions. In particular we will often refer to symmetric α-stable Lévy process, i.e., infinitely divisible processes with characteristic triple $(0, 0, \nu)$ where the Lévy measure ν has density with respect to Lebesgue measure on $\mathbb{R}$ of the form

$$\rho(dx) = c|x|^{-1-\alpha}dx, \quad x \in \mathbb{R}, \tag{2.2.4}$$

where $c \geq 0$ is a constant and $\alpha \in (0, 2]$.

Definition 2.2.4. *A stochastic process $(X_t)_{t\geq 0}$ is stable if its finite-dimensional distributions are stable.*

2.3 Random measures and stochastic integrals with respect to these measures

In this section we will give a brief introduction to the random measures which will be later used extensively for representations of limit processes. We begin with a rather general setting and then consider stable and Gaussian cases separately.

Let $(S, \mathcal{S})$ be a measurable space and consider a σ-finite measure m on $(S, \mathcal{S})$. By $\mathcal{S}_0$ denote all $A \in \mathcal{S}$ with $m(A) < \infty$. The measure m is usually called the *control measure*. A random measure M is a set function on $\mathcal{S}_0$ such that for each $A \in \mathcal{S}_0$, $M(A)$ is a complex-valued random variable. One may think of $\mathcal{S}_0$ as an index set. We are usually interested in random measures that satisfy two additional requirements.

Definition 2.3.1. *An* independently scattered *σ-finite random measure M is a random measure on $(S, \mathcal{S})$ such that:*

(i) $M(\bigcup_{n=1}^{\infty}) = \sum_{n=1}^{\infty} M(A_n)$ *a.s. for any pairwise disjoint A_n in $\mathcal{S}_0$;*

(ii) if $A_1, \ldots, A_n$ in $\mathcal{S}_0$ are pairwise disjoint, $M(A_1), \ldots, M(A_n)$ are independent.

In the thesis we will consider random measures such that, additionally to (i) and (ii), for $A \in \mathcal{S}_0$ $M(A)$ is real-valued and $\mathbb{E}(e^{i\theta M(A)}) = e^{-m(A)\psi(\theta)}$, where $\psi(\theta)$ is a (fixed) Lévy exponent as in (2.1.6). More specifically, we will consider Poisson ($\psi(\theta) = 1 - e^{i\theta}$), symmetric α-stable ($\psi(\theta) = |\theta|^\alpha$) and Gaussian ($\psi(\theta) = \theta^2$) cases.

Integration with respect to random measures is a fundamental building block of representations of a number of stochastic processes. Although one can be very general and construct an integral with respect to general infinitely-divisible random measures, we choose to consider the Gaussian and stable cases separately. This choice is made because in the results that follow the introductory section we are always in only one of this two domains.

Defining an integral of a simple function is straightforward. If

$$f(x) = \sum_{k=1}^{n} f_k \mathbf{1}_{A_k}(x), \tag{2.3.1}$$

for $f_k \in \mathbb{C}$, $A_k \in \mathcal{S}_0$, $k = 1, \dots, n$, we define the integral of f with respect to M by

$$I(f) = \int_S f(x) M(dx) := \sum_{k=1}^{n} f_k M(A_k). \tag{2.3.2}$$

Then the definition of the integral can be extended to a larger class of functions, depending on the properties of M. The general theory of integration with respect to infinitely divisible random measures was developed in [38].

2.3.1 Integrals with respect to Gaussian measures

Gaussian random measures are a subclass random measures with *orthogonal increments*. A random measure with orthogonal increments is σ-additive random measure with control measure m such that

$$\mathbb{E}Z(A) = 0, \tag{2.3.3}$$
$$\mathbb{E}\big(Z(A_1)\overline{Z(A_2)}\big) = m(A_1 \cap A_2), \tag{2.3.4}$$

for all $A, A_1, A_2 \in \mathcal{S}_0$. For orthogonal random measure M and simple functions f, g one has

$$\mathbb{E}(I(f)\overline{I(g)}) = \int_S f(x)\overline{g(x)} m(dx) = (f, g)_{L^2(S,m)}. \tag{2.3.5}$$

Given (2.3.5) it is easy to extend the definition of $I(f)$ to all $f \in L^2(S, m)$. More precisely, for any Cauchy sequence (f_n) in $\in L^2(S, m)$, $(I(f_n))$ is Cauchy in $L^2(\Omega)$ and if $f_n \overset{L^2(S,m)}{\to} f$, with the f_n being simple functions, we may define $I(f)$ as an L^2-limit of $(I(f_n))$.

Definition 2.3.2. *A random measure W on $(S, \mathcal{S})$ with orthogonal increments is called* Gaussian *if, for any $A_1, \ldots, A_n$ in $\mathcal{S}_0$, the random vector $(W(A_1), \ldots, W(A_n))$ has multivariate Gaussian distribution.*

It follows directly that for any functions $f_1, \ldots, f_n \in L^2(S, m)$, the random vector $(I(f_1), \ldots, I(f_n))$ has multivariate Gaussian distribution. Of special interest, especially for representing Hermite processes (see Section 3.1.1), are *Hermitian Gaussian random measures.* These are complex-valued Gaussian random measures Z on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with symmetric control measure, which additionally satisfy

$$\overline{Z(A)} = Z(-A), \quad A \in \mathcal{B}(\mathbb{R})_0.$$

They are also called (see [28]) *spectral Gaussian random measures.* For a short introduction to this topic see Section 2.5.4.

2.3.2 Integrals with respect to Poisson and stable symmetric random measures

We briefly cover Poisson random measures which are used extensively in the construction of many random objects presented in the thesis. Using Poisson random measures, we then describe an intuitive construction of stable random measures. Finally we provide a brief overview of the general case when the infinitely divisible random measure does not have a Gaussian component.

Poisson random measures

The content of this section is based on [25].

Definition 2.3.3. *Let $(S, \mathcal{S})$ be a measurable space and consider a σ-finite measure m on $(S, \mathcal{S})$. A Poisson random measure N on $(S, \mathcal{S})$ with intensity measure m is an independently scattered real-valued random measure (in the sense of Definition 2.3.1) such that for each $A \in \mathcal{S}_0$ $N(A)$ has Poisson distribution with parameter $m(A)$.*

It can be shown (see [24, Theorem 3.6]) that on any σ-finite measure space $(S, \mathcal{S}, m)$ there exists a Poisson random measure. Briefly, if $m(S) < \infty$, then for κ - Poisson distribution with parameter $m(S)$ and $X_1, X_2, \ldots$ i.i.d. distributed with X_1 having $m(\cdot)/m(S)$ as its distribution, the quantity

$$\sum_{k=1}^{\kappa} \delta_{X_k},$$

has all the required properties. When $m(S) = \infty$, then one can write m as a sum of finite measures, repeat the above construction independently for each of them and finally add them up.

For simple functions of the form (2.3.1) and $I(f)$ as in (2.3.2), we have

$$\mathbb{E}\exp(i\theta I(f)) = \exp\Big(\int_S (e^{i\theta f(x)} - 1)m(dx)\Big), \quad \theta \in \mathbb{R}. \tag{2.3.6}$$

Then, one shows that the definition of $I(f)$ may be extended to functions f satisfying

$$\int_S \big(|f(x)| \wedge 1\big)m(dx) < \infty.$$

The formula (2.3.6) continues to hold for such functions.

We will also consider stochastic integrals with respect to compensated Poisson random measures. If N is a Poisson random measure with intensity measure m, then we define its compensated form by $\widetilde{N} := N - m$. For f_k and A_k as in (2.3.2) and $I(f) := \sum_{k=1}^n f_k \widetilde{N}(A_k)$, one can easily show, using the characteristic function of Poisson random variable, that

$$\mathbb{E}\exp(i\theta I(f)) = \exp\Big(\sum_{k=1}^n m(A_k)(e^{if_k\theta} - 1 - if_k\theta)\Big), \quad \theta \in \mathbb{R}. \tag{2.3.7}$$

We also denote $I(f)$ by $\int_S f(s)\widetilde{N}(ds)$. In general one can define $I(f)$ for functions in $L^2(S, m(ds))$ by taking a sequence (f^n) of simple functions converging to f in $L^2(S, m(ds))$ and noticing that for f simple

$$\mathbb{E}(I(f)^2) = \int_S |f(s)|^2 m(ds).$$

This means that the sequence of random variables $(I(f^n))_{n\geq 1}$ converges in L^2 and the limit is independent of the particular choice of the sequence (f^n). Moreover, for $f \in L^2(S, m(ds))$ we have

$$\mathbb{E}\exp\Big(i\theta \int_S f(s)\widetilde{N}(ds)\Big) = \exp\Big(\int_S \big(e^{i\theta f(s)} - 1 - i\theta f(s)\big)m(ds)\Big). \tag{2.3.8}$$

Symmetric stable random measures

The content of this section is based mostly on [42, Chapter 3].

Definition 2.3.4. *We say that an independently scattered random measure M_α on $(S, \mathcal{S})$ is a symmetric α stable random measure with control measure m if for* *each $A \in \mathcal{S}$, of finite μ-measure, $M_\alpha(A)$ has symmetric α stable distribution, i.e.,*

$$\mathbb{E}\exp(i\theta M_\alpha(A)) = \exp\big(- m(A)|\theta|^\alpha\big), \quad \theta \in \mathbb{R}.$$

When $S = \mathbb{R}^d$ one can construct M_α in a relatively easy way. Assume that $\alpha \in (0, 2)$. Following section 2.1.9 in [13] we will define the symmetric α-stable random measure M_α as a limit of integrals with respect to compensated Poisson random measures. From now on we take $(S, \mathcal{S}) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for some positive integer d. Let N be a Poisson random measure on $(\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}))$ with intensity measure

$$\frac{c_\alpha m(ds)du}{|u|^{1+\alpha}},$$

where c_α is such that $c_\alpha \int_{\mathbb{R}} \frac{1-cosu}{|u|^{1+\alpha}}du = 1$. Then, for $A \in \mathcal{S}$ such that $m(A) < \infty$, we can set

$$M(A) := \lim_{\epsilon \to 0+} \int_A \int_{\{u:|u|>\epsilon\}} u\widetilde{N}(ds, du), \tag{2.3.9}$$

where the limit is in the distribution. For simple functions of the form (2.3.1) and $I(f)$ given by (2.3.2) one has

$$\mathbb{E}\exp(i\theta I(f)) = \exp\big(- |\theta|^\alpha \int_{\mathbb{R}^d} |f(s)|^\alpha m(ds)\big). \tag{2.3.10}$$

One shows that the definition of $I(f)$ may be extended to measurable functions satisfying

$$\int_S |f(s)|^\alpha m(ds) < \infty. \tag{2.3.11}$$

(2.3.10) is still valid for this extension. $I(f)$ may be described in terms of the underlying Poisson random measure as

$$I(f) = \lim_{\epsilon \to 0+} \int_{\mathbb{R}^d} \int_{\{u:|u|>\epsilon\}} f(s)u\widetilde{N}(ds, du), \tag{2.3.12}$$

with the limit in the sense as in (2.3.9).

Many of our limit processes will have the form

$$(X_t)_{t\in T} = \left(\int_S f(t, s)M(ds) \right)_{t\in T} \tag{2.3.13}$$

for suitable families of functions $\{f(t, \cdot) : t \in T\}$ and index sets T. It is important to notice that $(X_t)_{t\in T}$ is a stable process and its finite-dimensional distributions have the form

$$\mathbb{E}\exp(i\sum_{k=1}^m \theta_k X_{t_k}) = \exp\big(\int_S \Big| \sum_{k=1}^m \theta_k f(t_k, s)\Big|^\alpha m(ds)\big). \tag{2.3.14}$$

See [42, Section 3.3] for more details.

2.3.3 Examples

In this section we present some well known examples of stochastic processes which have a representation of the form (2.3.13) with M being a symmetric α-stable random measure. Recall that in the Gaussian case the only (up to a multiplicative constant) symmetric H-sssi process is fractional Brownian motion. However, in the stable case, for any given $\alpha \in (0, 2)$ and feasible[1] H, there are, in general many $H - sssi$ α-stable processes.

Example 2.3.1. Let Λ be standard (unit-scale) symmetric α-stable random measure on $\mathbb{R}$ with Lebesgue control measure. Then the process

$$X_t := \int_{\mathbb{R}} \mathbf{1}_{[0,t]}(x)\Lambda(dx), \quad t \geq 0$$

is a symmetric α-stable Lévy process.

Example 2.3.2. Let $H \in (0, 1)$. One can show that the function

$$R_H(t,s) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \quad t, s \in \mathbb{R},$$

is positive semi-definite (see [37, Proposition 2.6.1]). By part (iv) of Proposition 2.1.4, if the process $X = (X_t)_{t\in\mathbb{R}}$ is H-sssi, then, provided $\mathbb{E}(X_1^2) < \infty$, it must have the covariance function of the above form. Since the distribution of any centred Gaussian process is entirely determined by its covariance function, we conclude that there can be only one Gaussian H-sssi process (up to a multiplicative constant). We call this process a *fractional Brownian motion* (fBM for short). We say that X is a *standard* fBm if $\mathbb{E}(X_1^2) = 1$.

Fractional Brownian motion admits many representations using Gaussian random measures. We will first describe the one encountered most frequently in the literature. Let B be a Gaussian random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with Lebesgue intensity measure. Then (see Proposition 2.6.5 in [37]) the process

$$(B_H(t))_{t\in\mathbb{R}} = \left(\int_{\mathbb{R}} \Big((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}\Big)\, B(du)\right)_{t\in\mathbb{R}}, \tag{2.3.15}$$

is a fractional Brownian motion. Another representation can be obtained by using Hermitian Gaussian random measures. Let $\widehat{B}$ be a Hermitian Gaussian random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with Lebesgue intensity. Then ([37, Proposition 2.6.11]) the process defined by

$$\Big(\widehat{B_H}(t)\Big)_{t\in\mathbb{R}} = \left(\int_{\mathbb{R}} \frac{e^{itx}-1}{ix}|x|^{1/2-H}\widehat{B}(du)\right)_{t\in\mathbb{R}}, \tag{2.3.16}$$

is also a fractional Brownian motion with Hurst parameter H.

[1] $H \in (0, 1/\alpha)$ for $0 < \alpha < 1$ and $H \in (0, 1)$ for $1 \leq \alpha < 2$

Example 2.3.3 (Linear fractional symmetric stable motion)**.** Let Λ be standard symmetric α-stable random measure on $\mathbb{R}$ with Lebesgue control measure. Assume that $H \in (0,1)$ and $H \neq 1/\alpha$. Consider the following kernel

$$\begin{aligned} g_{c_1,c_2}(t,x) &:= c_1\Big(\big((t-x)_+\big)^{H-1/\alpha} - \big((-x)_+\big)^{H-1/\alpha}\Big) \\ &\quad + c_2\Big(\big((t-x)_-\big)^{H-1/\alpha} - \big((-x)_-\big)^{H-1/\alpha}\Big), \quad t,x \in \mathbb{R} \end{aligned} \tag{2.3.17}$$

where $c_1, c_2 \in \mathbb{R}$ are constants. One might show for any choice of these constants and every t, the function $x \mapsto g_{c_1,c_2}(t,x)$ is in $L^\alpha(\mathbb{R}, dx)$. We may then define the process X by

$$X(t) := \int_{\mathbb{R}} g_{c_1,c_2}(t,x)\Lambda(dx), \quad t \in \mathbb{R}.$$

Using the fact that $g_{c_1,c_2}(t,x) - g_{c_1,c_2}(s,x) = g_{c_1,c_2}(t-s, x-s)$ and $g_{c_1,c_2}(ct,cx) = c^{H-1/\alpha} g_{c_1,c_2}(t,x)$ for $s < t$ and $c > 0$, we might easily show that thus defined X is H-sssi. In a complete opposition to the Gaussian case, different choices of c_1 and c_2 lead to different processes (see [42, Proposition 3.5.3]).

2.4 Multiple Wiener-Itô integrals

In the exposition below we follow a very clear and readable presentation of Pipiras and Taqqu in [37]. Let $(S, \mathcal{S})$ be a measurable space and W be a Gaussian random measure on this space with control measure m as in Definition 2.3.2. In this section we consider only real-valued W. We want to give a brief overview of the construction and properties of integrals of the form

$$I_k(f) := \int_{S^k} f(u_1, \dots, u_k) W(du_1) \dots W(du_k), \tag{2.4.1}$$

for positive integers k and suitable real-valued functions f. We assume that m is non-atomic and define the class of simple functions as functions of the form

$$f(u_1, \dots, u_k) = \sum_{i_1, \dots, i_k = 1}^{n} a_{i_1,\dots,i_k} \mathbf{1}_{A_{i_1} \times \dots \times A_{i_k}}(u_1, \dots, u_k), \tag{2.4.2}$$

where $A_{i_1}, \dots, A_{i_k} \in \mathcal{S}_0$ are pairwise disjoint and $a_{i_1,\dots,i_k} \in \mathbb{R}$. Any such function vanishes on the sets $A_{ij} = \{(u_1, \dots, u_k) : u_i = u_j \text{ for some } i \neq j\}$. For f of the form (2.4.2) one defines

$$I_k(f) = \sum_{i_1, \dots, i_k = 1}^{n} a_{i_1,\dots,i_k} W(A_{i_1}) \dots W(A_{i_k}). \tag{2.4.3}$$

As expected, for simple function f, g and real a, b we have

$$I_k(af + bg) = aI_k(f) + bI_k(g).$$

More importantly for f simple

$$I_k(f) = I_k(\widetilde{f}),$$

where $\widetilde{f}$ denotes the *symmetrization* of f. Additionally,

$$\mathbb{E}I_k(f) = 0$$

and

$$\mathbb{E}\left(I_k(f)I_q(g)\right) = \begin{cases} 0, & \text{if } k \neq q, \\ k!(\widetilde{f},\widetilde{g})_{L^2(S^k,m^k)}, & \text{if } k = q, \end{cases} \tag{2.4.4}$$

for positive integers k, q and f, g simple (see [37, Appendix B]). This means that for f simple

$$\mathbb{E}I_k(f)^2 = k! \left\|\widetilde{f}\right\|_{L^2(S^k,m^k)} \leq k! \left\|f\right\|_{L^2(S^k,m^k)}.$$

The inequality above follows from triangle inequality in L^2. Now it is easy to extend the definition of $I_k(f)$ to all f in $L^2(S^k, m^k)$, i.e., for any Cauchy sequence (f_n) in $L^2(S^k, m^k)$, $I_k(f_n)$ converges in $L^2(\Omega)$.

2.5 Generalized random fields, random spectral measures and Itô's formula

2.5.1 Schwartz space, generalized functions and Fourier transform

The space of *generalized functions* $\mathcal{S}'(\mathbb{R}^d)$ is defined to be the linear space of all continuous linear maps $F : \mathcal{S}(\mathbb{R}^d) \to \mathbb{C}$. Using the dual bracket notation we will write (F, ϕ) to denote the value of the functional F evaluated at $\phi \in \mathcal{S}(\mathbb{R}^d)$. One can define the Fourier transform of $F \in \mathcal{S}'(\mathbb{R}^d)$ by extending the Parseval formula. Recall that for $f, g \in L^2(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \overline{f(x)}g(x)dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \overline{\widehat{f}(x)}\widehat{g}(x)dx. \tag{2.5.1}$$

It is not hard to show, using Fubini theorem, that the equality (2.5.1) also holds if we assume that f is in $L^1(\mathbb{R}^d)$ and g is in $\mathcal{S}^c(\mathbb{R}^d)$. Let us note that the Fourier transform is bicontinuous map from $\mathcal{S}^c$ to $\mathcal{S}^c$ and maps $\mathcal{S}$ to the subspace of $\mathcal{S}^c(\mathbb{R}^d)$ containing functions f satisfying $f(-x) = \overline{f(x)}$ for all $x \in \mathbb{R}^d$.

Definition 2.5.1. *For any $F \in \mathcal{S}'$ we define its Fourier transform $\widehat{F}$ by*

$$(\widehat{F}, \widehat{\phi}) = (2\pi)^d(F, \phi), \quad \phi \in \mathcal{S}^c(\mathbb{R}^d). \tag{2.5.2}$$

2.5.2 Hermite polynomials, Wick polynomials and Wick products

In this section we provide a brief overview of Gaussian Hilbert spaces and Wick products which are used extensively throughout this chapter. In this exposition we follow [19] and [28].

Definition 2.5.2. *A* Gaussian linear space *is a real linear space of random variables, defined on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, such that each random variable in this space has centred Gaussian distribution. A* Gaussian Hilbert space *is a a Gaussian linear space which is complete (i.e., a closed subspace of* $L^2_{\mathbb{R}}(\Omega, \mathcal{F}, \mathbb{P})$*).*

Let us briefly introduce the setting in which we are going to define the most important concepts of this section. Assume that $(X_t)_{t\in T}$ are jointly Gaussian random variables in some index set T. By $\mathcal{H}$ define the Hilbert space of all square integrable random variables measurable with respect to $\mathcal{B}(X_t, t \in T)$ (the Borel sigma algebra generated by the random variables X_t, $t \in T$) with the scalar product given by $(X, Y) := \mathbb{E}(XY)$ for $X, Y \in \mathcal{H}$. Moreover, let $\mathcal{H}_1$ denote the closed subspace of $\mathcal{H}$ generated by finite linear combinations $\sum_j a_j X_{t_j}$ with $a_j \in \mathbb{R}$ and t_j in T. Let $Y_1, Y_2, \ldots$ be an orthonormal basis in $\mathcal{H}_1$. Since these random variables are jointly Gaussian, they are independent and, moreover, we have $\mathcal{B}(Y_1, Y_2, \ldots) = \mathcal{B}(X_t, t \in T)$. In order to define Wick products we have to first define *Hermite polynomials*. For a non-negative integer n the n-th Hermite polynomial is the polynomial defined by

$$H_n(x) := (-1)^n \exp^{x^2/2} \frac{d^n}{dx^n}(\exp^{-x^2/2}). \tag{2.5.3}$$

Hermite polynomials form a complete orthogonal system in $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \frac{1}{\sqrt{2\pi}} \exp^{-x^2/2} dx)$. Furthermore, one can show (see [28, Theorem 2.1]) that for the process X and the random variables $Y_1, Y_2, \ldots$, the set of all possible finite products $H_{j_1}(Y_{l_1}) \ldots H_{j_k}(Y_{l_k})$ is a complete orthogonal system in the Hilbert space $\mathcal{H}$.

Given the above let us define $\mathcal{H}_{\leq n}$ to be the Hilbert space which is the closure of the linear space consisting of all the random variables of the form $P_n(X_{t_1}, \ldots, X_{t_m})$, where P_n is a polynomial of degree at most n and $t_1, \ldots, t_m \in T$ for some finite m. To complete the picture we define $\mathcal{H}_0$ to consist of real constants. Finally we define

$$\mathcal{H}_n := \mathcal{H}_{\leq n} \ominus \mathcal{H}_{\leq n-1}$$

for $n \geq 1$. The symbol $\ominus$ denotes the orthogonal completion. Notice that by [28, Theorem 2.1]

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots,$$

i.e., $\mathcal{H}$ is a direct sum of $\mathcal{H}_0, \mathcal{H}_1, \ldots$.

In this setting we define for each $n \geq 0$ the map $\pi_n : \mathcal{H} \to \mathcal{H}_n$ to be the orthogonal projection of an element of $\mathcal{H}$ onto $\mathcal{H}_n$. This means that any random variable $X \in \mathcal{H}$ satisfies

$$X = \sum_{n=0}^{\infty} \pi_n(X),$$

with the sum converging in L^2.

Definition 2.5.3. *Given any polynomial $P(x_1, \dots, x_n)$ of degree n and random variables $\xi_1, \dots, \xi_n \in \mathcal{H}_1$ we define the* Wick polynomial $: \xi_1 \dots \xi_n :$ *by*

$$: P(\xi_1, \dots, \xi_n) : := \pi_n(P(\xi_1, \dots, \xi_n)). \tag{2.5.4}$$

Notice that Wick polynomials of different degrees are orthogonal. Since the definition above applies only to Gaussian random variables we also provide an alternative definition, which is equivalent in the Gaussian case, but is much more general.

Definition 2.5.4. *Assume that random variables $\xi_1, \dots, \xi_n$ are centered and square integrable. We define*

$$: \xi_1 \dots \xi_n : := \sum_{A \in \mathcal{M}} (-1)^{|A|} \prod_{\{s,t\} \in A} \mathbb{E}(\xi_s \xi_t) \prod_{r \notin \cup A} \xi_r, \tag{2.5.5}$$

where $\mathcal{M}$ is the set of unordered pairs $\{s, t\} \subset \{1, \dots, n\}$, such that all the elements in these pairs are distinct. In particular $|\bigcup A| = 2|A|$. The sum above is over all distinct sets A of this form including the empty set.

For example we have

$$: \xi_1 : = \xi_1,$$
$$: \xi_1\xi_2 : = \xi_1\xi_2 - \mathbb{E}(\xi_1\xi_2),$$
$$: \xi_1\xi_2\xi_3 : = \xi_1\xi_2\xi_3 - \xi_1\mathbb{E}(\xi_2\xi_3) - \xi_2\mathbb{E}(\xi_1\xi_3) - \xi_3\mathbb{E}(\xi_1\xi_2).$$

The next two results (see [28, Corollary 2.3]) provide a link between Hermite and Wick polynomials.

Proposition 2.5.5. *Let $\xi_1, \dots, \xi_m$ be an orthonormal system in $\mathcal{H}_1$ and let*

$$P(x_1, \dots, x_m) = \sum c_{j_1,\dots,j_m} x_1^{j_1} \dots x_m^{j_m}$$

be a homogeneous polynomial of degree n. Then

$$: P(\xi_1, \dots, \xi_m) : = \sum c_{j_1,\dots,j_m} H_{j_1}(\xi_1) \dots H_{j_m}(\xi_m).$$

In particular

$$: \xi^n : = H_n(\xi),$$

for $\xi \in \mathcal{H}_1$ with $\mathbb{E}\xi^2 = 1$.

Proposition 2.5.6. *Let $\xi_1, \xi_2, \dots$ be an orthonormal basis in $\mathcal{H}_1$. Then the random variables $H_{j_1}(\xi_1) \dots H_{j_k}(\xi_k)$, $k - 1, 2, \dots$, $j_1 + \dots + j_k = n$, form a complete orthogonal basis in $\mathcal{H}_n$.*

2.5.3 Generalized random fields

In the previous section the set of indices T was arbitrary. In this chapter we are going to consider stochastic processes indexed by function in the Schwarz space (see Section 2.5.1). In the definition below we restrict our attention to $\mathcal{S} = \mathcal{S}(\mathbb{R})$, but it extends naturally to $\mathcal{S}(\mathbb{R}^d)$.

Definition 2.5.7. *We say that the set of random variables $(X(\phi))_{\phi\in\mathcal{S}}$ is a* generalized random field *over $\mathcal{S}$ if*

(i) $X(a_1\phi_1 + a_2\phi_2) = a_1X(\phi_1) + a_2X(\phi_2)$ *almost surely for all $a_1, a_2 \in \mathbb{R}$ and $\phi_1, \phi_2 \in \mathcal{S}$. The exceptional zero-measure set may depend on a_1, a_2, ϕ_1 and ϕ_2.*

(ii) $X(\phi_n) \to X(\phi)$ *in probability if $\phi_n \to \phi$ in the topology of $\mathcal{S}$.*

By the regularization theorem of Itô (see [21, Theorem 3.1.2]), if a generalized random field X is continuous in $L^2(\Omega, ,\mathbb{P})$, then there exists an $\mathcal{S}'$-valued random variable $\overline{X}$ such that for any $f \in \mathcal{S}$

$$\langle \overline{X}, \phi\rangle = X(\phi)$$

almost surely, where $\langle\cdot,\cdot\rangle$ is the dual bracket.

The general idea behind using generalized random fields is as follows. Very often, given a generalized random field X over $\mathcal{S}$ we can extend it in a unique and meaningful way to a more general linear class of functions $\mathcal{T}$ and working first only with indices ϕ from $\mathcal{S}$ is much less cumbersome. We will see a lot of examples of this idea later on. Now we will briefly sketch the construction of random spectral measures as given in [28, Chapter 3].

2.5.4 Random Spectral Measures and integrals with respect to these measures

In this section we provide an overview of the construction and integration with respect to random spectral measures in a way most pertinent to Chapter 3. "Random spectral measure" is a different name for a Hermitian random measure which was already discussed in Section 2.3.1. We base our exposition on the excellent book of Major [28].

A large number of generalized random fields have the representation of the form

$$X(\phi) = \int_{\mathbb{R}^d} \widehat{\phi}(x) Z_G(dx), \quad \phi \in \mathcal{S}, \tag{2.5.6}$$

where Z_G is the so called *random spectral measure*, which will be described shortly. The above representation can be very useful, especially in connection with the *Itô's formula* which we will also formulate. Now we will briefly sketch the construction of random spectral measures as given in [28, Chapter 3].

First, the Bochner-Schwartz theorem states that for any generalized random field over $\mathcal{S}$ there exists a unique σ-finite measure G on $\mathbb{R}^d$ such that

$$\mathbb{E}\left(X(\phi)X(\psi)\right) = \int_{\mathbb{R}^d} \widehat{\phi}(x)\overline{\widehat{\psi}}(x)G(dx), \quad \phi, \psi \in \mathcal{S}. \tag{2.5.7}$$

The measure G satisfies $G(A) = G(-A)$ for $A \in \mathcal{B}(\mathbb{R}^d)$ and there exists some $r > 0$ such that

$$\int_{\mathbb{R}^d} (1 + \|x\|_2)^{-r} G(dx) < \infty.$$

Fix a generalized random field X over $\mathcal{S}$. Let $\mathcal{H}_1$ be the Gaussian Hilbert space as defined in the discussion following Definition 2.5.2 and $\mathcal{H}_1^c$ its complexification. One may define the mapping $I : \mathcal{S}^c \to \mathcal{H}_1^c$ by

$$I(\widehat{\phi + i\psi}) = X(\phi) + iX(\psi), \quad \phi, \psi \in \mathcal{S} \tag{2.5.8}$$

and then easily show that

$$\left\|\widehat{\phi + i\psi}\right\|^2_{L^2(G(dx))} = \mathbb{E}\left|X(\phi) + iX(\psi)\right|^2,$$

which means that I is defined on a linear subspace of $L^2(G(dx))$ and it is norm-preserving. Thus it can be uniquely extended to a unitary transformation the whole of $L^2(G(dx))$ which satisfies

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}G(dx) = \mathbb{E}\left(I(f)\overline{I(g)}\right), \quad f, g \in L^2(G(dx)).$$

We now may define the *random spectral measure* using the formula

$$Z_G(A) := I(\mathbf{1}_A), \tag{2.5.9}$$

for any $A \in \mathcal{B}(\mathbb{R}^d)$ with $G(A) < \infty$. These are complex-valued Gaussian random measures Z on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, with symmetric control measure, which additionally satisfy

$$\overline{Z(A)} = Z(-A), \quad A \in \mathcal{B}(\mathbb{R})_0,$$

where $\mathcal{B}(\mathbb{R})_0$ denotes Borel subsets of $\mathbb{R}$ of finite G-measure. They are also called *Hermitian Gaussian random measures* in the literature. We recall their basic properties in the following proposition.

Proposition 2.5.8. *Let Z be a Hermitian Gaussian random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with control measure m.*

(i) *The random variables* $Z(A)$ *are jointly Gaussian in the sense that* $ReZ(A)$ *and* $\Im Z(B)$ *are jointly Gaussian for* $A, B \in \mathcal{B}(\mathbb{R})$ *with* $G(A), G(B) < \infty$*.*

(ii) $\mathbb{E}Z(A) = 0$*.*

(iii) $\mathbb{E}Z(A)\overline{Z(B)} = G(A \cap B)$*.*

(iv) $Z(A) = \overline{Z(-A)}$*.*

(v) $\sum_{j=1}^{n} Z(A_j) = Z(\cup_{j=1}^{n} A_j)$ *almost surely for* A_j *disjoint.*

Constructing the integral as usual with the help of simple functions and then extending it by means of isometry we see that

$$\mathbb{E}\left(\int_{\mathbb{R}} f(x)Z_G(dx)\overline{\int_{\mathbb{R}} g(x)Z_G(dx)}\right) = \int_{\mathbb{R}} f(x)\overline{g(x)}G(dx),$$

for any $f, g \in L^2(G(dx))$. It is easy to show that for $g \in L^2(\mathbb{R}, G(dx))$ satisfying $\overline{g(x)} = g(-x), x \in \mathbb{R}$, $\int_{\mathbb{R}} g(x)Z_G(dx)$ is real-valued. Together with (2.5.7) this implies that

$$X(\phi) = \int_{\mathbb{R}^d} \widehat{\phi}(x)Z_G(dx), \quad \phi \in \mathcal{S}, \tag{2.5.10}$$

which leads to the conclusion that for any stationary generalized Gaussian random field there exists, on the same probability space as X, a unique random spectral measure Z_G corresponding to the spectral measure G such that (2.5.10) holds (see [28, Theorem 3.1]. Recalling Definition 2.5.1, one might notice that in view of Itô regularization theorem (see [21, Theorem 3.1.2]), Z_G can be seen as a Fourier transform of an $\mathcal{S}'(\mathbb{R})$-valued random variable.

2.5.5 Multiple integrals with respect to spectral random measure

Our goal in this section is to provide an overview of construction of multiple Wiener-Itô integrals in the spectral case. We will only consider the case when the spectral measure Z_G is defined on $\mathbb{R}$. We want to provide meaning to the integrals of the form

$$\int_{\mathbb{R}^n}'' f_n(x_1, \ldots, x_n)Z_G(dx_1)\ldots Z_G(dx_n), \tag{2.5.11}$$

where $f_n : \mathbb{R}^n :\to \mathbb{R}$ is a measurable function satisfying some further properties that will be specified later. The $''$ sign over the integral signifies that the integral does not take into account the diagonals in $\mathbb{R}^d$. The exact definition will be provided later.

Assume for the rest of this section that G is a spectral measure (see (2.5.7)) of a stationary generalized Gaussian field. For $n \geq 1$ define $\overline{\mathcal{H}_G^n}$ to be the space of complex-valued functions f_n satisfying $f_n(-x_1, \ldots, -x_n) = \overline{f_n(x_1, \ldots, x_n)}$ and

$$\int_{\mathbb{R}^n} \left|f_n(x_1, \ldots, x_n)\right|^2 G(dx_1) \ldots G(dx_n) < \infty. \tag{2.5.12}$$

Moreover, by $\mathcal{H}_G^n$ denote all the functions of $\overline{\mathcal{H}_G^n}$ invariant under the permutations of their arguments. In particular, since G is symmetric, we see that the scalar product (f, g) inherited from $L^2(G(dx_1) \otimes \ldots \otimes G(dx_n))$ is real-valued as long as f and g are in $\overline{\mathcal{H}_G^n}$. For any function $f_n \in \overline{\mathcal{H}_G^n}$ we define its symmetrized version by

$$Sym(f)(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{\pi \in \Pi_n} f(x_{\pi(1)}) \ldots f(x_{\pi(n)}), \tag{2.5.13}$$

where Π_n is the set of all permutations of n elements. As usual, we are first going to define integrals for simple functions. As in the case of classic Multiple Wiener-Itô (see Section 2.4) we first define integrals on the set of simple functions. The details are as follows.

Fix a positive integer $n \geq 0$. A collection $\mathcal{D}$ of 2^N bounded measurable sets in $\mathbb{R}$ is called a *regular system* (we borrow here the nomenclature from [28]) if it is of the form

$$\mathcal{D} = \{A_j : j = \pm 1, \ldots, \pm N\}, \tag{2.5.14}$$

for some $N \geq 0$, with $A_j = -A_{-j}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. We say that a function f in $\overline{\mathcal{H}_G^n}$ is *adapted* to this system if it is constant on the sets of the form $A_{j_1} \times \ldots A_{j_n}$, is equal to zero outside of this Cartesian product and and on the products for which $j_l = \pm j_k$ for some $l \neq k$. This basically means that the function f *stays off the diagonals* of $\mathbb{R}^n$ of the form $x_l = \pm x_{l'}$ for $l \neq l'$.

Definition 2.5.9. *Assume that a function $f \in \overline{\mathcal{H}_G^n}$ is adapted to a regular system as in* (2.5.14). *We define its multiple integral with respect to the spectral measure Z_G by*

$$\begin{aligned} I(f) &:= \frac{1}{n!} \int_{\mathbb{R}^n} f(x_1, \ldots, x_n) Z_G(dx_1) \ldots Z_G(dx_n) \\ &:= \frac{1}{n!} \sum_{\substack{j_l = \pm 1, \ldots, \pm N, \\ l = 1, \ldots, N}} f(x_{j_1}, \ldots, x_{j_n}) Z_G(A_{j_1}) \ldots Z_G(A_{j_n}) \end{aligned} \tag{2.5.15}$$

where $x_{j_l} \in A_{j_l}$.

Notice that in the above definition all products the form $Z_G(A_{j_1}) \ldots Z_G(A_{j_n})$ are products of independent random variables since Z_G is independently scattered. It is also easy to notice (using Proposition 2.5.8) that for a simple function (in the sense of remarks above Definition 2.5.9) $f \in \overline{\mathcal{H}_G^n}$, $I(f)$ is real-valued and $\mathbb{E}I(f) = 0$. Since the sum in (2.5.15) is for simple functions essentially a sum over all permutations of their arguments we also have

$$I(f) = I(Sym(f)) \tag{2.5.16}$$

for f simple in $\overline{\mathcal{H}_G^n}$. One of the more fundamental properties of thus defined $I(f)$ is the fact that the L^2 scalar product between $I(f)$ and $I(g)$,

$$\mathbb{E}I(f)I(g) = \frac{1}{n!}\int_{\mathbb{R}^n} f(x_1,\ldots,x_n)\overline{g(x_1,\ldots,x_n)}G(dx_1)\ldots G(dx_n), \tag{2.5.17}$$

for simple symmetric functions in $\in \overline{\mathcal{H}_G^n}$ (for proof see the discussion after equation (4.7) in [28]). Furthermore for $n \neq m$ and f, g simple functions in $\overline{\mathcal{H}_G^n}$ and $\overline{\mathcal{H}_G^m}$, respectively, we have

$$\mathbb{E}I(f)I(g) = 0. \tag{2.5.18}$$

We may extend the definition of $I(f)$ using the following result (see [28, Lemma 4.1] for proof.

Lemma 2.5.10. *The class of simple functions is dense in $\overline{\mathcal{H}_G^n}$ and the class of symmetric simple functions is dense in $\mathcal{H}_G^n$.*

We are now ready to provide a general definition of multiple spectral integrals.

Definition 2.5.11. *For a function $f \in \overline{\mathcal{H}_G^n}$ choose a sequence (f_N) of simple functions in $\overline{\mathcal{H}_G^n}$ converging to f in $\overline{\mathcal{H}_G^n}$ (which exists by Lemma 2.5.10). Then, we define $I(f)$ to be the L^2 limit of $I(f_N)$.*

The most important result to us is the following consequence of Itô formula (see [28, Theorem 4.7]). Recall that $: X_1 \ldots X_k :$ denotes the Wick product of square integrable variables $X_1, \ldots, X_k$.

Theorem 2.5.12. *Assume that X is a generalized Gaussian random field over $\mathcal{S}(\mathbb{R})$ with spectral measure G. For $n \geq 1$ and $\phi_1, \ldots, \phi_n \in \mathcal{S}(\mathbb{R})$ we have*

$$: \langle X, \phi_1\rangle \ldots \langle X, \phi_n\rangle : = \int_{\mathbb{R}^n} \widehat{\phi(x_1)}\ldots\widehat{\phi(x_n)} Z_G(dx_1)\ldots Z_G(dx_n), \tag{2.5.19}$$

almost surely, where Z_G is the random spectral measure as constructed in Section 2.5.4.

2.5.6 Relationship between the time-domain and spectral domain approaches

Often one makes a choice to work with stochastic integrals described in 2.4 and 2.5.5 based on convenience. For our purposes we find the spectral domain approach more natural, based on tools developed by Major in [28]. However, the two approaches are closely related, which is clarified by the proposition below. For proof see [36, Proposition 9.3.1].

Proposition 2.5.13. *Assume that B is a Gaussian random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with Lebesgue control measure du. Let $\widehat{B}$ be a random spectral measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with Lebesgue control measure dx. Denote the corresponding multiple stochastic integrals by I_k and $\widehat{I_k}$, respectively. Let f_n be in $L^2(\mathbb{R}^n, du_1 \otimes \ldots \otimes du_n)$ for $n = 1, \ldots, k$ and by $\widehat{f_n}$ denote their L^2 Fourier transforms. Then,*

$$(I_1(f_1), \ldots, I_k(f_k)) \stackrel{d}{=} \left(\frac{1}{(2\pi)^{1/2}} \widehat{I}_1(\widehat{f}_1), \ldots, \frac{1}{(2\pi)^{1/2}} \widehat{I}_k(\widehat{f}_k) \right). \tag{2.5.20}$$

2.5.7 Generalized Gaussian Random Fields and Fractional Brownian Motion

One of the important tools which we will be using in this paper are $\mathcal{S}'$-valued random variables. In particular, we will be working with centered Gaussian $\mathcal{S}'$-valued random variables . For each $\alpha < 1$ there exists (by the Itó regularization theorem) a centered Gaussian $\mathcal{S}'$-random variable X with covariance functional given by

$$\mathbb{E}\langle X, \phi \rangle \langle X, \psi \rangle = \frac{1}{\pi} \int_{\mathbb{R}} \widehat{\phi}(x) \overline{\widehat{\psi}(x)} |x|^{-\alpha} dx, \quad \phi, \psi \in \mathcal{S}(\mathbb{R}), \tag{2.5.21}$$

The spectral measure of this field is given by $G(dx) = \frac{1}{\pi}|x|^{-\alpha}dx$. As noted in [8], for $\alpha \in (0, 1)$, X can be approximated by the normalized total charge occupation of our particle system from Subsection 1.2.2 on the interval $[0, T]$, that is by a functional given by

$$\langle X_T, \phi \rangle = \frac{1}{\sqrt{T}} \sum_j \sigma_j \int_0^T \phi(x_j + \xi_s^j) ds, \quad \phi \in \mathcal{S}(\mathbb{R}). \tag{2.5.22}$$

By an L^2 extension we may evaluate X on functions from a much wider class than $\mathcal{S}(\mathbb{R})$ and in fact $(\langle X, \mathbf{1}_{[0,t]} \rangle)_{t \geq 0}$ is up to a constant the fractional Brownian motion with Hurst coeficient equal to $H = \frac{1+\alpha}{2}$. The particle system we are working with can be used to approximate this process as in the theorem below.

Theorem 2.5.14 (Theorem 2.1 in [8])**.** *For $\alpha \in (0, 1)$, as $T \to \infty$, we have:*

(i) $X_T \Rightarrow X$ *in* $\mathcal{S}'(\mathbb{R})$,

(ii) $(\langle X_T, \mathbf{1}_{[0,t]} \rangle)_{t \geq 0}$ *converges in the sense of finite dimensional distributions to* $(KB_t^H)_{t \geq 0}$ *where* K *is a constant and* B^H *is a fractional Brownian motion with Hurst coefficient* $H = \frac{1+\alpha}{2}$.

Remark 2.5.15. *The random field $(\langle X, \phi\rangle)_{\phi\in\mathcal{S}(\mathbb{R})}$ can be used (see Section 2.5.4 in the introduction to this chapter) to construct a random spectral measure Z_G associated with this field such that $\langle X, \phi\rangle = \int_{\mathbb{R}} \widehat{\phi}(x) Z_G(dx)$ for $\phi \in \mathcal{S}$.*

2.6 Local times

Assume that $(X_t)_{t\in\mathbb{R}}$ is a measurable stochastic process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., a stochastic process such that the mapping $(t, \omega) \mapsto X_t(\omega)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable. In what is to come we will follow a very intuitive exposition given in [42, Chapter 10.4]. Let D be in $\mathcal{B}(\mathbb{R}^2)$. We define the *occupation measure* of X by

$$\mu_X(D) := \lambda\left(\{t \in \mathbb{R} : (t, X_t) \in D\}\right),$$

where λ is one-dimensional Lebesgue measure. Intuitively, for $A, B \in (\mathbb{R})$, $\mu_X(A \times B)$ corresponds to the amount of time in the time period A that the process X spends in the set B. Given some fixed $A \in \mathbb{R}$ we may also define

$$\mu_{X,A}(B) := \mu_X(A \times B), \quad B \in \mathcal{B}(\mathbb{R}). \tag{2.6.1}$$

This allows as to write the following identity.

$$\int_A f(X_t)dt = \int_{\mathbb{R}} f(x)\mu_{X,A}(dx) \tag{2.6.2}$$

for any non-negative measurable function f.

Definition 2.6.1. *Let $A \in \mathcal{B}(\mathbb{R})$ and assume that X is a measurable process, i.e, a process such that the mapping $(t, \omega) \mapsto \mathbb{R}$ is $\mathcal{B}([0, \infty)) \otimes \mathcal{F}$-measurable. We say that X has a* local time *over A if the measure $\mu_{X,A}$ defined in* (2.6.1) *is absolutely continuous with respect to one-dimensional Lebesgue measure. We then write*

$$L_{X,A}(x) := \frac{d\mu_{X,A}}{d\lambda}(x), \quad x \in \mathbb{R}. \tag{2.6.3}$$

Equipped with the definition above we may rewrite (2.6.2) as

$$\int_A f(X_t)dt = \int_{\mathbb{R}} f(x)L_{X,A}(x)dx. \tag{2.6.4}$$

Whenever a local time of X at A exists it may also be computed as a limit

$$L_{X,A}(x) = \lim_{\epsilon\searrow 0} \frac{1}{2\epsilon} \int_A \mathbf{1}_{[x-\epsilon,x+\epsilon]}(X_t)dt \tag{2.6.5}$$

and the limit exists almost surely. The almost sure existence of the limit in (2.6.5) can also be taken as a definition of local time, see for example [29, Chapter 3.6]. If a process X has a local time over $[0,t]$ for every $t > 0$ then we may use the following notation which is also the most popular one:

$$L_X(t,x) := L_{X,[0,t]}(x), \quad t > 0, x \in \mathbb{R}. \tag{2.6.6}$$

The classical condition for existence of a local time is due to Berman. For proof see [42, Proposition 10.4.5].

Proposition 2.6.2. *Let X be measurable and $A \in \mathcal{B}(\mathbb{R})$ have finite Lebesgue measure.*

(i) If

$$\int_{\mathbb{R}} \left(\int_A \int_A \mathbb{E}\left(e^{i\theta(X_t - X_s)}\right) dtds \right) d\theta < \infty,$$

then the local time $L_{X,A}$ exists and satisfies

$$\int_{\mathbb{R}} L_{X,A}(x)^2 dx < \infty.$$

(ii) If

$$\int_{\mathbb{R}} \left(\int_A \int_A \mathbb{E}\left(e^{i\theta(X_t - X_s)}\right) dtds \right)^{1/2} d\theta < \infty,$$

then the local time $L_{X,A}$ exists and is bounded and uniformly continuous in x.

It is often desirable for the local time $L_X(t,x)$ defined by (2.6.6) to be *jointly continuous* in t and x. This, fortunately, is the case for a number of fundamental examples like:

- fractional Brownian motion with $H \in (0,1)$;
- α-stable Lévy motion for $\alpha \in (1,2)$ (see for example [23, Section 3]);
- linear fractional symmetric stable motion with $\alpha > 1$, $1/\alpha < H < 1$ and $c_2 = 0$ in Example 2.3.3.

Unsurprisingly, properties like self-similarity and stationarity translate into equivalent properties for the local time.

Proposition 2.6.3. *Let $(X_t)_{t\in\mathbb{R}}$ be a measurable process and assume that it has a jointly continuous local time $(L_X(t,x))_{t\geq 0, x\in\mathbb{R}}$.*

(i) If X is H-self-similar, then for every $c > 0$

$$\left(L_X(ct, c^H x)\right)_{t\geq 0, x\in\mathbb{R}} \stackrel{d}{=} \left(c^{1-H} L_X(t,x)\right)_{t\geq 0, x\in\mathbb{R}}. \tag{2.6.7}$$

(ii) If X is stationary, then for each $h > 0$

$$(L_X(t+h,x) - L_X(h,x))_{t\geq 0, x\in\mathbb{R}} \stackrel{d}{=} (L_X(t,x))_{t\geq 0, x\in\mathbb{R}}\,. \tag{2.6.8}$$

The proof of the Proposition 2.6.3 is straightforward (see for instance [42, Proposition 10.4.8]).

2.7 Weak convergence and tightness criteria for continuous processes

In the thesis we will often need the following theorem which characterizes weak convergence in the space of continuous functions with the supremum metric, from now on denoted by $\mathcal{C}[0,T]$, $0 < T < \infty$. For proof see [5, Theorem 8.1].

Theorem 2.7.1. *Let $(\mathbb{P}_n)$ be a sequence of probability measures on $\mathcal{C}[0,T]$ for some finite positive T. $\mathbb{P}_n$ converges weakly in $\mathcal{C}[0,T]$ to some probability measure $\mathbb{P}$ on $\mathcal{C}[0,T]$ if and only if the sequence $(\mathbb{P}_n)$ is tight and $(\mathbb{P}_n)$ converges to $\mathbb{P}$ in the sense of finite-dimensional distributions.*

One can often prove the convergence of finite-dimensional distributions of some process $(X_t)_{t\in[0,T]}$ by examining the characteristic functions of the vectors $(X_{t_1},\ldots,X_{t_n})$.

The criterion for tightness, which is often very useful and is used extensively in the thesis is given in the following theorem (see [5, Theorem 12.3]).

Theorem 2.7.2. *A sequence of stochastic processes $(X_n)_{n\in\mathbb{N}}$ ($X_n = (X_n(t))_{t\geq 0}$) is tight in $\mathcal{C}[0,1]$ if it satisfies the following two conditions:*

(i) The sequence of random variables $(X_n(0))_{n\in\mathbb{N}}$ is tight.

(ii) There exist constants $\gamma \geq 0$ and $\alpha > 1$ and a non-decreasing, continuous function $F : [0,1] \to \mathbb{R}$, such that

$$\mathbb{P}(|X_n(t) - X_n(s)| \geq \lambda) \leq \frac{1}{\lambda^\gamma}|F(t) - F(s)|^\alpha, \tag{2.7.1}$$

holds for all $s,t \in [0,1]$ and all $\lambda > 0$.

In particular (2.7.1) is implied by

$$\mathbb{E}|X_n(t) - X_n(s)|^\gamma \leq |F(t) - F(s)|^\alpha. \tag{2.7.2}$$

2.8 Regular variation

In this section we include some of the basic results on regular variation. They are extensively used in most of our work and we have decided to include them for the sake of reader's convenience.

Definition 2.8.1. *A measurable function $f : [0, \infty) \to \mathbb{R}$ is regularly varying at infinity with exponent $\gamma \in \mathbb{R}$ if f is eventually positive or negative and for each $a > 0$*

$$\lim_{x\to\infty} \frac{f(ax)}{f(x)} = a^{\gamma}. \tag{2.8.1}$$

If a function is regularly varying at infinity with exponent 0, then we say that it is *slowly varying at infinity.* The next result is immediate and provides a convenient way of working with regularly varying functions.

Lemma 2.8.2. *If a function f is regularly varying at infinity with exponent γ, then it can be represented in the form*

$$f(x) = x^{\gamma} L(x), \quad x \geq 0,$$

for some slowly varying function L.

For brevity, we will often denote the fact that a function f is regularly varying at infinity with exponent γ by writing $f \in RV_{\infty}(\gamma)$. One frequently considers also functions regularly varying at 0. We say that a measurable function $f : (0, \infty) \to \mathbb{R}$ is regularly varying at 0 if the function $z \mapsto f(1/z)$ is regularly varying at infinity.

Next, we recall some fundamental facts regarding regular variation which will be used frequently later on. The first result states that the convergence in (2.8.1) is uniform on compact subsets of $\mathbb{R}_+$ which do not contain 0. For its proof see [42, Proposition 10.5.5].

Proposition 2.8.3. *Let L be a slowly varying function. Then for every $0 < a < b < \infty$ and any $\epsilon > 0$ there exists $x_0 \in (0, \infty)$ such that for all $x \geq x_0$ and all $c \in [a, b]$,*

$$\left| \frac{L(cx)}{L(x)} - 1 \right| \leq \epsilon.$$

The next result says that regularly varying functions behave very much like power functions under integration (see Theorem 10.5.6 in [42]).

Theorem 2.8.4. *Assume that $f \in RV_{\infty}(\gamma)$ with $\gamma \geq -1$ and that f is locally integrable. Then the function defined by $F(x) := \int_0^x f(x)dx, x \geq 0$, is in $RV_{\infty}(\gamma + 1)$ and, moreover,*

$$\lim_{x\to\infty} \frac{F(x)}{xf(x)} = \frac{1}{1+\gamma}.$$

The following Proposition (see [42, Proposition 10.5.5]) states that if L is slowly varying at infinity, then for all c from a compact interval in $(0, \infty)$ for all x large enough the quantity $L(cx)/L(x)$ is arbitrarily close to 1.

Proposition 2.8.5. *Let L be slowly varying at infinity. Then for every $0 < a < b < \infty$ and $\delta > 0$, there exists $x_0 < \infty$ such that for all $x \geq x_0$,*

$$\left| \frac{L(cx)}{L(x)} - 1 \right| \leq \delta \tag{2.8.2}$$

for all $c \in [a, b]$.

The last result we are going to formulate is usually known under the name *Potter bounds* and follows from Karamata's representation theorem (see [42, Theorem 10.5.7]).

Proposition 2.8.6 (Potter bounds)**.** *Assume that $f \in RV_\infty(\gamma)$ for some $\gamma \in \mathbb{R}$. For every $\epsilon \in (0, 1)$, there exists $x_0 \in (0, \infty)$ such that for each $x \geq x_0$ and $a \geq 1$ we have*

$$(1 - \epsilon)a^{\gamma-\epsilon} \leq \frac{f(ax)}{f(x)} \leq (1 + \epsilon)a^{\gamma+\epsilon}.$$

In the course of the thesis we will also need the following result. For proof see [42, Theorem 10.5.6].

Theorem 2.8.7. *Let f be a positive function regularly varying at infinity with exponent $\beta \geq -1$. Assume that f is locally integrable, i.e., $\int_0^a f(x)dx < \infty$ for every $0 < a < \infty$. Then the function $F(x) = \int_0^x f(t)dt$, $x \geq 0$, is regularly varying at infinity with exponent $\beta + 1$ and satisfies*

$$\lim_{x\to\infty} \frac{F(x)}{xf(x)} = \frac{1}{\beta + 1}, \tag{2.8.3}$$

with $1/0$ defined as $+\infty$.

Part II

Results

Chapter 3

Particle picture interpretation for non-symmetric Rosenblatt process and Hermite processes

3.1 Introduction

In this chapter we discuss the particle picture interpretation of Hermite processes and non-symmetric Rosenblatt process. We give detailed proofs of the results described in Section 1.2.2. We first recall the most important concepts, then state the results in their full generality and finally provide their proofs.

The presentation in this chapter is based on the article [48] by the author of the thesis.

3.1.1 Hermite Processes and Generalized Hermite Processes

We follow section 7.1 in [34] and sketch the original framework which leads to Hermite processes. Let $(\xi_n)_{n\in\mathbb{Z}}$ be a centered stationary Gaussian sequence with variance equal to 1 such that

$$r(n) := \mathbb{E}(\xi_n \xi_0) = n^{\frac{2H-2}{k}} L(n), \tag{3.1.1}$$

with $H \in (\frac{1}{2}, 1)$, $k \geq 1$ and L - a function slowly varying at infinity. Take any function $g : \mathbb{R} \to \mathbb{R}$ satisfying $\mathbb{E}g(\xi_0) = 0$ and $\mathbb{E}g(\xi_0)^2 < \infty$, which has the following expansion in Hermite polynomials:

$$g(x) = \sum_{j=0}^{\infty} c_j H_j(x), \tag{3.1.2}$$

where H_j is the j-th Hermite polynomial, $c_j = \frac{1}{j!}\mathbb{E}(g(\xi_0)H_j(\xi_0))$ and k is the smallest j with $c_j \neq 0$. If we introduce the following sequence of stochastic processes:

$$Z_H^{k,n}(t) := \frac{1}{n^H} \sum_{l=1}^{[nt]} g(\xi_l), \quad , n \in \mathbb{N}, t \geq 0, \tag{3.1.3}$$

then

$$Z_H^{k,n} \overset{f.d.d.}{\Rightarrow} c_k Z_H^k, \tag{3.1.4}$$

where Z_H^k is the k-Hermite process and the convergence holds in the sense of finite-dimensional distributions. This is how Hermite processes were obtained in the first place.

Recall that for $k \in \mathbb{N}$ (using the notation from [3]) one can represent a k-Hermite process as

$$Z_H^k(t) := a_{k,d} \int_{\mathbb{R}^k}' \int_0^t \prod_{j=1}^k (s - x_j)_+^{d-1} ds W(dx_1) \dots W(dx_k), \tag{3.1.5}$$

where W is a two-sided Brownian motion, $\frac{1}{2}(1 - \frac{1}{k}) < d < \frac{1}{2}$ is the number satisfying $H = kd - k/2 + 1$ so that $1/2 < H < 1$, $a_{k,d}$ is a positive constant chosen so that $\mathbf{Var}(Z_H^k(1)) = 1$ and "′" above the integral sign indicates that the diagonal is excluded from integration. We use the so called spectral representation using multiple Wiener-Itô integrals in the sense of [28] (see Section 2.5.5):

$$Z_H^k(t) = c_{k,d} \int_{\mathbb{R}^k}'' \frac{e^{i(u_1+\ldots+u_k)t} - 1}{i(u_1 + \ldots + u_k)} |u_1|^{-d} \ldots |u_k|^{-d} \widehat{W}(du_1) \ldots \widehat{W}(du_k); \tag{3.1.6}$$

here the constant $c_{k,d}$ serves the same purpose as $a_{k,d}$ in (3.1.5) and $\widehat{W}$ is the random complex Gaussian white noise measure on $\mathbb{R}$, where d is given as before. For $k = 1$ the 1-Hermite process is just a fractional Brownian motion and the 2-Hermite process is called the *Rosenblatt process*. For all $k \in \mathbb{N}$ k-Hermite processes have the same covariance given by

$$\mathbb{E}(Z_H^k(s) Z_H^k(t)) =: R(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |s - t|^{2H}).$$

Recall from Section 1.2.2 that if we replace the kernel $\prod_{j=1}^{k}(s-x_j)_+^{d-1}$ in equation (3.1.5) (for $k=2$) by

$$g(x,y) = x^{-1+\alpha/2}y^{-1+\beta/2}\mathbf{1}_{\{x>0,y>0\}}, \tag{3.1.7}$$

with $\alpha, \beta \in (0,1)$ and $\alpha+\beta>1$, then the corresponding process as in 3.1.6 is the non-symmetric Rosenblatt process.

3.1.2 Representation of Hermite Processes

Hermite processes arise naturally as limits of normalized partial sums of stationary sequences as in (3.1.3). Recently, in [8] a different type of limit theorem was proved. It was shown that the Rosenblatt process can be obtained from a Poisson system of particles evolving according to α-stable processes. Our aim is to extend this representation to the general k-Hermite processes and the non-symmetric Rosenblatt process. Let us briefly sketch the particle system we are going to use.

Let (x^j) be a Poisson system with Lebesgue intensity measure on $\mathbb{R}$ and let $(\xi^j)_{j=1}^{\infty}$ be independent symmetric α-stable Levy processes with the index of stability $\alpha \in (0,1)$. Notice that we only consider the values of the parameter α for which $(\xi_t)_{t\geq 0}$ is transient. We also assume that these processes are independent of the points (x^j). In the end, assume that $(\sigma_j)_{j=1}^{\infty}$ are i.i.d random variables such that $\mathbb{P}(\sigma_1=1)=\mathbb{P}(\sigma_1=-1)=\frac{1}{2}$ and that these variables are independent of everything else. The particle system is given by $(x^j+\xi_t^j)_{t\geq 0}$. Thus the initial position of the particles is given by the points (x^j) and they evolve independently according to the symmetric α-stable processes. Furthermore, we independently assign charges σ_j to these particles.

In [8] it was shown that the process given by

$$\xi_t^T = \frac{1}{T}\sum_{j\neq k}\sigma_j\sigma_k\langle\Lambda(x^j+\xi^j, x^k+\xi^k;T), \mathbf{1}_{[0,t]}\rangle, \quad t\geq 0, \tag{3.1.8}$$

where Λ is the *intersection local time* of two independent α-stable Lévy processes ([8, Section 2.3], [1] and (1.2.4)), converges, as $T\to\infty$, (up to a constant) for $\alpha \in (1/2,1)$, in $C([0,\tau])$ for $\tau\in(0,\infty)$, to the Rosenblatt process with the Hurst coeficient $H=\alpha$. We will show how a non-symmetric Rosenblatt process is obtained from the same particle system. We will also extend the result of [8] to k-Hermite processes for $k\geq 3$.

3.2 Main results

First we will state a limit theorem leading to a non-symmetric Rosenblatt process. Consider the particle system described in Section 1.2.2 and let $\beta > \alpha$. Define

$$\eta_t^T = \frac{1}{T}\sum_{j\neq k}\sigma_j\sigma_k\int_0^T\int_0^T \mathbf{1}_{[0,t]}(x^j+\xi_r^j)\frac{1}{|x^k+\xi_s^k-x^j-\xi_r^j|^{1-\frac{\beta-\alpha}{2}}}drds \quad (3.2.1)$$

for $T > 0, t \geq 0$. The fact that the above functional is well defined and has a continuous modification is the content of the following proposition which is proved in Section 3.4.1. In the sequel we will use η^T to denote this continuous modification.

Proposition 3.2.1. *The process in* (3.2.1) *is well defined, in the sense that the sum in* (3.2.1) *converges in* $L^2(\Omega)$*, and has a continuous modification.*

The first of the two main results in this chapter is the related to the non-symmetric Rosenblatt process.

Theorem 3.2.2. *Let* α *and* β *be such that* $1 > \beta > \alpha > 0$*,* $\alpha + \beta > 1$*. Then, as* $T \to \infty$*, the process* $(\eta_t^T)_{t\geq 0}$ *converges, in* $\mathcal{C}[0,\tau]$ *for every* $\tau > 0$*, to the non-symmetric Rosenblatt process with parameters* (α,β)*, up to a multiplicative constant.*

The second question we set out to answer was whether the representation given by (1.2.3) can be extended to k-Hermite processes for general $k \geq 2$. To formulate our result we must first introduce the so called *k-intersection local time* (k-ILT), which is an extension of the notion of *intersection local time* (ILT) and was first considered in [44]. Informally, k-ILT of cadlag processes $\rho^1,\dots,\rho^k$ at time $T \geq 0$ (denoted here by $\Lambda^{(k)}$) can be defined by

$$\left\langle \Lambda^{(k)}(\rho^1,\dots,\rho^k;T),\phi\right\rangle = \\ = \int_{[0,T]^k}\phi(\rho_{s_1}^1)\delta_0(\rho_{s_2}^2-\rho_{s_1}^1)\dots\delta_0(\rho_{s_k}^k-\rho_{s_1}^1)ds_1\dots ds_k, \quad (3.2.2)$$

where δ_0 is the Dirac distribution at 0 and $\phi \in \mathcal{S}$ (the Schwartz space of rapidly decreasing function). One gives a meaning to (3.2.2) by approximating δ_0 by smooth fuctions. The precise definition and the proof of existence of ILT in the case of independent symmetric α-stable Lévy processes is given in Section 3.4.2, which is an extension of Proposition 5.1 in [11]. The answer to the second question is provided by the following theorem.

Theorem 3.2.3. *Let* $k \geq 2$ *be a natural number and* $\alpha \in (1-\frac{1}{k},1)$*. For* $T > 0$

we denote

$$\rho_t^T := \frac{1}{T^{k/2}} \sum_{j_1 \neq \ldots \neq j_k} \sigma_{j_1} \ldots \sigma_{j_k} \langle \Lambda^{(k)}(x^{j_1} + \xi^{j_1}, \ldots, x^{j_k} + \xi^{j_k}; T), \mathbf{1}_{[0,t]} \rangle, t \geq 0. \tag{3.2.3}$$

Then, the process ρ^T is well defined and has a continuous modification. Moreover, as $T \to \infty$, it converges, in $\mathcal{C}[0,\tau]$ for every $\tau > 0$, to the k-Hermite process Z_H^k with the Hurst coefficient equal to $H = 1 - (1-\alpha)k/2$.

The main scheme of the proofs of Theorems 3.2.2 and 3.2.3 is similar to the one employed in [8], in particular the idea of using Wick products of an appropriate $\mathcal{S}'$-valued random variable. To stress some of the main differences and difficulties that had to be overcome in our case let us point out that in the case of Theorem 3.2.2 it was at first not at all clear what functional of a particle system can be used to approximate the non-symmetric Rosenblatt process. Also, since the functional is different, we need to use different approximations.

In case of Theorem 3.2.3 it was more or less clear that one should use (3.2.3) as the approximating process. However, in the proof of the representation of the symmetric Rosenblatt process in [8] the identification of the limiting distribution was done using the cumulants and the fact that the finite-dimendional distributions of this process are determined by its moments. In our thesis we will take a different route and utilize the Itô formula for multiple Wiener-Itô integrals (Theorem 4.3 in [28]).

Additionally, since we have to deal with k-intersection local times and Wick products of order k, there are some non-trivial technical difficulties (see the proof of (3.4.54)).

This chapter is organized as follows. In Section 3.3 we fix the notation. Section 3.4.1 contains proof of Theorem 3.2.2 and in Section 3.4.2 we discuss the existence of k-intersection local time and prove Theorem 3.2.3.

3.3 Notation

Let $\mathcal{F}$ denote the class of non-negative symmetric, infinitely differentiable functions on $\mathbb{R}$ with with support in $B(0,1) = \{x \in \mathbb{R} : |x| < 1\}$ satisfying $\int_{\mathbb{R}} f(x)dx = 1$. These functions will be used to to approximate Dirac delta distributions. For any $f \in \mathcal{F}$, $\epsilon > 0$ put

$$f_\epsilon(x) = \epsilon^{-d} f(\frac{x}{\epsilon}), \qquad x \in \mathbb{R}. \tag{3.3.1}$$

λ_1 will denote one-dimensional Lebesgue measure on $\mathbb{R}$ and $\Rightarrow$ will denote convergence in law.

Since we are interested in the convergence of stochastic processes in the sense of finite dimensional distributions it is convenient to introduce the following class of functions. Let $\mathcal{A}$ be the family of functions of the form

$$\psi = \sum_{j=1}^{m} a_j \mathbf{1}_{I_j}, \tag{3.3.2}$$

where $a_j \in \mathbb{R}$ and I_j is a bounded interval for each $j = 1, \ldots, m$. For $g \in \mathcal{F}$ and $\psi \in \mathcal{A}$ we will write $\psi_\kappa := \psi * g_\kappa$, without explicitly referring to the function g to make the notation more transparent. As it happens, we will always require that in the limit the particular choice of g is irrelevant as far as our purposes are concerned. Notice that

$$\widehat{\psi_\kappa}(x) = \widehat{\psi}(x)\widehat{g_\kappa}(x) = \widehat{\psi}(x)\widehat{g}(\kappa x), \tag{3.3.3}$$

so that

$$|\widehat{\psi_\kappa}(x)| \leq |\widehat{\psi}(x)|, \quad x \in \mathbb{R}, \tag{3.3.4}$$

since $|\widehat{g}(z)| \leq 1$ for all $z \in \mathbb{R}$.

3.4 Proofs for Section 3.2

3.4.1 Proof of Theorem 3.2.2

Preliminary lemmas

We would like to have the generalizations of some of the facts used in [11] and [8]. First we state the generalized version of Lemma 8.1 from [11], which is the simplified version of the so called Mecke-Palm formula (see for instance equation (2.10) in [24]).

Lemma 3.4.1. *Let (x^j) be a Poisson system with intensity μ on $\mathbb{R}^d, d \geq 1$. If F is in $L^1(\mathbb{R}^{kd}, \mu^{\otimes k})$, then $\mathbb{E}\left(\sum_{j_1 \neq j_2 \neq \ldots \neq j_k} |F(x^{j_1}, \ldots, x^{j_k})|\right) < \infty$ and*

$$\mathbb{E}\left(\sum_{j_1 \neq j_2 \neq \ldots \neq j_k} G(x^{j_1}, \ldots, x^{j_k})\right) = \int_{\mathbb{R}^{kd}} F(x_1, \ldots, x_k)\mu(dx_1)\ldots\mu(dx_k).$$

Assume that $\xi^1, \ldots, \xi^k$ are independent symmetric α-stbale Lévy processes with $\alpha \in (1 - \frac{1}{k}, 1)$ and $k \geq 2$ is an integer. Moreover, let $(\sigma_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. random variables independent of $\xi^1, \ldots, \xi^k$ and such that $\mathbb{P}(\sigma_1 = 1) = \mathbb{P}(\sigma_1 = -1)$. We then have the following.

Lemma 3.4.2. *For any $F(\cdot+\xi^1,\dots,\cdot+\xi^k) \in L^2(\mathbb{R}^k \times \Omega, \lambda_k \otimes \mathbb{P})$ the series*

$$\sum_{j_1 \neq \ldots \neq j_k} \sigma_{j_1} \ldots \sigma_{j_k} F(x^{j_1}+\xi^{j_1},\dots,x^{j_k}+\xi^{j_k})$$

converges in $L^2(\Omega)$, and

$$\mathbb{E}\left(\sum_{j_1 \neq \ldots \neq j_k} \sigma_{j_1} \ldots \sigma_{j_k} F(x^{j_1}+\xi^{j_1},\dots,x^{j_k}+\xi^{j_k})\right)^2 =$$

$$= \int_{\mathbb{R}^k} \mathbb{E}\left(\sum_{\pi \in \Pi(k)} F(x_1+\xi^1,\dots,x_k+\xi^k) \times F(x_{\pi_1}+\xi^{\pi_1},\dots,x_{\pi_k}+\xi^{\pi_k})\right) dx_1 \ldots dx_k,$$

where the summation in the second integral is over all permutations π of $\{1,2,\dots,k\}$.

This lemma follows immediately from Lemma 3.4.1 and the fact that the σ_js are independent of $\xi^1,\dots,\xi^k$.

Proof of Proposition 3.2.1

Let us denote

$$\langle \Delta(x+\xi^1, y+\xi^2; T), \phi\rangle = \int_0^T \int_0^T \phi(x+\xi^1_r)|y+\xi^2_s - x-\xi^1_r|^{\frac{\beta-\alpha}{2}-1} dr ds, \quad (3.4.1)$$

where ξ^1, ξ^2 are independent symmetric α-stable Lévy processes. According to Lemma 3.4.2, in order to show that (3.4.1) is well defined (in the sense that the series converges in $L^2(\Omega)$) it suffices to show that for any $t>0$ and independent symmetric α-stable processes ξ^1, ξ^2 we have

$$\langle \Delta(\cdot+\xi^1, \cdot+\xi^2; T), \mathbf{1}_{[0,t]}\rangle \in L^2(\mathbb{R}\times\mathbb{R}\times\Omega, \lambda_1 \otimes \lambda_1 \otimes \mathbb{P}), \quad (3.4.2)$$

which is done in the following lemma.

Lemma 3.4.3. *Let $\alpha, \beta \in (0,1)$ be such that $\alpha+\beta>1$, $\beta>\alpha$ and let $\phi \in L^1(\mathbb{R})$ be bounded. Then*

$$I := \mathbb{E}\left(\int_{\mathbb{R}^2}\left(\int_0^T \int_0^T \phi(x+\xi^1_r)|y+\xi^2_s - x - \xi^1_r|^{\gamma-1} ds dr\right)^2 dx dy\right) \le CT^2 < \infty, \quad (3.4.3)$$

where $\gamma := \frac{\beta-\alpha}{2}$ and C is a constant.

Proof. Without loss of generality we may assume that $\phi \geq 0$. We have

$$\begin{aligned} I &= \mathbb{E}\int_{\mathbb{R}^2}\int_{[0,T]^4}\phi(x+\xi^1_r)|y+\xi^2_s-x-\xi^1_r|^{\gamma-1} \\ &\quad \phi(x+\xi^1_u)|y+\xi^2_v-x-\xi^1_u|^{\gamma-1}drdsdudvdxdy \qquad (3.4.4)\\ &= \mathbb{E}\int_{\mathbb{R}^2}\int_{[0,T]^4}\phi(x)|y-x|^{\gamma-1}\phi(x+\xi^1_u-\xi^1_r) \\ &\quad |y+\xi^2_v-\xi^2_s-x-(\xi^1_u-\xi^1_r)|^{\gamma-1}drdsdudvdxdy \qquad (3.4.5)\\ &= \int_{\mathbb{R}^4}\int_{[0,T]^4}\phi(x)|y-x|^{\gamma-1}\phi(z)|w-z|^{\gamma-1} \\ &\quad p_{|u-r|}(z-x)p_{|v-s|}(w-y)drdudsdvdxdydzdw, \qquad (3.4.6)\end{aligned}$$

where p is the α-stable transition density. The second equality uses Fubini theorem. Since for $x \neq 0$, $\int_0^\infty p_s(x)ds = C|x|^{\alpha-1}$ for some constant $C = C(\alpha)$, $\alpha < 1$ and

$$\int_0^T\int_0^T p_{|u-r|}(x)drdu \leq \frac{2CT}{|x|^{1-\alpha}}, \qquad (3.4.7)$$

we see that I can be bounded by

$$C^2T^24\int_{\mathbb{R}^2}\phi(x)|y-x|^{\gamma-1}\phi(z)|w-z|^{\gamma-1}\frac{1}{|z-x|^{1-\alpha}}\frac{1}{|w-y|^{1-\alpha}}dxdydzw. \qquad (3.4.8)$$

Notice that

$$\int_{\mathbb{R}}|y-x|^{\gamma-1}|w-y|^{\alpha-1}dy = C'(\alpha,\beta)|w-x|^{\frac{\alpha+\beta}{2}-1}, \qquad (3.4.9)$$

with $C'(\alpha,\beta)$ being a constant. Similarly

$$\int_{\mathbb{R}}|w-x|^{\frac{\alpha+\beta}{2}-1}|w-z|^{\gamma-1}dw = C''(\alpha,\beta)|z-x|^{\beta-1}. \qquad (3.4.10)$$

Thus

$$I \leq C'''(\alpha,\beta)T^2\int_{\mathbb{R}^2}\phi(x)\phi(z)|z-x|^{\alpha+\beta-2}. \qquad (3.4.11)$$

The right-hand side of (3.4.11) is finite since ϕ is bounded and in $L^1(\mathbb{R})$ and $\alpha+\beta > 1$. □

Corollary 3.4.4. *The process η^T given by* (3.2.1) *has a continuous modification.*

Proof. From (3.4.11), for $\phi = \mathbf{1}_{(s,t]}$, $0 \leq s < t$, we have that

$$\begin{aligned}\mathbb{E}\left(\eta^T_t-\eta^T_s\right)^2 &\leq T^2C(\alpha,\beta)\int_{\mathbb{R}^2}\mathbf{1}_{(s,t]}(x)\mathbf{1}_{(s,t]}(z)|x-z|^{\alpha+\beta-2}dxdz \qquad (3.4.12)\\ &= T^2C(\alpha,\beta)(t-s)^{\alpha+\beta}\int_{\mathbb{R}^2}\mathbf{1}_{(0,1]}(x)\mathbf{1}_{(0,1]}(z)|x-z|^{\alpha+\beta-2}dxdz \\ &= T^2C'(\alpha,\beta)(t-s)^{\alpha+\beta},\end{aligned}$$

where the constants $C(\alpha,\beta)$ and $C'(\alpha,\beta)$ are independent of $T>0$. Since $\alpha+\beta>1$, this shows that for each $T>0$, η^T has a continuous modification. □

Idea of the proof of Theorem 3.2.2

Let us briefly discuss the main ideas behind the proof of convergence of finite-dimensional distributions of η^T to those of the Rosenblatt process. We will show that the functional η^T_t is close to $\langle : X_T \otimes X_T :, \Phi_t\rangle$, where $: X_T \otimes X_T :$ is the Wick product of the process X_T defined by (2.5.22) and $\Phi_t(x,y) = \mathbf{1}_{[0,t]}(x)|x-y|^{\frac{\beta-\alpha}{2}-1}$, $x,y\in\mathbb{R}$. Recall that the Wick product $: X_T \otimes X_T :$ is defined in the following way (here we use the more general Definition 2.5.4). First, for $\Phi\in\mathcal{S}(\mathbb{R}^2)$ of the form

$$\Phi = \sum_{j=1}^{m} \phi_j \otimes \psi_j, \tag{3.4.13}$$

where $\phi_j,\psi_j\in\mathcal{S}$, for $j=1,\ldots,m$, we set

$$\langle : X_T \otimes X_T :, \Phi\rangle := \sum_{j=1}^{m} \Big(\langle X_T,\phi_j\rangle\langle X_T,\psi_j\rangle - \mathbb{E}(\langle X_T,\phi_j\rangle\langle X_T,\psi_j\rangle)\Big). \tag{3.4.14}$$

(3.4.14) can then be extended to arbitrary $\Phi\in\mathcal{S}(\mathbb{R}^2)$. Next we would like to use Theorem 2.5.14 to obtain that $\langle : X_T \otimes X_T :, \Phi_t\rangle$ converges, as $T\Rightarrow\infty$, in distribution to $\langle : X\otimes X :, \Phi_t\rangle$. Finally we will show that $(\langle : X\otimes X :, \Phi_t\rangle)_{t\geq 0}$ is up to a constant a non-symmetric Rosenblatt process. One of the difficulties lies in the fact that Φ_t is not in $\mathcal{S}(\mathbb{R}^2)$ and we must approximate it by functions from the Schwartz space. $\langle : X\otimes X :, \Phi_t\rangle$ is then understood as a limit under these approximations.

Approximating functionals

We now proceed to discussing our approximating functions and some of their properties. For convenience let us set $\gamma = \frac{\beta-\alpha}{2}$. We are going to approximate the function $y\mapsto |y|^{\gamma-1}$ by the convolution $\int_{\mathbb{R}} |y-z|^{\gamma-1} f_\epsilon(z)dz$ (where $f\in\mathcal{F}$ and $\epsilon>0$), and then use the fact that as $\epsilon\to 0$ the integral converges (up to a constant depending only on γ) to $|y|^{-\gamma}$. recall that f_ϵ is defined in 3.3.1. However, the function $y\mapsto \int_{\mathbb{R}} |y-z|^{\gamma-1} f_\epsilon(z)dz$ still does not belong to $\mathcal{S}(\mathbb{R})$ as it vanishes slowly. To overcome this obstacle take $\delta\in(0,1)$, let $h_\delta(x) := |x|^{\gamma-1}\mathbf{1}_{\delta<|x|<\frac{1}{\delta}}$ and put

$$V^\delta\phi(x) := \int_{\mathbb{R}} h_\delta(x-y)\phi(y)dy, \quad \phi\in\mathcal{S}, x\in\mathbb{R}. \tag{3.4.15}$$

We approximate the function $y\mapsto |y|^{\gamma-1}$ by $V^\delta f_\epsilon$. Let us also define $V\phi(x) := \lim_{\delta\to 0_+} V^\delta\phi(x)$. This limit exists as long as $\int_{\mathbb{R}} |x-y|^{\gamma-1}|\phi(y)|dy<\infty$. It is easy to see that for $f \in C_c^\infty(\mathbb{R})$, $V^\delta f \in C_c^\infty(\mathbb{R})$ for any $\delta \in (0,1)$. The Fourier transform of $V^\delta \phi$ is given by

$$\widehat{V^\delta \phi}(x) = \widehat{h_\delta}(x)\widehat{\phi}(x), \quad x \in \mathbb{R}, \tag{3.4.16}$$

for $\phi \in \mathcal{S}$. In the sequel we will need several of properties of operators V^δ. We list them in the lemma below.

Lemma 3.4.5. *Let f be in $\mathcal{F}$. The operators V^δ defined by* (3.4.15) *have the following properties:*

(i) for each $\delta \in (0,1)$ $V^\delta f$ is nondecreasing as $\delta \searrow 0$ and

$$Vf(x) = \lim_{\delta \to 0_+} V^\delta f(x); \tag{3.4.17}$$

(ii)

$$\lim_{\epsilon \to 0_+} Vf_\epsilon(x) = |x|^{\gamma-1}, \quad x \neq 0; \tag{3.4.18}$$

(iii) for each $\epsilon > 0$

$$Vf_\epsilon(x) \leq \|f\|_\infty \frac{2^{2-\gamma}}{\gamma}|x|^{\gamma-1}. \tag{3.4.19}$$

Proof. Parts (i) and (ii) are obvious. To prove part (iii) fix $\epsilon > 0$ and consider two cases. Assume first that $|x| \geq 2\epsilon$. Then $Vf_\epsilon(x) \leq 1/|x-\epsilon|^{1-\gamma} \leq 2^{1-\gamma}|x|^{\gamma-1}$. If $|x| < 2\epsilon$ then

$$Vf_\epsilon(x) = \int_{-\epsilon}^{\epsilon} \frac{1}{|x-y|^{1-\gamma}} \frac{1}{\epsilon} f\left(\frac{y}{\epsilon}\right) dy \leq \|f\|_\infty \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} \frac{1}{|y|^{1-\gamma}} dy = \|f\|_\infty \frac{2^{2-\gamma}}{\gamma}|x|^{\gamma-1}, \tag{3.4.20}$$

and this finishes the proof. □

Now we can define the approximating functional which will be at the center of our investigation. Let $\epsilon, \delta \in (0,1), f \in \mathcal{F}$. Mimicking (3.4.1), for any $\phi \in \mathcal{S}(\mathbb{R})$ and a pair of real-valued cadlag processes η, ξ we put

$$\langle \Delta^f_{\epsilon,\delta}(\eta, \xi; T), \phi \rangle := \int_0^T \int_0^T \phi(\eta_u)(V^\delta f_\epsilon)(\xi_v - \eta_u) du dv. \tag{3.4.21}$$

For $\phi \geq 0$ the above integral converges pointwise as $\delta \to 0$ and then $\epsilon \to 0$ by Lemma 3.4.5 (to a possibly infinite limit) independently of the choice of f. This limit is given by

$$\langle \Delta(\eta, \xi; T), \phi \rangle = \int_0^T \int_0^T \phi(\eta_u)|\xi_v - \eta_u|^{\gamma-1} du dv, \quad \phi \in \mathcal{S}. \tag{3.4.22}$$

Now we proceed to show that in the setting that interests us most ($\eta = x+\xi^1, \xi = y + \xi^2$ for independent symmetric α-stable Lévy processes ξ^1 ,ξ^2 and $x, y \in \mathbb{R}$) the random variables given by (3.4.21) and (3.4.22) are meaningful. We can only show that $\Delta(\eta_1, \eta_2; T)$ exists as an $\mathcal{S}'$-valued random variable in the setting in which $\alpha, \beta \in (1/2, 1)$. The convergece of $\langle \Delta^f_{\epsilon,\delta}(\eta_1, \eta_2; T), \phi\rangle$ in $L^2(\Omega)$ for fixed $x, y \in \mathbb{R}$ remains an open question. However, we will only need the following.

Lemma 3.4.6. *Let ξ^1, ξ^2 be independent symmetric α-stable processes with $\alpha \in (0, 1)$.*

(i) For every $\phi \in \mathcal{S}(\mathbb{R}), T > 0$ the function given by

$$(x, y, \omega) \mapsto \langle \Delta^f_{\epsilon,\delta}(x + \xi^1, y + \xi^2; T), \phi\rangle \tag{3.4.23}$$

converges in $L^2(\mathbb{R}^2 \times \Omega, \lambda_1 \otimes \lambda_1 \otimes \mathbb{P})$, as ϵ, δ go to 0, to the function

$$(x, y, \omega) \mapsto \Delta(x, y, \omega) = \int_0^T \int_0^T \phi(x+\xi^1_r(\omega))|y+\xi^2_s(\omega)-x-\xi^2_r(\omega)|^{\frac{\beta-\alpha}{2}-1}drds. \tag{3.4.24}$$

(ii) The L^2-convergence as in (i) holds also if we replace ϕ by any function $\psi \in \mathcal{A}$. Moreover, for ϕ of the form $\phi = \psi_\kappa = \psi * g_\kappa$, where $\kappa \in (0, 1)$ and $g \in \mathcal{F}$, the convergence is uniform in κ.

Here λ_1 is the one dimensional Lebesgue measure and $\mathbb{P}$ is the underlying probability measure.

Proof of Lemma 3.4.6. The proof is quite straightforward once we have established Lemma 3.4.3. We have

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^2} \Big(\langle \Delta^f_{\epsilon,\delta}(x + \xi^1, y + \xi^2; T), \phi\rangle - \langle \Delta(x + \xi^1, y + \xi^2; T), \phi\rangle\Big)^2 dxdy \\
&\qquad \leq \mathbb{E}\int_{\mathbb{R}^2} \Bigg(\int_{[0,T]^2} |\phi(x + \xi^1_u)|\Big|V^\delta_{f_\epsilon}(y + \xi^2_v - x - \xi^1_u) \\
&\qquad\qquad - |y + \xi^2_v - x - \xi^1_u|^{\gamma-1}\Big|dudv\Bigg)^2 dxdy \\
&\quad \leq \mathbb{E}\int_{\mathbb{R}^2} \Bigg(\int_{[0,T]^2} |\phi(x + \xi^1_u)|C_1(f, \gamma)|y + \xi^2_v - x - \xi^1_u|^{\gamma-1}dudv\Bigg)^2 dxdy,
\end{aligned} \tag{3.4.25}$$

which is finite by Lemma 3.4.3. In the second inequality in (3.4.25) we have used part (iii) of Lemma 3.4.5. Now, using parts (i) and (ii) of the same Lemma and dominated convergence theorem we get the desired convergence. Evidently, the particular choice of f from $\mathcal{F}$ is irrelevant. The rest of the proof is now straightforward. □

Main body of the proof

Proof of Theorem 3.2.2. The proof of Theorem 3.2.2 more or less follows the line of reasoning of the proof of Theorem 3.5 in [8]. We will study the behavior of the following functional, which approximates the functional in the statement of Theorem 3.2.2:

$$\eta^T_{f,\phi,\epsilon,\delta} := \frac{1}{T}\sum_{j\neq k}\sigma_j\sigma_k\langle\Delta^f_{\epsilon,\delta}(x^j+\xi^j,x^k+\xi^k;T),\phi\rangle, \tag{3.4.26}$$

where, as before $\epsilon,\delta\in(0,1), f\in\mathcal{F},\phi\in\mathcal{S}(\mathbb{R}),T>0$. It is well defined by Lemma 3.4.2. Moreover, the above functional converges in $L^2(\Omega)$, as $\epsilon,\delta\to 0$ (uniformly in $T\geq 1$ and independently of the choice of $f\in\mathcal{F}$) to a random variable η^T_ϕ given by

$$\eta^T_\phi := \frac{1}{T}\sum_{j\neq k}\sigma_j\sigma_k\langle\Delta(x^j+\xi^j,x^k+\xi^k;T),\phi\rangle, \tag{3.4.27}$$

which again follows from Lemmas 3.4.6 and 3.4.2. Recall that $\gamma=(\beta-\alpha)/2$. Let ψ be any function from $\mathcal{A}$ and $\Psi^f_{\epsilon,\delta,\phi}(x,y):=\phi(x)(V^\delta f_\epsilon)(y-x)$, where V^δ is defined by (3.4.15). We will show the convergence, as $T\to\infty$, of the characteristic function of η_{T,ψ_κ} to the characteristic function of the finite-dimensional distributions of the non-symmetric Rosenblatt process. Using inequality $|\mathbb{E}(e^{iX})-\mathbb{E}(e^{i\widetilde{X}})|\leq 2\mathbb{E}|X-\widetilde{X}|\leq 2\left(\mathbb{E}|X-\widetilde{X}|^2\right)^{\frac{1}{2}}$ (for real valued random variables X and $\widetilde{X}$, it is enough to show that:

$$\lim_{\kappa\to 0}\ \sup_{T\geq 1}\ \mathbb{E}\left|\eta^T_\psi-\eta^T_{\psi_\kappa}\right|^2=0, \tag{3.4.28}$$

$$\lim_{\epsilon,\delta\to 0}\ \sup_{T\geq 1}\sup_{\kappa\in(0,1)}\mathbb{E}\left|\eta^T_{f,\psi_\kappa,\epsilon,\delta}-\eta^T_{\psi_\kappa}\right|^2=0, \tag{3.4.29}$$

$$\lim_{T\to\infty}\sup_{\kappa\in(0,1)}\mathbb{E}\left|\langle :X_T\otimes X_T:,\Psi^f_{\epsilon,\delta,\phi}\rangle-\eta^T_{f,\phi,\epsilon,\delta}\right|^2=0,\quad \epsilon,\delta>0,\phi\in\mathcal{S}, \tag{3.4.30}$$

$$\langle :X_T\otimes X_T:,\Psi^f_{\epsilon,\delta,\phi}\rangle \underset{T\to\infty}{\Rightarrow} \langle :X\otimes X:,\Psi^f_{\epsilon,\delta,\phi}\rangle,\quad \epsilon,\delta>0,\phi\in\mathcal{S}, \tag{3.4.31}$$

$$\lim_{\epsilon,\delta,\kappa\to\infty}\mathbb{E}\left|\langle :X\otimes X:,\Psi^f_{\epsilon,\delta,\psi_\kappa}\rangle-\int''_{\mathbb{R}^2}\widehat{\psi}(x+y)|y|^{-\gamma}Z_G(dx)Z_G(dy)\right|^2=0, \tag{3.4.32}$$

where Z_G is the random spectral measure as in Remark 2.5.15.

Similarly as in the proof of Lemma 3.4.6 it is easy to show (using dominated convergence theorem) that (3.4.28) holds. It is enough to notice that $|\psi_\kappa| \leq |\psi|$. Lemma 3.4.6 and Lemma 8.1 from [11] give us (3.4.29) (for details see the proof of equation (6.26) in [8]). The proof of (3.4.30) is very similar to the proof of equation (6.27) in [8] so we will only sketch it. Recalling (2.5.22) we may write

$$\begin{aligned}\langle : X_T \otimes X_T :, \Psi^f_{\epsilon,\delta,\phi}\rangle &= \frac{1}{T}\int_{[0,T]^2}\Big(\sum_{j,k}\sigma_j\sigma_k\Psi^f_{\epsilon,\delta,\phi}(x^j+\xi^j_s, x^k+\xi^k_u)\\ &\quad - \int_{\mathbb{R}}\mathbb{E}\Psi^f_{\epsilon,\delta,\phi}(x+\xi^1_s, x+\xi^1_u)dx\Big)dsdu\\ &= \eta^f_{T,\phi,\epsilon,\delta} + \frac{1}{T}\int_{[0,T]^2}\Big(\sum_j \Psi^f_{\epsilon,\delta,\phi}(x^j+\xi^j_s, x^j+\xi^j_u)\\ &\quad - \int_{\mathbb{R}}\mathbb{E}\Psi^f_{\epsilon,\delta,\phi}(x+\xi^1_s, x+\xi^1_u)dx\Big)dsdu. \qquad (3.4.33)\end{aligned}$$

This implies that, again by Lemma 3.4.1 in the Appendix,

$$\mathbb{E}\left|\langle : X_T \otimes X_T :, \Psi^f_{\epsilon,\delta,\phi}\rangle - \eta^f_{T,\phi,\epsilon,\delta}\right|^2 = \\ \frac{1}{T}\int_{[0,T]^4}\int_{\mathbb{R}}\mathbb{E}\Psi^f_{\epsilon,\delta,\phi}(x+\xi^1_s, x+\xi^1_u)\Psi^f_{\epsilon,\delta,\phi}(x+\xi^1_r, x+\xi^1_v)dxdsdudrdv. \quad (3.4.34)$$

The rest of the argument is exactly the same as in the proof of equation (6.26) in [8] because $V^\delta f_\epsilon \in \mathcal{S}(\mathbb{R})$ implies that $\Psi^f_{\epsilon,\delta,\phi}$ is in $\mathcal{S}(\mathbb{R}^2)$. The convergence in (3.4.31) follows from Lemma 6.3 in [8]. For the proof of (3.4.32) we will look at Hermite processes from the point of view of multiple Wiener-Ito integrals. From Theorem 4.7 in [28] it follows that for any $\Phi \in \mathcal{S}(\mathbb{R}^2)$

$$\langle : X \otimes X :, \Phi\rangle = \int''_{\mathbb{R}^2}\widehat{\Phi}(x,y)Z_G(dx)Z_G(dy), \qquad (3.4.35)$$

where Z_G is the random spectral measure, as in Remark 2.5.15, corresponding to the spectral measure $G(dx) = |x|^{-\alpha}dx$. Hence, using (3.4.16),

$$\langle : X \otimes X :, \Psi^f_{\epsilon,\delta,\psi_\kappa}\rangle = \frac{1}{2\pi}\int''_{\mathbb{R}^2}\widehat{\psi_\kappa}(x+y)\widehat{f_\epsilon}(y)\widehat{h_\delta}(y)Z_G(dx)Z_G(dy). \qquad (3.4.36)$$

By dominated convergence and L^2 isometry $\langle : X \otimes X :, \Psi^f_{\epsilon,\delta,\psi_\kappa}\rangle$ converges in $L^2(\Omega)$ as $\epsilon, \delta, \kappa \to 0$ to $\frac{1}{2\pi}\int''_{\mathbb{R}^2}\widehat{\psi}(x+y)|y|^{-\gamma}Z_G(dx)Z_G(dy)$, which by the change of variables formula for multiple Wiener-Itô integrals (Theorem 4.4 in [28]), is equal to

$$\frac{1}{2\pi}\int''_{\mathbb{R}^2}\widehat{\psi}(x+y)|x|^{-\frac{\alpha}{2}}|y|^{-\frac{\beta}{2}}\widehat{W}(dx)\widehat{W}(dy), \qquad (3.4.37)$$

where $\widehat{W}$ is a complex-valued Fourier transform of white noise (which is another name for a gaussian random spectral measure with Lebesgue spectral density). When we replace ψ with $\mathbf{1}_{[0,t]}$, then (using definition 3.8 in [3] and spectral representation discussed in the following remarks) we can define

$$Z_t^T := \eta_{T,\mathbf{1}_{[0,t]}}, \quad t \geq 0, \tag{3.4.38}$$

then $(Z_t^T)_{t\geq 0}$ converges (up to a constant), as $T \to \infty$, in the sense of finite dimensional distributions, to the non-symmetric (α, β)-Rosenblatt process.

Tightness in $\mathcal{C}[0,\tau]$ for every $\tau > 0$ follows from Corollary 3.4.4. More precisely, it is a consequence of (3.4.12) and Theorem 12.3 in [5]. This finishes the proof. □

3.4.2 k-intersection local times and the proof of Theorem 3.2.3

k-intersection local time of independent α-stable processes

Following [11] we would like to extend the definition of intersection local time to *k-intersection local time.* For $\epsilon > 0, \phi \in \mathcal{S}(\mathbb{R}), f \in \mathcal{F}$ and an integer $k \geq 2$ put

$$\Phi^f_{\epsilon,\phi}(x_1, \dots, x_k) := \phi(x_1) f_\epsilon(x_2 - x_1) \dots f_\epsilon(x_k - x_1). \tag{3.4.39}$$

Note that $\Phi^f_{\epsilon,\phi} \in \mathcal{S}(\mathbb{R}^k)$ and

$$\widehat{\Phi^f_{\epsilon,\phi}}(x_1, \dots, x_k) = \widehat{\phi}(x_1 + \dots + x_k) \widehat{f_\epsilon}(x_2) \dots \widehat{f_\epsilon}(x_k). \tag{3.4.40}$$

Using (3.4.39) we can define the *approximate intersection local time* of k real valued cadlag stochastic processes. For any cadlag processes $\rho_1, \dots, \rho_k$ taking values in $\mathbb{R}$, $\phi \in \mathcal{S}(\mathbb{R})$ and $f \in \mathcal{F}$ we denote the approximate intersection local time at time $T > 0$ by

$$\left\langle \Lambda^f_\epsilon(\rho^1, \dots, \rho^k; T), \phi \right\rangle = \\ = \int_{[0,T]^k} \phi(\rho^1_{s_1}) f_\epsilon(\rho^2_{s_2} - \rho^1_{s_1}) \dots f_\epsilon(\rho^k_{s_k} - \rho^1_{s_1}) ds_1 \dots ds_k. \tag{3.4.41}$$

Definition 3.4.7. *If there exists an $\mathcal{S}'$-valued random variable $\Lambda^{(k)}(\rho_1, \dots, \rho_k)$ such that for each $\phi \in \mathcal{S}$ and $f \in \mathcal{F}$ $\left\langle \Lambda^f_\epsilon(\rho^1, \dots, \rho^k; T), \phi \right\rangle$ converges to $\langle \Lambda^{(k)}(\rho_1, \dots, \rho_k), \phi \rangle$ in $L^2(\Omega)$and the limit is independent of the choice of $f \in \mathcal{F}$ then $\Lambda^{(k)}$ is called the k-intersection local time of $\rho_1, \dots, \rho_k$.*

We have the following extension of Proposition 5.1 in [11].

Lemma 3.4.8. *Let $\xi_1, \ldots, \xi_k$ be independent α-stable Lévy processes with $\alpha \in (1-\frac{1}{k}, 1)$. Then for any starting points $x_1, \ldots, x_k \in \mathbb{R}$ and $\phi \in \mathcal{S}$ the k-intersection local time $\left\langle \Lambda^{(k)}(x_1+\xi^1, \ldots, x_k+\xi^k; T), \phi \right\rangle$ exists. Moreover $\Lambda^{(k)}$ can be evaluated for any function ϕ in $\mathcal{A}$.*

Proof. To prove the lemma it is enough to show that for any $f, g \in \mathcal{F}$, $x_1, \ldots, x_k \in \mathbb{R}$ and each $\phi \in \mathcal{S}$, the limit

$$\lim_{\epsilon,\delta\to 0} \mathbb{E}\left(\left\langle \Lambda^f_\epsilon(x_1+\xi^1, \ldots, x_k+\xi^k; T), \phi \right\rangle \left\langle \Lambda^g_\delta(x_1+\xi^1, \ldots, x_k+\xi^k; T), \phi \right\rangle \right) \tag{3.4.42}$$

exists and does not depend on the choice of f and g. The proof is at first very similar to the proof of Proposition 3.3 and 5.1 in [11]. Writing out the expectation in (3.4.42) using the α-stable transition densities, passing to the Fourier transform, using Plancherel formula and then using the estimate

$$\int_{[0,T]^2} |\widehat{\mu}_{s,u}(z,z')| ds du \le C(T) \frac{1}{1+|z+z'|^\alpha} \left(\frac{1}{1+|z|^\alpha} + \frac{1}{1+|z'|^\alpha} \right),$$

where $C(T)$ is a constant and $\mu_{s,u}$ is the law of (ξ^1_s, ξ^1_u), the proof is reduced to showing that

$$\begin{aligned} I &= \int_{\mathbb{R}^{2k}} |\widehat{\phi}(z_1+\ldots+z_k)\widehat{\phi}(w_1+\ldots+w_k)| \\ &\quad \times b(z_1,w_1)\ldots b(z_k,w_k) dz_1 \ldots dz_k dw_1 \ldots dw_k < \infty, \end{aligned} \tag{3.4.43}$$

where $b(z,w) = C(T)\frac{1}{1+|z+w|^\alpha}(\frac{1}{1+|z|^\alpha} + \frac{1}{1+|w|^\alpha})$. To show that we will use Hölder inequality. First, fix some $\lambda \in (0,1)$. Rewrite equation (3.4.43) as follows.

$$\begin{aligned} I &= \int_{\mathbb{R}^{2k}} |\widehat{\phi}(z_1+\ldots+z_k)^{\frac{k}{k}} \widehat{\phi}(w_1+\ldots+w_k)^{\frac{k}{k}}| \\ &\quad \times b(z_1,w_1)^{\lambda\frac{k}{k}} \ldots b(z_k,w_k)^{\lambda\frac{k}{k}} \\ &\quad \times b(z_1,w_1)^{(1-\lambda)\frac{k-1}{k-1}} \ldots b(z_k,w_k)^{(1-\lambda)\frac{k-1}{k-1}} dz_1 \ldots dz_k dw_1 \ldots dw_k. \end{aligned} \tag{3.4.44}$$

The integrand can be written as $g_1(\mathbf{z},\mathbf{w}) \ldots g_k(\mathbf{z},\mathbf{w})$, where

$$\begin{aligned} g_j(\mathbf{z},\mathbf{w}) &= |\widehat{\phi}(z_1+\ldots+z_k)\widehat{\phi}(w_1+\ldots+w_k)|^{\frac{1}{k}} h(\mathbf{z},\mathbf{w})^{\frac{\lambda}{k}} \\ &\quad \times b(z_1,w_1)^{(1-\lambda)\frac{1}{k-1}} \ldots b(z_{j-1},w_{j-1})^{(1-\lambda)\frac{1}{k-1}} \\ &\quad \times b(z_{j+1},w_{j+1})^{(1-\lambda)\frac{1}{k-1}} \ldots b(z_k,w_k)^{(1-\lambda)\frac{1}{k-1}}, \end{aligned} \tag{3.4.45}$$

and $h(\mathbf{z},\mathbf{w}) = b(z_1,w_1)\ldots b(z_k,w_k)$. By Hölder inequality

$$I \le \prod_{j=1}^k \left(\int_{\mathbb{R}^{2k}} g(\mathbf{z},\mathbf{w})^k d\mathbf{z} d\mathbf{w} \right)^{\frac{1}{k}} \tag{3.4.46}$$

For $\phi \in \mathcal{S}$ or $\mathcal{A}$, $|\widehat{\phi}(x)| \le \frac{C}{1+|x|}$, $\quad x \in \mathbb{R}$, where C is a constant. Now, taking λ close enough to 0, we have $\frac{(1-\lambda)k\alpha}{k-1} > 1$. This implies that each factor in (3.4.46) is finite. $\square$

In fact we will not need this "pointwise" sort of convergence and we will only utilize a weaker result (which is an analogue of Lemma 3.4.6) to be able to formulate the main theorem of this section in a rigorous way.

Lemma 3.4.9. *Assume that* $\alpha \in (1-\frac{1}{k}, 1)$*. Then* $\langle \Lambda^f_\epsilon(\cdot+\xi^1, \dots, \cdot+\xi^k; T), \phi \rangle$ *converges in* $L^2(\mathbb{R}^k \times \Omega, \lambda_k \otimes \mathbb{P})$ *as* $\epsilon \to 0$ *for any* $\phi \in \mathcal{S}(\mathbb{R})$ *and the limit is independent of he choice of* $f \in \mathcal{F}$*. Moreover, if we replace* ϕ *by any function of the form* ψ_κ *as in* (3.3.2)*, the convergence is uniform in* $\kappa \in (0,1)$*. We also denote this limit by* $\langle \Lambda^{(k)}(\cdot+\xi^1, \dots, \cdot+\xi^k; T), \phi \rangle$*.*

The proof of this lemma is similar to the case when $k = 2$ and amounts to showing that that

$$\int_{\mathbb{R}^k} |\widehat{\phi}(x_1+\ldots+x_k)|^2 |x_1|^{-\alpha} \ldots |x_k|^{-\alpha} dx_1 \ldots dx_k < \infty, \tag{3.4.47}$$

for $\phi \in \mathcal{S}(\mathbb{R})$ or $\mathcal{A}$. In the symmetric case above (3.4.47) follows by Hölder inequality, similarly as in Lemma 3.4.8. Indeed, putting

$$s_j(x_1, \dots, x_k) := \prod_{i=1, i\neq j}^{k} \Big(\frac{1}{1+|x_i|^\alpha} \Big)^{1/(k-1)}$$

$j = 1, \dots, k$, we get

$$\begin{aligned} \int_{\mathbb{R}^k} |\widehat{\phi}(x_1+\ldots+x_k)|^2 \frac{1}{1+|x_1|^\alpha} \cdots \frac{1}{1+|x_k|^\alpha} dx_1 \ldots dx_k \leq \\ \leq \prod_{j=1}^{k} \left(\int_{\mathbb{R}^k} |\widehat{\phi}(x_1+\ldots+x_k)|^2 s_j(x_1, \dots, x_k)^k dx_1 \ldots dx_k \right)^{\frac{1}{k}}, \end{aligned} \tag{3.4.48}$$

which is finite since, by assumption, $\alpha > 1 - \frac{1}{k}$.

Remark 3.4.10. *In fact one can prove that for any choice of* $\alpha_1, \dots, \alpha_k \in (0,1)$ *satisfying* $\alpha_1 + \ldots + \alpha_k > k-1$ *and any* $f \in L^1(\mathbb{R})$

$$\int_{\mathbb{R}^k} |\widehat{f}(x_1+\ldots+x_k)|^2 |x_1|^{-\alpha_1} \ldots |x_k|^{-\alpha_k} dx_1 \ldots dx_k < \infty, \tag{3.4.49}$$

but the proof is a little more complicated.

In order to use the properties of $\mathcal{S}'$-valued random variables we introduce the following approximating functional:

$$\rho^T_{f,\epsilon,\phi} := \frac{1}{T^{k/2}} \sum_{j_1 \neq \ldots \neq j_k} \sigma_{j_1} \ldots \sigma_{j_k} \langle \Lambda^f_\epsilon(x^{j_1}+\xi^{j_1}, \dots, x^{j_k}+\xi^{j_k}; T), \phi \rangle, \tag{3.4.50}$$

where $T > 0, \epsilon > 0, f \in \mathcal{F}, \phi \in \mathcal{S}(\mathbb{R})$. It is well defined by Lemmas 3.4.2 and 3.4.9. The same Lemmas also show that the functional given by

$$\rho_{\psi_\kappa}^T := \frac{1}{T^{k/2}} \sum_{j_1 \neq \ldots \neq j_k} \sigma_{j_1} \ldots \sigma_{j_k} \langle \Lambda^{(k)}(x^{j_1} + \xi^{j_1}, \ldots, x^{j_k} + \xi^{j_k}; T), \psi_\kappa \rangle, \quad (3.4.51)$$

is well defined for any $\kappa \in (0,1)$, $T > 0$, $\psi \in \mathcal{A}$ and is an $L^2(\Omega)$-limit of the functional in 3.4.50 with ϕ replaced by ψ_κ.

Main body of the proof

Proof of Theorem 3.2.3. The proof follows the footsteps of the proof of Theorem 3.5 in [8] with some necessary generalizations. From now on we fix $\alpha \in (1 - \frac{1}{k}, 1)$ and $f \in \mathcal{F}$. We are going to prove the following claims (which imply the convergence of finite-dimensional distributions):

$$\mathop{lim}_{\kappa \to 0} \mathop{sup}_{T \geq 1} \mathbb{E} \left| \rho_\psi^T - \rho_{\psi_\kappa}^T \right|^2 = 0, \quad (3.4.52)$$

$$\mathop{lim}_{\epsilon \to 0} \mathop{sup}_{T \geq 1} \mathop{sup}_{\kappa \in (0,1)} \mathbb{E} \left| \rho_{\psi_\kappa}^T - \rho_{f,\epsilon,\psi_\kappa}^T \right|^2 = 0, \quad (3.4.53)$$

$$\lim_{T \to \infty} \mathop{sup}_{\kappa \in (0,1)} \mathbb{E} \left| \langle : X_T \otimes \ldots \otimes X_T :, \Phi_{\epsilon,\psi_\kappa}^f \rangle - \rho_{f,\epsilon,\psi_\kappa}^T \right|^2 = 0, \quad \epsilon > 0, \quad (3.4.54)$$

$$\langle : X_T \otimes \ldots \otimes X_T :, \Phi_{\epsilon,\psi_\kappa}^f \rangle \Rightarrow \langle : X \otimes \ldots \otimes X :, \Phi_{\epsilon,\psi_\kappa}^f \rangle, \quad , \epsilon > 0, \kappa > 0, \quad (3.4.55)$$

$$\lim_{\epsilon \to 0} \mathop{sup}_{\kappa \in (0,1)} \mathbb{E} \left| \int_{\mathbb{R}^k}'' \widehat{\psi_\kappa}(x_1 + \ldots + x_k) Z_G(dx_1) \ldots Z_G(dx_k) - \langle : X \otimes \ldots \otimes X :, \Phi_{\epsilon,\psi_\kappa}^f \rangle \right|^2 = 0, \quad (3.4.56)$$

$$\lim_{\kappa \to 0} \mathbb{E} \left| \int_{\mathbb{R}^k}'' \widehat{\psi_\kappa}(x_1 + \ldots + x_k) Z_G(dx_1) \ldots Z_G(dx_k) - \int_{\mathbb{R}^k}'' \widehat{\psi}(x_1 + \ldots + x_k) Z_G(dx_1) \ldots Z_G(dx_k) \right|^2 = 0. \quad (3.4.57)$$

Here $: Z \otimes \ldots \otimes Z :$ stands for the k-th Wick product of the random variable Z. For definition see equation (3.4.60). From Lemma 3.4.2 we have (with F replaced by $\langle(\Lambda^f_\epsilon - \Lambda), \psi_\kappa\rangle$) the following inequality:

$$\begin{aligned}
\mathbb{E}|\rho^T_{\psi_\kappa} - \rho^T_{f,\epsilon,\psi_\kappa}|^2 &\leq \frac{2k!}{T^k}\int_{\mathbb{R}^k} \mathbb{E}|\langle(\Lambda^f_\epsilon - \Lambda^{(k)})(x_1+\xi^1,\ldots,x_k+\xi^k;T),\psi_\kappa\rangle|^2 \\
&\leq 2k!2^k\int_{\mathbb{R}^k} |\widehat{\psi_\kappa}(x_1+\ldots+x_k)|^2|\widehat{f}_\epsilon(x_1+\ldots+x_k)-1|^2 \\
&\quad \times |x_1|^{-\alpha}\ldots|x_k|^{-\alpha}dx_1\ldots dx_k. \qquad (3.4.58)
\end{aligned}$$

By (3.4.47), dominated convergence theorem and the fact that $\widehat{\psi_\kappa}(z) \leq \widehat{\psi}(z)$ for $z \in \mathbb{R}$ we get (3.4.53). The proof of (3.4.52) is very similar and we skip it.

The hardest part is to prove (3.4.54). From [8] we know that (iii) holds for $k = 2$. Let Φ be of the form

$$\Phi = \sum_{j=1}^m \phi^{(1,j)} \otimes \ldots \otimes \phi^{(k,j)}, \qquad (3.4.59)$$

where each $\phi^{(s,t)}$ is in $\mathcal{S}(\mathbb{R})$ for $s = 1,\ldots,k, \quad t = 1,\ldots,m$. By definition

$$\begin{aligned}
&\langle : X_T \otimes \ldots \otimes X_T :, \Phi\rangle = \\
&\quad \sum_{j=1}^m \sum_{A\in\mathcal{M}} (-1)^{|A|} \prod_{\{s,t\}\in A} \mathbb{E}(\langle X_T, \phi^{(s,j)}\rangle)\mathbb{E}(\langle X_T, \phi^{(t,j)}\rangle) \prod_{n\notin\cup A} \langle X_T, \phi^{(n,j)}\rangle,
\end{aligned} \qquad (3.4.60)$$

where $\mathcal{M}$ is the set of unordered pairs $\{s,t\} \subset \{1,\ldots,k\}$, such that all the elements in these pairs are distinct. In particular $|\bigcup A| = 2|A|$. The sum above is over all distinct sets A of this form including the empty set. If we define the approximating functional by

$$\rho^T_\Phi := \frac{1}{T^{k/2}} \sum_{j_1\neq\ldots\neq j_k} \sigma_{j_1}\ldots\sigma_{j_k} \int_{[0,T]^k} \Phi(x^{j_1}+\xi^{j_1}_{s_1},\ldots,x^{j_k}+\xi^{j_k}_{s_k})ds_1\ldots ds_k, \qquad (3.4.61)$$

then one can easily see that

$$\mathbb{E}(\langle : X_T \otimes \ldots \otimes X_T :, \Phi\rangle - \rho^T_\Phi)^2 = \mathbb{E}\langle : X_T \otimes \ldots \otimes X_T :, \Phi\rangle^2 - \mathbb{E}(\rho^T_\Phi)^2. \qquad (3.4.62)$$

This follows from the fact that in ρ^T_Φ we have summation over distinct indices $\{j_1,\ldots,j_k\} \in \mathbb{N}$ and so the only nonzero terms in $\mathbb{E}(\langle : X_T \otimes \ldots \otimes X_T :, \Phi\rangle\rho^T_\Phi$ are those that correspond to $A = \emptyset$ in (3.4.60). Furthermore, if we recall the sum in (2.5.22) defining $\langle X_T, \phi\rangle$ for $\phi \in \mathcal{S}(\mathbb{R})$, then it is obvious that $\mathbb{E}(\langle : X_T \otimes \ldots \otimes X_T :, \Phi\rangle\rho^T_\Phi = \mathbb{E}(\rho^T_\Phi)^2$. Let us denote $\mathbb{E}\langle : X_T \otimes \ldots \otimes X_T :, \Phi\rangle^2$ by I. Then

$$I = \sum_{j=1}^{m}\sum_{j'=1}^{m}\sum_{A,A'\in\mathcal{M}}(-1)^{|A|}(-1)^{|A'|}\prod_{(s,t)\in A}\mathbb{E}\langle X_T,\phi^{(s,j)}\rangle\langle X_T,\phi^{(t,j)}\rangle$$
$$\times\prod_{(s',t')\in A'}\mathbb{E}\langle X_T,\phi^{(s',j')}\rangle\langle X_T,\phi^{(t',j')}\rangle$$
$$\times\,\mathbb{E}\left(\prod_{n\notin\cup A}\langle X_T,\phi^{(n,j)}\rangle\prod_{n'\notin\cup A'}\langle X_T,\phi^{(n',j')}\rangle\right). \tag{3.4.63}$$

Computing the last expected value in (3.4.63) amounts to summation over different choices of the *diagonals* just as in the proof of Lemma 3.4.2. To illustrate it consider first the case $A=\emptyset=A'$. Then, we have no covariances and are left with

$$\begin{aligned}I_\emptyset &= \mathbb{E}\Big(\sum_{j=1}^{m}\sum_{j'=1}^{m}\langle X_T,\phi^{(1,j)}\rangle\dots\langle X_T,\phi^{(k,j)}\rangle\\ &\quad\times\langle X_T,\phi^{(1,j)}\rangle\dots\langle X_T,\phi^{(k,j)}\rangle\Big) \qquad (3.4.64)\\ &= \mathbb{E}\Big(\sum_{\substack{j_1,\dots,j_k\\ j_{k+1},\dots,j_{2k}}}\sigma_{j_1}\dots\sigma_{j_k}\sigma_{j_{k+1}}\dots\sigma_{j_{2k}}F_T(x^{j_1}+\xi^{j_1},\dots,x^{j_k}+\xi^{j_k})\\ &\quad\times F_T(x^{j_{k+1}}+\xi^{j_{k+1}},\dots,x^{j_{2k}}+\xi^{j_{2k}}),\Big),\end{aligned}$$

where

$$F_T(x_1+\xi^1,\dots,x_k+\xi^k):=\frac{1}{T^{k/2}}\int_{[0,T]^k}\Phi(x_1+\xi^1_{s_1},\dots,x_k+\xi^k_{s_k})ds_1\dots ds_k,\quad x_1,\dots,x_k\in\mathbb{R}. \tag{3.4.65}$$

The only terms in the sum in (3.4.64), whose expected values are non-zero, are those for which for every $l\in\{j_1,\dots,j_k,j_{k+1},\dots,j_{2k}\}$ there is an *even* number of indices taking that value. This sum can be split into a finite number of sums over different *diagonals*. To be precise, by a *diagonal* $\mathcal{C}$ we mean a partition of $\{1,2,\dots,2k\}$ into a disjoint family of subsets $C_1,\dots,C_m$ of $\{1,2,\dots,2k\}$ such that $|C_l|$ is even for $l=1,\dots,m$. Then the term in (3.4.64) corresponding to this diagonal is given by

$$\sum_{\substack{j_{v^1_1}=\dots=j_{v^1_{k_1}},\quad v^1_1,\dots,v^1_{k_1}\in C_1\\ \dots\\ j_{v^m_1}=\dots=j_{v^m_{k_m}},\quad v^m_1,\dots,v^m_{k_m}\in C_m}}\sigma_{j_1}\dots\sigma_{j_k}\sigma_{j_{k+1}}\dots\sigma_{j_{2k}}$$
$$\times F_T(x^{j_1}+\xi^{j_1},\dots,x^{j_k}+\xi^{j_k})F_T(x^{j_{k+1}}+\xi^{j_{k+1}},\dots,x^{j_{2k}}+\xi^{j_{2k}}). \tag{3.4.66}$$

Now, a diagonal $\mathcal{C}$ is *large* if any of the sets $C_1, \ldots, C_m$ has more than two elements. In due course we will see that the sums over these diagonals behave as $\frac{1}{T}$ as $T \to \infty$. All other diagonals are *pairings* of different charges. This means that for these diagonals all C_l's have exactly two elements. We will say that a pairing is *normal* if for every $C \in \mathcal{C}$, C has exactly one element from $\{1, \ldots, k\}$ and one element from $\{k+1, \ldots, 2k\}$. All other non-large diagonals will be called *non-normal pairings*. Notice that the choice of A, A' in (3.4.63) corresponds to fixing some particular part of the diagonal over which summation is being done. Looking at (3.4.63) from this perspective, there will only be normal pairings in the sums corresponding to $A, A' = \emptyset$. We can use the same notation for sums that will emerge from the term $\mathbb{E}\left(\prod_{n \notin \cup A} \langle X_T, \phi^{(n,j)} \rangle \prod_{n' \notin \cup A'} \langle X_T, \phi^{(n',j')} \rangle\right)$ in (3.4.63). Putting all this together we can write

$$\mathbb{E}\langle : X_T \otimes \ldots \otimes X_T :, \Phi \rangle^2 = I_1 + I_2 + R, \tag{3.4.67}$$

where in I_1 we have only the sums over the diagonals which correspond to normal pairings. We see immediately that $I_1 = \mathbb{E}\eta^2_{T,\Phi})$. I_2 corresponds to the sums over non-large non-normal pairings (notice that all the sums in (3.4.63) with $A \neq \emptyset$ or $A' \neq \emptyset$ will be in I_2) and R contains only sums over large diagonals. Notice that the terms in (3.4.67) may be written with the help of the function F_T given by (3.4.65) and extend continuously to general $\Phi \in \mathcal{S}(\mathbb{R}^2)$, not necessarily of the form (3.4.59).

We are going to show that $I_2 = 0$. Fix a non-large non-normal pairing $\mathcal{C}$. Assume that in the fixed diagonal over which the summation is being performed there are n non-normal pairs B formed between members of the sequence $(j_1, \ldots, j_k)$ and n non-normal pairs B' formed between members of the sequence $(j_{k+1}, \ldots, i_{2k})$. All the other pairs (there are $k - 2n$ of them) are normal. The sum over our fixed diagonal is going to appear in terms from (3.4.63) with $|A|, |A'| \leq n$. Fix $c, d \leq n$ and consider the summands in I_2 for which $|A| = c, |A'| = d$. The sum over $\mathcal{C}$ is going to appear in exactly $\binom{n}{c}\binom{n}{d}$ summands with $c + d = n$ with the sign equal to $(-1)^{c+d}$. This can be justified as follows. The sum over $\mathcal{C}$ can only appear in the terms with $A \subset B$ and $A' \subset B'$ and there will be $\binom{n}{c}$ choices of A with $A \subset B$ and $\binom{n}{d}$ choices of A' with $A' \subset B'$. We see that for each $0 \leq m \leq 2n$ with $m = |A| + |A'|$ the sum corresponding to our fixed diagonal will appear exactly $\sum_{l=0}^{m} \binom{n}{m-l}\binom{n}{l} = \binom{2n}{m}$ with sign equal to $(-1)^m$. Hence, the number of times (with signs taken into account) the sum over our fixed diagonal will appear in I_2 is exactly $\sum_{m=0}^{2n} (-1)^m \binom{2n}{m} = 0$. This proves that $I_2 = 0$.

R can be split into a finite number of sums over large diagonals, all of which have the property that the summation is taken over indices $(j_1, \ldots, j_k, j_{k+1}, \ldots, j_{2k})$ such that at least four of them are equal. To finish the proof of (3.4.54) we fix $\epsilon \in (0, 1)$ and take $\Phi = \Phi^f_{\epsilon,\psi_\kappa}$ (see (3.4.39)). It remains to show that $R = R_{T,\Phi^f_{\epsilon,\psi_\kappa}}$ converges to 0 as $T \to \infty$ uniformly in $\kappa \in (0, 1)$. Put $\theta(x) := \frac{1}{1+|x|^2}, x \in \mathbb{R}$.

Notice that

$$|\Phi(x_1,\dots,x_k)| \le p(\Phi)\frac{1}{1+|x_1|^2}\cdots\frac{1}{1+|x_k|^2} = p(\Phi)\theta(x_1)\dots\theta(x_k), \quad (3.4.68)$$

where $p(\Phi)$ is a continuous seminorm on $\mathcal{S}(\mathbb{R}^k)$, given by

$$p(\Phi) = \sup_{x_1\in\mathbb{R},\dots,x_k\in\mathbb{R}} \left|(1+|x_1|^2)\dots(1+|x_k|^2)\Phi(x_1,\dots,x_k)\right|. \quad (3.4.69)$$

Thanks to this

$$\sup_{\kappa\in(0,1)} p(\Phi^f_{\epsilon,\psi_\kappa}) \le C(\epsilon,f), \quad (3.4.70)$$

with $C(\epsilon,f)$ being a constant depending only on f and ϵ and independent of κ. To fix our attention, let us consider an example of a large diagonal with $k=3$. This diagonal is given by requiring that $j_1=j_2=j_5=j_6$ and $j_3=j_4$. Then the expected value of the sum corresponding to this diagonal is given by

$$\frac{1}{T^{\frac{3}{2}}}\int_{\mathbb{R}^2}\mathbb{E}\left(F_T(x_1+\xi^1,x_1+\xi^1,x_2+\xi^2)F_T(x_2+\xi^2,x_1+\xi^1,x_1+\xi^1)\right)dx_1dx_2. \quad (3.4.71)$$

The absolute value of the above integral is no bigger than

$$\frac{1}{T^{\frac{3}{2}}}p(\Phi)^2\int_{\mathbb{R}^2}\int_{[0,T]^6}\mathbb{E}\Big(\theta(x_1+\xi^1_{u_1})\theta(x_1+\xi^1_{u_2})\theta(x_2+\xi^2_{u_3}) \times\theta(x_2+\xi^2_{u_4})\theta(x_1+\xi^1_{u_5})\theta(x_1+\xi^1_{u_6})\Big)du_1\dots du_6dx_1dx_2. \quad (3.4.72)$$

By (3.4.70), for $\Phi=\Phi^f_{\epsilon,\psi_\kappa}$, the integral in (3.4.72) can be bounded uniformly in κ by an integral which (by independence) can be written as a product of two integrals times a constant $C(\epsilon,f)$. One of the factors of this product (the one correspoding to the pairing $j_3=j_4$) is bounded by a constant. The other is given by

$$\frac{1}{T^2}\int_{[0,T]^4}\int_{\mathbb{R}}\mathbb{E}\left(\theta(x+\xi^1_s)\theta(x+\xi^1_u)\theta(x+\xi^1_r)\theta(x+\xi^1_v)\right)dxdsdudrdv.$$

Following the proof of Theorem 3.5 in [8] we see that the above is bounded by

$$\frac{1}{T^2}\int_0^T\int_{\mathbb{R}}\theta(x)U(\theta U(\theta U\theta))(x)dxds \le \frac{1}{T}C_2, \quad (3.4.73)$$

where C_2 is another constant, and U is the potential of an α-stable semigroup. The second inequality above follows from the fact that $U\psi$ is bounded. In the case of blocks larger than four the argument is very similar. To conclude, $R\le\frac{C}{T}p(\Phi)^2$, where C is a constant, which depends only on k,ϵ and f (the bound $\frac{1}{T}$ was given in [8] only for the diagonal with the largest element consisting of four equal indexes, but having larger diagonals is even better which can easily be inferred from the proof of equation (6.27) in [8]). This means that we can write $\mathbb{E}\left(\langle: X_T \otimes \ldots \otimes X_T :, \Phi\rangle - \rho_{T,\Phi}\right)^2 \leq \frac{C}{T} p(\Phi)^2$ for $\Phi \in \mathcal{S}(\mathbb{R}^k)$.

To prove (3.4.55) we will need the following generalization of lemma 6.3 in [8]

Lemma 3.4.11. *Let $(X_T)_{T\geq 1}$ be a family of $\mathcal{S}'$-valued random variables such that*

$$\sup_{T\geq 1} \mathbb{E}\langle X_T, \phi\rangle^2 \leq p^2(\phi), \quad \phi \in \mathcal{S}(\mathbb{R}),$$

for some continuous Hilbertian seminorm p on $\mathcal{S}(\mathbb{R})$. Suppose that $X_T \Rightarrow X$ and $\mathbb{E}\langle X_T, \phi\rangle^2 \to \mathbb{E}\langle X, \phi\rangle^2$ for $\phi \in \mathcal{S}(\mathbb{R})$ as $T \to \infty$. Then $X_T \otimes \ldots \otimes X_T$ and $X \otimes \ldots \otimes X$ are well defined and $: X_T \otimes \ldots \otimes X_T :\Rightarrow: X \otimes \ldots \otimes X :$ as $T \to \infty$.

This together with Theorem 2.5.14 implies (3.4.55). We proceed to prove (3.4.56) and (3.4.57). Fix $k \in \mathbb{N}, k \geq 2$. Let $\alpha \in (1 - \frac{1}{k}, 1)$ and let $(X_\phi)_{\phi\in\mathcal{S}(\mathbb{R})}$ be a generalized centered Gaussian random field over the Schwartz space with spectral measure $G(dx) = |x|^{-\alpha}dx$. Notice that, as before, using Theorem 4.7 in [28], we might write

$$\langle: X \otimes \ldots \otimes X :, \Phi\rangle = \int''_{\mathbb{R}^k} \widehat{\Phi}(x_1, \ldots, x_k) Z_G(dx_1) \ldots Z_G(dx_k), \tag{3.4.74}$$

for any $\Phi \in \mathcal{S}(\mathbb{R}^k)$. Whenever $\int''_{\mathbb{R}^k} |\widehat{\phi}(x_1 + \ldots + x_k)|^2 G(dx_1) \ldots G(dx_k) < \infty$, we see that by dominated convergence theorem,

$$\langle: X \otimes \ldots \otimes X :, \Phi^f_{\epsilon,\phi}\rangle \overset{L^2(\Omega)}{\to} \int''_{\mathbb{R}^k} \widehat{\phi}(x_1 + \ldots + x_k) Z_G(dx_1) \ldots Z_G(dx_k). \tag{3.4.75}$$

Given the above (3.4.56) and (3.4.57) follow immediately. Establishing (3.4.52) - (3.4.57) shows that the finite-dimensional distributions of $(\eta^T_t)_{t\geq 0}$ converge to the finite dimensional distributions of the k-Hermite process given by (1.2.1).

Showing tightness, again, is relatively straightforward. It follows (along with the fact that ρ^T has a continuous modification) from the following. Using Lemmas 3.4.9 and 3.4.2, similarly as in (3.4.58) one can show that for any $0 \leq s \leq t < \infty$ and $T > 0$

$$\begin{aligned}
\mathbb{E}(\rho^T_t - \rho^T_s)^2 &\leq C(k) \int_{\mathbb{R}^k} \left|\widehat{\mathbf{1}_{(s,t]}}(x_1 + \ldots x_k)\right|^2 |x_1|^{-\alpha} \ldots |x_k|^{-\alpha} dx_1 \ldots dx_k \\
&\leq C(k) \int_{\mathbb{R}^k} \left|\frac{e^{i(t-s)(x_1+\ldots x_k)} - 1}{i(x_1 + \ldots x_k)}\right|^2 |x_1|^{-\alpha} \ldots |x_k|^{-\alpha} dx_1 \ldots dx_k \\
&\leq C'(\alpha, k)(t-s)^{2+k(\alpha-1)},
\end{aligned}$$

where $C(k)$ and $C'(\alpha, k)$ are constants independent of $T > 0$, depending only on k and k, α, respectively. Since $\alpha \in (1 - \frac{1}{k}, 1)$ by assumption, $2 + k(\alpha - 1) > 1$ and as in the proof of Theorem 3.2.2 we conclude that the sequnce $(\rho^T)_{T \geq 1}$ is tight in $\mathcal{C}[0, \tau]$ for any $\tau > 0$. This finishes the proof.

□

Remark 3.4.12. *In fact we have shown that for* $\Phi \in \mathcal{S}(\mathbb{R}^k)$ *the functional defined by*

$$\rho_{T,\Phi} := \frac{1}{T^{k/2}} \sum_{j_1 \neq \ldots \neq j_k} \sigma_{j_1} \ldots \sigma_{j_k} \int_{[0,T]^k} \Phi(x^{j_1} + \xi^{j_1}_{s_1}, \ldots, x^{j_k} + \xi^{j_k}_{s_k}) ds_1 \ldots ds_k \tag{3.4.76}$$

converges (up to a constant) in distribution to

$$\int''_{\mathbb{R}^k} \widehat{\Phi}(z_1, \ldots, z_k) Z_G(dz_1) \ldots Z_G(dz_k), \tag{3.4.77}$$

as $T \to \infty$.

Chapter 4

Infinite variance H-sssi processes as limits of particle systems

4.1 Introduction

Quite recently new classes of α-stable H-sssi processes were introduced by Dombry et al. in [15] and Samorodnitsky et al. in [35] and [20], which we will describe in a moment (see Section 4.1.1). The aim of this chapter is to show that these processes appear as limit processes related to occupation time of a Poisson system of particles moving acording to independent Lévy processes with charges and heavy-tailed weights. This provides a particle picture interpretation the processes studied in [15], [35] and [20], thus leading to a better understanding of these processes, which at first sight (see (4.1.1) and (4.1.2)) may seem somewhat artificial. Our second objective was to study the behaviour of the particle system considered in [12] (and earilier in [14] in a special case) but with a modification that the particles have weights, which seems to be quite natural. Additionally we obtain a new class of α-stable H-sssi processes. Moreover, we extend the results of Rosen in [41] on occupation times of stable Lévy processes in order to present out results in greater generality.

The particle system under investigation is similar to the one considered in Chapter 3, with two major differences: we attach heavy tailed weights to the particles and we consider particles moving according to the law more general recurrent Lévy process.

4.1.1 Processes obtained as limits of our particle system

First we describe the processes which appear as limits in our setup. Most of them have already been introduced in literature. See Section 4.3 for a more detailed description of the context in which they appeared.

All of the processes that we obtain as limits have representations as stochastic integrals with respect to symmetric α-stable random measures. For a brief introduction to stable random variables, stable random measures and integrals with respect to these measures see Section 2.2.2 and 2.3.2, respectively.

The first class of processes which we discuss is the so called *β-stable local time fractional $S\alpha S$ motion.* It was first considered in [15] and later was obtained as a limit process in [35]. In the latter work it was obtained as limit in law of sums of symmetric stationary infinitely-divisible processes represented as integrals with respect to infinitely-divisible Lévy measure with regularly varying tails. In the former work it appeared in the context of random walks in random scenery.

It has the following integral representation:

$$X = \left(\int_{\mathbb{R}\times\Omega'} L_t^{\beta}(x,\omega') M_\alpha(dx, d\omega') \right)_{t\geq 0}, \tag{4.1.1}$$

where $(L_t^\beta(x,\omega'))_{t\geq 0, x\in\mathbb{R}}$ is a jointly continuous version of the local time of a symmetric β-stable Lévy process, with $\beta \in (1,2)$ (defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$) and M_α is a symmetric α-stable random measure on $\mathbb{R}\times\Omega'$ with control measure $\lambda_1 \otimes \mathbb{P}'$, which is itself defined on some other probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For a brief note on local times see the Section 2.6. For the properties of local times of stable Lévy processes see Appendix 4.11.1.

The second class of processes we investigate was recently introduced by Jung, Owada and Samorodnitsky in [20]. Members of this class have the following integral representation

$$(Y_{\alpha,\beta,\gamma}(t))_{t\geq 0} = \left(\int_{\Omega'\times\mathbb{R}} S_\gamma(L_t^\beta(x,\omega'), \omega') M_\alpha(sx, d\omega') \right)_{t\geq 0}, \tag{4.1.2}$$

where $(L_t^\beta(x))_{t\geq 0}$ is as in (4.1.1) and S_γ is an independent symmetric γ-stable Lévy process. Both of these processes are defined on some probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and M_α is as in (4.1.1). Finally, the parameters α, β and γ satisfy $1 < \beta < 2$ and $1 < \alpha < \gamma \leq 2$.

In the framework presented in [20] all the processes obtained as limits had $\gamma = 2$. In our recent work (see [47] and Chapter 5) we have provided an augmented random walk in random scenery model for which the functional limit spans the whole range of parameters in (4.1.2).

In the limit of the occupation time of our particle system we also obtain a new class α-stable H-sssi processes that have the following form. The first one is given by

$$V = \left(\int_{\mathbb{R}\times\Omega'} Z_t(x,\omega') M_\alpha(dx, d\omega') \right)_{t\geq 0}, \tag{4.1.3}$$

where M_α is a symmetric α-stable random measure on $\mathbb{R}\times\Omega'$ with control measure $\lambda_1 \otimes \mathbb{P}'$ and

$$Z_t(x,\omega') = \int_0^\infty |y|^{-\gamma} \left(L_t^\beta(x+y,\omega') - L_t^\beta(x-y,\omega') \right) dy, \quad x\in\mathbb{R}, \tag{4.1.4}$$

with $(L_t^\beta(x))$ as in (4.1.1) with $\beta \in (1,2)$, defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$. The parameter γ is in $(1, 1+(\beta-1)/2)$. The random measure M_α itself is defined on another probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The second one is given by

$$\widetilde{V} = \left(\int_{\mathbb{R}\times\Omega'} \widetilde{Z}_t(x,\omega') M_\alpha(dx, d\omega') \right)_{t\geq 0}, \tag{4.1.5}$$

where

$$\widetilde{Z}_t(x,\omega') = \int_0^\infty |y|^{-\gamma_1} \left(L_t^\beta(x+y,\omega') - L_t^\beta(x,\omega') \right) dy, \quad x\in\mathbb{R}, \tag{4.1.6}$$

with M_α and L^β as in the first case. The processes (Z_t) and $(\widetilde{Z}_t)$ are fractional derivatives of stable local times and their properties were extensively studied in [17].

4.1.2 Particle system

We consider the following particle system on $\mathbb{R}$. The initial positions of particles $(x_j)_{j=1}^\infty$ are given by a Poisson random measure with Lebesgue intensity measure and each of the particles is independently assigned a ± 1 charge and weight from the same distribution. We denote these signed weights by $(z_j)_{j=1}^\infty$.The particles evolve according to i.i.d. Lévy processes $(\eta^j)_{j=1}^\infty$, that is, each η^j is a process with stationary independent increments (see Section 2.2.1 for more details). In the simplest setting considered here η^1 is a symmetric, β-stable Lévy process, i.e., process with independent increments such that $\mathbb{E}\exp(i\theta\eta_t^1) = \exp(-t|\theta|^\beta)$ for $\theta\in\mathbb{R}$, but we also consider more general Lévy processes, see Section 4.4.

For a test function $\phi \in L^1(\mathbb{R})$ we consider the charged occupation time with time scaled by $T > 0$

$$G_t^T := \frac{1}{F_T} \sum_j z_j \int_0^{Tt} \phi(x^j + \eta_u^j) du, \tag{4.1.7}$$

where F_T is an appropriate normalization which will be chosen such that the process G^T converges in law. This model with $\mathbb{P}(z_j = 1) = \mathbb{P}(z_j = -1) = 1/2$ and η^j are symmetric β-stable processes was considered in [12]. Therein it was shown that a suitably normalized functional of the form 4.1.7, considered as an $\mathcal{S}'(\mathbb{R}^d)$-valued process ($d \in \mathbb{N}$) converges in law to $K\lambda\xi$ where $K > 0$, λ is the Lebesgue measure and ξ is a fractional Brownian motion. In fact in the original setting considered earlier in [14] (with η^j's being independent Brownian motions) z_j's were all taken equal to one but Y_T was centred by subtracting the mean.

Analogous models concerning occupation limits for particle systems with branching, again with $z_j \equiv 1$ and centering were studied by Bojdecki et al in [6] and a series of other papers by the same authors.

We only consider systems without branching but the crucial difference in our setup is that we assume that the law of the z_j's has heavy tails. This changes the behaviour of the process G and leads in the limit to stable self-similar processes described in the previous section with different limits depending on whether $\int_{\mathbb{R}} \phi(y)dy = 0$ or not, and how ϕ behaves at infinity.

4.2 Organisation of the chapter

The chapter is organised in the following way. In Section 4.3 we provide some more background information regarding the context in which the processes described in Sec tion 4.1.1 first appeared. Section 4.4 provides the major assumptions we use for the rest of the chapter. In Section 4.5 we formulate the extensions of the results of [41] on occupation times of stable processes which we later use to prove the main results. Section 4.6 provides all the main results formulated in their full generality. In Section 4.7 we provide proofs of Propositions 4.5.1 and 4.5.2 which we later use to prove results from Section 4.6. Section 4.8 is devoted to the proof of Theorem 4.6.1.In Section 4.9 we prove Theorem 4.6.2. Finally in Section 4.10 we prove 4.6.4.

Appendices provide some additional technical results on the properties of stable local times and aspects related to regular variation which are used throughout the chapter.

4.3 Additional information on limit processes

In [15] the process (4.1.1) was obtained in the so called *random rewards schema* (see [13]) or *random walks in random scenery models* (see [15]). Following [15] these models can be described in the following way. Assume that there is a *user* moving randomly on the *network* which earns random rewards (governed by the random scenery) associated to the points in the network that they visit. The quantity of interest is then the total amount of rewards collected. The concrete model considered in [15] goes as follows. Assume that that the movement of the user is a random walk on $\mathbb{Z}$ which after suitable scaling converges to the β-stable Lévy process with $\beta \in (1, 2]$. Furthermore, let the random scenery be given by i.i.d. random variables $(\xi_j)_{j\in\mathbb{Z}}$ which belong to the normal domain of attraction of a strictly stable distribution with index of stability $\alpha \in (0, 2]$. Then the *random walk in random scenery* is given by

$$Z_n = \sum_{k=1}^{n} \xi_{S_k}, \tag{4.3.1}$$

where $S_k = \sum_{j=1}^{k} X_k$ is the random walk determining the movement of the user. If we consider a large number of independent *random walkers* moving in independent random sceneries, then the scaling limit in the corresponding functional limit theorem (see Theorem 1.2 in [15]) leads to the process (4.1.1).

The process (4.1.1) was then investigated in [35] where it arose as a limit of partial sums of a stationary and infinitely divisible process $(X_n)_{n=1}^{\infty}$ given by

$$X_n = \int_E f_n(x) dM(x) \tag{4.3.2}$$

where M is a symmetric homogeneous infinitely divisible random measure on some measurable space $(E, \mathcal{E})$ with a σ-finite control measure μ and local Lévy measure which is regularly varying at infinity with index $\alpha \in (0, 2)$. The f_n's are deterministic functions such that $f_n(x) = f(T^n(x))$ for some ergodic conservative measure preserving map on $(E, \mathcal{E}, \mu)$ possessing a Darling-Kac set with a normalizing sequence regularly varying with exponent $\widetilde{\beta} \in (0, 1)$. Crucially, it was also assumed that $\int_E f(x)\mu(dx) \neq 0$. For details see [35, Theorem 5.1] and for general ergodic-theoretical introduction to this setting see [42, Chapter 3]. The parameter α here is the same as in the random walk in random scenery model and the parameters β and $\widetilde{\beta}$ satisfy $\beta = 1/(1 - \widetilde{\beta})$.

The model presented in [20] is basically the same as the one presented in [35] with one crucial difference being that the function f from the discussion below (4.3.2) is such that $\int_E f(x)\mu(dx) = 0$. Under some conditions on the function f (see [20, Chapter 4]), the limit process of a suitably normalized sequence as in (4.3.2) was shown to belong to a class of H-sssi stable processes which have an integral representation given by

$$Y_{\alpha,\widetilde{\beta},\gamma}(t) := \int_{\Omega'\times[0,\infty)} S_\gamma(M_{\widetilde{\beta}}((t-x)_+,\omega'),\omega')dZ_{\alpha,\widetilde{\beta}}(\omega',x), \quad t\geq 0, \tag{4.3.3}$$

where

$$0<\alpha<\gamma\leq 2, 0\leq\widetilde{\beta}<1,$$

$(S_\gamma(t,\omega'))_{t\geq 0}$ is a symmetric γ-stable Lévy motion and $(M_{\widetilde{\beta}}(t,\omega'))_{t\geq 0}$ is an independent $\widetilde{\beta}$-Mittag-Leffler process (see section 3 in [35] for more on the latter). Both of these processes are defined on a probability space $(\Omega',\mathcal{F}',\mathbb{P}')$. Finally $Z_{\alpha,\beta}$ is a $S\alpha S$ random measure on $\Omega'\times[0,\infty)$ with control measure $\mathbb{P}'\otimes\nu_{\widetilde{\beta}}$, where $\nu_{\widetilde{\beta}}(dx) = (1-\widetilde{\beta})x^{-\widetilde{\beta}}\mathbf{1}_{x\geq 0}dx$. By Proposition 3.2 in [20] the process $Y_{\alpha,\widetilde{\beta},\gamma}$ is H-sssi with Hurst coefficient $H=\widetilde{\beta}/\gamma+(1-\widetilde{\beta})/\alpha$. Here we use $\widetilde{\beta}$ instead of β so as not to confuse it with the notation we have adopted for this paper. Similarly as in the proof of (3.10) in [35] we can show that for $\widetilde{\beta}\in(0,\frac{1}{2})$ the process (4.3.3) has the same law as (4.1.2) with $\beta=(1-\widetilde{\beta})^{-1}$. The limit process obtained in [20] corresponds to $\gamma=2$ in (4.3.3). It should be said that the process (4.3.3) or, equivalently, (4.1.2) have not appeared as limit processes for $\gamma<2$. We provide a scheme in which it does so in Chapter 5.

4.4 Assumptions

Assumption (A). *Let η be a Lévy process without drift and diffusion components with Lévy measure*

$$\nu(dx) = c(\beta)^{-1}f(x)|x|^{-1-\beta}dx,$$

where f is a symmetric eventually positive function slowly varying at infinity.

In Section 4.11.2 in the appendix to this chapter we prove that under these assumptions, the characteristic exponent ψ of η converges, up to a multiplicative constant to $|\cdot|^\beta$, which is equivalent to the fact that $\left(\frac{1}{F_T}\eta_{Tt}\right)_{t\geq 0} \overset{f.d.d}{\Rightarrow} \rho$, as $T\to\infty$ where ρ is a β-stable symmetric Lévy process and F_T is a suitable normalization (see Lemma 4.11.8 in the Appendix). We always assume that ψ satisfies

$$\int_1^\infty \psi(z)^{-1}dz<\infty. \tag{4.4.1}$$

Note that Assumption (**A**) is clearly satisfied for symmetric β-stable Lévy processes with $\beta\in(1,2)$. Moreover it also admits a larger class of Lévy processes whose 1-dimensional distributions are in the domain of attraction of symmetric β-stable law.

It turns out that it is more convenient to consider an equivalent formulation of our particle system according to which (x^j, z_j) are points of a Poisson point process on $\mathbb{R} \times \mathbb{R}$ with intensity measure $dx \otimes \nu_\alpha(dz)$, where $\nu_\alpha(dz)$ is a probability measure-the law of the z_js. This equivalence follows easily by simple properties of Poisson random measures.

Assumption (B). *Let (x_j, z_j) be a Poisson point process on $\mathbb{R}^2$ with intensity measure $dx \otimes \nu_\alpha(dz)$, where $\nu_\alpha(dz) := \frac{1}{|z|^{1+\alpha}} l(z)dz$, where $\alpha \in (1,2)$ and l is eventually positive symmetric and slowly varying at infinity. We assume that $z \mapsto l(z)|z|^{-1-\alpha}$ is a probability density function on $\mathbb{R}$, although in general it only needs to be integrable.*

In some cases we will also need additional assumptions, which are stated below.

Assumption (C). *Assume that there exists $\kappa \in (0,1)$ such that the characteristic exponent from Assumption (**A**) satisfies*

$$\int_1^\infty \psi(w)^{-\kappa} dw < \infty. \tag{4.4.2}$$

Assumption (D). *Assume that the function ϕ satisfies*

$$\left|\widehat{\phi}(x+y) - \widehat{\phi}(x)\right| \leq C|y|^\kappa, \tag{4.4.3}$$

for all $x, y \in \mathbb{R}$ with $\kappa > (\beta - 1)/2$ and that C is a constant independent of x and y. Here

$$\widehat{\phi}(x) = \int_{\mathbb{R}} e^{ixy} \phi(y) dy.$$

Remark 4.4.1. *For (4.4.3) to hold it suffices to assume that*

$$\int_{\mathbb{R}} |\phi(y)||y|^\kappa < \infty \tag{4.4.4}$$

for some $\kappa > (\beta - 1)/2$.

4.5 Extension of the occupation time theorems for stable processes

First we provide some extensions of the results of Rosen in[41]. We later use them to prove our main theorems.

Assume that $(\xi_t)_{t\geq 0}$ is a symmetric β-stable Lévy motion with $\beta \in (1,2)$. Then for any $\phi \in L^1(\mathbb{R})$ we have

$$\left(T^{\frac{1-\beta}{\beta}} \int_0^{Tt} \phi(\xi_s - T^{1/\beta} x) ds\right)_{t\geq 0} \overset{\mathcal{C}[0,\infty)}{\Longrightarrow} \left(L_t^\beta(x) \int_{\mathbb{R}} \phi(y) dy\right)_{t\geq 0}, \tag{4.5.1}$$

as $T \to \infty$, where $(L_t^\beta(x))_{t\geq 0, x\in\mathbb{R}}$ is a jointly continuous version of a local time of symmetric β-stable Lévy process. If $\int_{\mathbb{R}} \phi(y)dy = 0$ then the limit process of left-hand side of (4.5.1) is trivial and a different normalization is more appropriate. In [41], Rosen proved that if ϕ is a bounded Borel function on $\mathbb{R}$ with compact support such that $\int_{\mathbb{R}} \phi(x)dx = 0$, then we have

$$\frac{1}{T^{\frac{\beta-1}{2\beta}}} \int_0^{Tt} \phi(\xi_s)ds \overset{C([0,\infty))}{\Longrightarrow} \sqrt{d(\phi,\beta)} W_{L_t(0)}, \tag{4.5.2}$$

as $T \to \infty$, where W is a Brownian motion independent of ξ and $d(\phi,\beta)$ is a constant.

The extensions of the above results are given below. As before, $L_t(x)$ stands for the jointly continuous version of the local time of a symmetric β-stable Lévy process.

Proposition 4.5.1. *Assume that η is a Lévy process satisfying Assumption (**A**) and $\phi \in L^1(\mathbb{R})$ satisfies $\int_{\mathbb{R}} \phi(y)dy \neq 0$. Then the following convergence holds*

$$\left(\frac{1}{F_T} \int_0^{Tf(T^{1/\beta})^{-1}t} \phi(\eta_s - T^{1/\beta}x)ds \right)_{t\geq 0} \overset{f.d.d}{\Longrightarrow} \left(L_t^\beta(x) \int_{\mathbb{R}} \phi(y)dy \right)_{t\geq 0}, \tag{4.5.3}$$

*as $T \to \infty$, with $F_T = T^{-1/\beta+1} f(T^{1/\beta})^{-1}$. Furthermore all thee corresponding moments of one-dimensional distributions also converge. Moreover, if Assumption (**C**) holds, then the convergence holds in $\mathcal{C}[0,\infty)$.*

Perhaps more interestingly we prove an extension of the main result of (4.5.2) which relaxes the stringent assumptions on the function ϕ made in the original formulation by Rosen in [41], but at the cost of weakening the convergence to finite-dimensional distributions in general. However, the conditions necessary for convergence in $\mathcal{C}[0,\infty)$ are relatively weak.

Proposition 4.5.2. *Assume that η is a Lévy process satisfying Assumption (**A**) and $\phi \in L^1(\mathbb{R})$ satisfies $\int_{\mathbb{R}} \phi(y)dy = 0$ and Assumption (**D**). Then the following convergence holds*

$$\left(\frac{1}{F_T^{1/2}} \int_0^{Tf(T^{1/\beta})^{-1}t} \phi(\eta_s - T^{1/\beta}x)ds \right)_{t\geq 0} \overset{f.d.d}{\Longrightarrow} c(\phi) \left(W_{L_t^\beta(x)} \right)_{t\geq 0}, \tag{4.5.4}$$

as $T \to \infty$, and let $F_T = T^{-1/\beta+1} f(T^{1/\beta})^{-1}$, where W is a standard Brownian motion independent of the local time process $(L_t^\beta(x))_{x\in\mathbb{R}, t\geq 0}$ and

$$c(\phi) = \frac{1}{\pi} \sqrt{\int_{\mathbb{R}} \left| \widehat{\phi}(w) \right|^2 \psi(w)^{-1} dw}. \tag{4.5.5}$$

*Moreover, if, additionally, Assumption (**C**) holds, then the convergence holds in $\mathcal{C}[0,\infty)$.*

This result seems relatively robust, in the sense that we cannot expect Proposition 4.5.2 to hold if the tails of ϕ are heavier than $y \mapsto |y|^{-1-(\beta-1)/2}$. If this happens then, at least for ϕ with regularly varying tails, the normalization on the left-hand side of (4.5.4) is no longer valid and the class of limit processes is different. See discussion at the beginning of Section 4.10 for details.

4.6 Main results

In this section we formulate the main results on the convergence of the the processes of the for (4.1.7). The exact assumptions are given in Section 4.4, although, typically, one can think about the setting in which η^js are independent symmetric β-stable Lévy processes and z_js have distribution ν_α with density of the form $c_1\mathbf{1}_{\{|x|>c_2\}}|x|^{-1-\alpha}dx$ for some $\alpha \in (1,2)$ and positive constants c_1, c_2. Then, in the theorems below we have $l \equiv 1$ and $f \equiv 1$.

In some of our results we assume that η^js are more general Lévy processes, which asymptotically behave as symmetric β-stable Lévy processes and z_js have heavy tailed symmetric distributions in the domain of attraction of α-stable law with $\alpha \in (1,2)$.

Section 4.6.1 presents first order limit theorems which cover the case when the test function ϕ in (4.1.7) satisfies $\int_{\mathbb{R}} \phi(y)dy \neq 0$. Section4.6.2 presents second order limit theorems, where $\int_{\mathbb{R}} \phi(y)dy = 0$.

4.6.1 First order limit theorem

Here we formulate the first main result of our paper in which we identify the limit process (as $T \to \infty$) of the functional (4.1.7), provided the function ϕ is integrable and the integral $\int_{\mathbb{R}} \phi(y)dy$ does not vanish. Since we consider here a fairly general setting, the functional (4.1.7) is slightly modified.

Theorem 4.6.1. *Assume that the Assumptions (**A**) and (**B**) hold. Consider the stochastic process given by*

$$G_t^T = \frac{1}{F_T}\sum_j z_j \int_0^{D_T t} \phi(C_T x^j + \eta_u^j)du, \quad t \geq 0, \tag{4.6.1}$$

where $T \geq 1$ and let

$$\begin{aligned} F_T &= T^{1-1/\beta+1/\alpha\beta} f(T^{1/\beta})^{-1}, & (4.6.2)\\ C_T &= l(T^{1/\alpha\beta}), & (4.6.3)\\ D_T &:= Tf(T^{1/\beta})^{-1}, & (4.6.4)\end{aligned}$$

where f and l are given in Assumptions ($\boldsymbol{A}$) and ($\boldsymbol{B}$), respectively. Then, for any integrable function ϕ,

$$G_T \xrightarrow{f.d.d} K\Big(\int_{\mathbb{R}} \phi(y)dy\Big)X,$$

where X is given by (4.1.1) and K is a positive constant depending only on α and β. Furthermore, if additionally ($\boldsymbol{C}$) holds, then convergence holds in law in $\mathcal{C}[0,\infty)$.

4.6.2 Second order limit theorems

When $\int_{\mathbb{R}} \phi(y)dy = 0$ the, the limit process given by Theorem 4.6.1 is the zero process. To obtain a non-trivial limit in this case one has to use a normalization different than F_T given by (4.6.2). This case being more complicated, we only consider the case where the particle motion is given by symmetric stable Lévy processes.

In the case of relatively *light* tails we have the following theorem which produces another representation of the process first described in [20].

Theorem 4.6.2. *Assume that (η^j) are independent symmetric β-stable Lévy process with $\beta \in (1,2)$ and the Assumption ($\boldsymbol{B}$) is satisfied. Let ϕ be an integrable function with $\int_{\mathbb{R}} \phi(y)dy = 0$, satisfying Assumption ($\boldsymbol{D}$) such that additionally*

$$\int_{\mathbb{R}} |\phi(y)||y|^{\frac{\beta-1}{2}} dy < \infty. \tag{4.6.5}$$

Moreover let G^T be given by (4.6.1) with $F_T = T^{\frac{\beta-1}{2\beta}+\frac{1}{\alpha\beta}}$, $C_T = l(T^{1/\alpha\beta})$ and $D_T = T$. Then

$$G^T \xrightarrow{f.d.d.} c(\phi,\beta)Y_{\alpha,\beta,2},$$

where $Y_{\alpha,\beta,2}$ is given by (4.1.2), i.e., with S_γ replaced by Brownian motion and

$$c(\phi,\beta) = \sqrt{\int_{\mathbb{R}} |\widehat{\phi}(y)|^2 |y|^{-\beta} dy}. \tag{4.6.6}$$

Remark 4.6.3. *The assumptions in Theorem 4.6.2 regarding the function ϕ can be written in a more concise form. For instance it suffices to assume that $\int_{\mathbb{R}} |\phi(y)||y|^\kappa < \infty$ for some $\kappa > (\beta-1)/2$ and $\int_{\mathbb{R}} \phi(y)dy = 0$.*

Things change significantly if we allow ϕ to have heavier tails. In this case we have to assume that it is more regular. More precisely, we assume that ϕ is regularly varying at $+\infty$ and $-\infty$. We show that in this case the limit process of the functional (4.1.7) is a stable H-sssi process (4.1.3) or (4.1.5), which, to our knowledge, has not appeared before. In order to avoid complicating already cumbersome notation, in Theorem 4.6.4 below we assume that the density $\nu_\alpha(dz)$ from Assumption (**B**) is of the form

$$\nu_\alpha(dz) = \frac{\alpha}{2}\left(\mathbf{1}_{\{z>1\}}|z|^{-1-\alpha} + \mathbf{1}_{\{z<-1\}}|z|^{-1-\alpha}\right) dz. \tag{4.6.7}$$

We believe that considering a more general symmetric regularly varying density would not have any qualitative effect.

Theorem 4.6.4. *Suppose that the particle system and their movements are as in the formulation of Theorem 4.6.2, assume* (4.6.7) *and let* ϕ *be an* $L^1(\mathbb{R})$*-function such that* $\int_{\mathbb{R}} \phi(y)dy = 0$ *and*

$$\phi(y) = \mathbf{1}_{\{y>0\}} f_1(y) - \mathbf{1}_{\{y<0\}} f_2(-y), \tag{4.6.8}$$

where the functions $f_1, f_2 : (0,\infty) \to \mathbb{R}$ *are integrable and can be written as* $f_1 = |\cdot|^{-\gamma_1} g_1$, $f_2 = |\cdot|^{-\gamma_2} g_2$ *with* g_1 *and* g_2 *being slowly varying at infinity and eventually positive. Furthermore, assume that*

$$\gamma_1, \gamma_2 > 1, \quad \min(\gamma_1, \gamma_2) < 1 + \frac{\beta - 1}{2}.$$

Let G^T *be given by by* (4.1.7)*. We consider two possible cases.*

(i) Assume first that $\gamma_1 = \gamma_2 = \gamma$ *and* $\lim_{T\to\infty} f_1(T)/f_2(T) = 1$*. Set the normalizing factors*

$$F_T = g_1(T^{1/\beta})T^{1+1/(\alpha\beta)-\gamma/\beta}, \tag{4.6.9}$$

$D_T = T$ *and* $C_T \equiv 1$ *in* (4.1.7)*. Then*

$$G^T \xrightarrow{f.d.d} K_1 V,$$

where V *given by* (4.1.3) *and* K_1 *is a positive constant.*

(ii) In the second case assume without loss of generality that $\gamma_1 < \gamma_2$*. Then*

$$G^T \xrightarrow{f.d.d.} K_2 \widetilde{V},$$

where F_T*,* D_T *and* C_T *are as in (i) with* $\gamma = \gamma_1$*. The process* $\widetilde{V}$ *is* H*-sssi with Hurst exponent* $H = 1 + 1/(\alpha\beta) - \gamma/\beta$*.*

As can be expected the new processes we obtain are H-sssi.

Proposition 4.6.5. *The processes* V *and* $\widetilde{V}$ *defined by* (4.1.3) *and* (4.1.5)*, respectively, are* H*-sssi with* $H = 1 + \frac{1}{\alpha\beta} - \frac{\gamma}{\beta}$.

In this setting H can take any value from the interval $(\frac{1}{2}, 1)$. In the part (ii) of Theorem 4.6.4, we see that in the limit the heavier tail (corresponding to γ_1) totally dominates the lighter one (corresponding to γ_2), even though the integral of ϕ is zero. Also notice that for functions of the form $\phi(x) = sgn(x)/(1+|x|^\gamma)$ if $0 < \gamma < 1 + (\beta-1)/2$ we are in the setting of Theorem 4.6.4 and if $\gamma > (\beta-1)/2$ then ϕ satisfies assumptions of Theorem 4.6.2. It should be possible to show, that if we replace z_j in (4.1.7) by i.i.d. symmetric random variables with finite variance, then theorems analogous to 4.6.1, 4.6.2 and 4.6.4 hold when one formally sets $\alpha = 2$. Since in this case all the processes are Gaussian sssi processes, the limit process in case $\int_{\mathbb{R}} \phi(y)dy \neq 0$ is fractional Brownian motion with Hurst exponent $H = 1 - 1/2\beta$, which agrees with the results of [12]. If $\int_{\mathbb{R}} \phi(y)dy = 0$ and ϕ vanishes sufficiently quickly at infinity, the limit would be Brownian motion, while if ϕ has heavier tails, the limit process should again be a fractional Brownian motion. We expect that such analogies hold but leave them uninvestigated.

4.7 Proofs for Section 4.5

4.7.1 Proof of Proposition 4.5.1

Outline of the proof

In the prof of Proposition 4.5.1 we use the method of moments to prove the convergence of finite-dimensional distributions. The proof is relatively straightforward once we use the Fourier transform to show the convergence of appropriate moments, therefore we only give a proof of the convergence of one-dimensional distributions. We will only show the convergence of moments for a fixed time t. It will be clear from the proof, that the same arguments can be used to show convergence of mixed moments.

Because of the form of the limit process it is enough to show the following lemma. Tightness will be evident from the proof of the lemma itself.

Lemma 4.7.1. *Let us denote*

$$P_t^T(x) = \frac{1}{F_T} \int_0^{D_T t} \phi(\eta_s - T^{1/\beta} x)\, \mathrm{d}s, \tag{4.7.1}$$

for $T, t \geq 0$, with $F_T = T^{1-1/\beta} f(T^{1/\beta})^{-1}$ and $D_T = T f(T^{1/\beta})^{-1}$. Let ψ_T denote the characteristic exponent of the rescaled Lévy process $\widetilde{\eta}_T := T^{-1/\beta} \eta_{tTf(T^{1/\beta})^{-1}}$ (see Corollary 4.11.9). Then,

$$\lim_{T\to\infty} \mathbb{E}\big(P_t^T(x)\big)^k = \Big(\int_{\mathbb{R}} \phi(y)dy\Big)^k \mathbb{E}\big(L_t^\beta(x)^k\big). \tag{4.7.2}$$

Very similarly one shows that mixed moments moments of the process $(P_t^T(x))_{t\geq 0}$ converge to the mixed moments of the limit process. Thus, we establish the convergence of finite-dimensional distributions.

Proof of Lemma 4.7.1

One can easily show that

$$\psi_T(z) = D_T\psi\Big(\frac{z}{T^{1/\beta}}\Big), \tag{4.7.3}$$

where ψ is the characteristic exponent of the Lévy process η.

An easy application of Plancherel formula and change of variables formula shows that for any positive integer k

$$\begin{aligned}\mathbb{E}\big(P_t^T(x)^k\big) \;=\; & k!\Big(\frac{1}{2\pi}\Big)^k \int_{0<s_1<\ldots<s_k<t}\int_{\mathbb{R}^k} \widehat{\phi}\Big(\frac{w_1-w_2}{T^{1/\beta}}\Big)\widehat{\phi}\Big(\frac{w_2-w_3}{T^{1/\beta}}\Big)\\ & \times\ldots\times\widehat{\phi}\Big(\frac{w_{k-1}-w_k}{T^{1/\beta}}\Big)\widehat{\phi}\Big(\frac{w_k}{T^{1/\beta}}\Big)\\ & \times e^{iw_1x}e^{-s_1\psi_T(w_1)}\ldots e^{-(s_k-s_{k-1})\psi_T(w_k)}\,\mathrm{d}w_1\ldots\mathrm{d}w_k\,\mathrm{d}s_1\ldots\mathrm{d}s_k.\end{aligned} \tag{4.7.4}$$

We have used Plancherel formula in the second equality of (4.7.4). To see how the formula (4.7.4) emerges, let us consider the case when $k=2$ for simplicity. Then

$$\begin{aligned}\mathbb{E}\big(P_t^T(x)^2\big) \;=\; & \Big(\frac{1}{F_T}\Big)^2\int_0^{E_Tt}\int_0^{E_Tt}\mathbb{E}\Big(\phi(\eta_{s_1}-T^{-1/\beta}x)\phi(\eta_{s_2}-T^{-1/\beta}x)\Big)\\ & \mathrm{d}s_1\,\mathrm{d}s_2\\ =\; & 2\Big(\frac{1}{F_T}\Big)^2\int_0^{E_Tt}\int_{s_1}^{E_Tt}\mathbb{E}\Big(\phi(\eta_{s_1}-T^{-1/\beta}x)\\ & :\phi(\eta_{s_2}-\eta_{s_1}+\eta_{s_1}-T^{-1/\beta}x)\Big)\,\mathrm{d}s_1\,\mathrm{d}s_2\\ =\; & 2\Big(\frac{1}{F_T}\Big)^2\int_0^{E_Tt}\int_{s_1}^{E_Tt}\int_{\mathbb{R}^2}\Big(\phi(y_1-T^{-1/\beta}x)\phi(y_2+y_1-T^{-1/\beta}x)\Big)\\ & \nu_{s_1}(\mathrm{d}y_1)\nu_{s_2-s_1}(\mathrm{d}y_2)\,\mathrm{d}s_1\,\mathrm{d}s_2,\end{aligned}$$

where $\nu_s(\cdot)$ is the probability distribution of η_s for $s\geq 0$. Now by Plancherel formula

$$\begin{aligned}
\mathbb{E}\big(P_t^T(x)^2\big) &= 2\Big(\frac{1}{2\pi F_T}\Big)^2 \int_0^{E_T t}\int_{s_1}^{E_T t}\int_{\mathbb{R}^2} e^{-iT^{-1/\beta}xy_1}\widehat{\phi}(y_1-y_2)\widehat{\phi}(y_2) \\
&\quad \widehat{\nu_{s_1}}(y_1)\widehat{\nu_{s_2-s_1}}(y_2)\,\mathrm{d}y_1\,\mathrm{d}y_2,\,\mathrm{d}s_1\,\mathrm{d}s_2 \\
&= 2\Big(\frac{1}{2\pi F_T}\Big)^2 \int_0^{E_T t}\int_{s_1}^{E_T t}\int_{\mathbb{R}^2} e^{-iT^{-1/\beta}xy_1}\widehat{\phi}(y_1-y_2)\widehat{\phi}(y_2) \\
&\quad e^{-s_1\psi(y_1)}e^{-(s_2-s_1)\psi(y_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}s_1\,\mathrm{d}s_2 \\
&= 2\Big(\frac{1}{2\pi}\Big)^2 \int_0^{t}\int_{s_1}^{t}\int_{\mathbb{R}^2} e^{-ixy_1}\widehat{\phi}\Big(\frac{y_1-y_2}{T^{1/\beta}}\Big)\widehat{\phi}\Big(\frac{y_2}{T^{1/\beta}}\Big) \\
&\quad e^{-s_1\psi_T(y_1)}e^{-(s_2-s_1)\psi_T(y_2)}\,\mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}s_1\,\mathrm{d}s_2, \qquad (4.7.5)
\end{aligned}$$

where the last equality in (4.7.5) follows from (4.7.3). The general form of 4.7.4 is obtained analogously.

Clearly $\widehat{\phi}$ is bounded. We would like to take the limit under the integral sign. However, due to the terms ψ_T the use of dominated convergence cannot be justified as simply as in the proof of the stable case. By Lemma 4.11.12 in the Appendix,

$$\int_0^t\int_{\mathbb{R}} e^{-u\psi_T(z)}\,\mathrm{d}z\,\mathrm{d}u \qquad (4.7.6)$$

is bounded uniformly in $T\geq 1$. Now, fix some $K>0$. The last integral in (4.7.4) with $\mathbb{R}^k$ replaced by $G_K := \{(w_1,\ldots,w_k) : |w_1|,\ldots,|w_k|\leq K\}$ converges, as $T\to\infty$, to

$$k!\Big(\frac{1}{2\pi}\Big)^k\Big(\int_{\mathbb{R}}\phi(y)\,\mathrm{d}y\Big)^k\int_{0<s_1<\ldots<s_k<t}\int_{G_K} e^{ixw_1}e^{-s_1|w_1|^\beta}\ldots e^{-(s_k-s_{k-1})|w_k|^\beta}$$
$$\mathrm{d}w_1\ldots\mathrm{d}w_k\,\mathrm{d}s_1\ldots\mathrm{d}s_k, \quad (4.7.7)$$

by dominated convergence theorem. In view of Lemma 4.11.10 in the Appendix, the integral in (4.7.4) with $\mathbb{R}^k$ replaced by $\mathbb{R}^k\setminus G_K$ can be made arbitrarily small for K large enough. Hence

$$\begin{aligned}
\lim_{T\to\infty}\mathbb{E}\big(P_t^T(x)\big)^k &= k!\Big(\frac{1}{2\pi}\Big)^k\int_{0<s_1<\ldots<s_k<t}\int_{\mathbb{R}^k} e^{ixw_1}e^{-s_1|w_1|^\beta}\ldots \\
&\quad \ldots e^{-(s_k-s_{k-1})|w_k|^\beta}\,\mathrm{d}w_1\ldots\mathrm{d}w_k\,\mathrm{d}s_1\ldots\mathrm{d}s_k, \quad (4.7.8)
\end{aligned}$$

which by Lemma 4.11.1 equals $\mathbb{E}L_t^\beta(x)^k$.

Tightness

Tightness under Assumption (**C**) follows almost immediately. One just has to notice that for $s<t$, a calculation similar to the one in (4.7.4) and Lemma 4.11.13

in the Appendix imply that for k sufficiently large $\mathbb{E}\big|P_t^T(x) - P_s^T(x)\big|^k \le C_k(t - s)^\gamma$ for some $\gamma > 1$, $C_k < \infty$ and then use Theorem 2.7.2.

4.7.2 Proof of Proposition 4.5.2

Outline of the proof

The proof uses the method of moments and can be outlined as follows: show the convergence of finite-dimensional distributions and then prove tightness. To prove the convergence of finite-dimensional distributions we use the method of moments. We will only show the convergence of moments for a fixed time t. It will be clear from the proof, that the same arguments can be used to show convergence of mixed moments.

Due to the form of the limit process on the right-hand side of (4.5.4) and the constant $c(\phi)$ given by (4.5.5), in order to do so, it suffices to show the following lemma.

Lemma 4.7.2. *Suppose that the assumptions of Proposition 4.5.2 are satisfied. For $T \ge 1, x \in \mathbb{R}$ and $t > 0$ put*

$$\widetilde{P}_t^T(x) = \frac{1}{F_T^{1/2}} \int_0^{Tf(T^{1/\beta})^{-1}t} \phi(T^{1/\beta}x - \eta_s)\,\mathrm{d}s, \tag{4.7.9}$$

with F_T as in the statement of the Proposition 4.5.2. For a fixed $t > 0, x \in \mathbb{R}$ we have:

(i) for an even positive integer k

$$\lim_{T\to\infty} \mathbb{E}\big(\widetilde{P}_t^T(x)\big)^k \;=\; \frac{k!}{(k/2)!}\Big(\frac{1}{\pi}\int_{\mathbb{R}} \big|\widehat{\phi}(w)\big|^2 \frac{1}{\psi(w)}\,\mathrm{d}w\Big)^{k/2} \mathbb{E}\Big(L_t^\beta(x)^{k/2}\Big), \tag{4.7.10}$$

(ii) if k is and odd positive integer then $\mathbb{E}\big(\widetilde{P}_t^T(x)\big)^k \to 0$ as $T \to \infty$.

Tightness follows from the following lemma.

Lemma 4.7.3. *Assume **(A)** and **(C)**. Under the assumptions of Proposition 4.5.2 and for $\widetilde{P}^T$ defined by 4.7.9 there exists a positive finite constant C_k, independent of T such that for k even and $0 \le s < t < \infty$ we have*

$$\mathbb{E}\big|\widetilde{P}_t^T(x) - \widetilde{P}_s^T(x)\big|^k \le C_k(t - s)^{k\delta/2} \tag{4.7.11}$$

Proof of Lemma 4.7.2

Similarly as in (4.7.4), after a change of variables we obtain that $\mathbb{E}\big(\widetilde{P}_t^T(x)\big)^k$ is equal to

$$\begin{aligned} k!\Big(\frac{1}{2\pi}\Big)^k \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \quad & \mathbf{1}_{\{0<u_1<D_T^{-1}u_2+u_1<u_3<\ldots<u_{k-1}<D_T^{-1}u_k+u_{k-1}<t\}} \\ & \times\widehat{\phi}\big(\frac{w_1}{T^{1/\beta}}-w_2\big)\widehat{\phi}\big(w_2-\frac{w_3}{T^{1/\beta}}\big)\ldots\widehat{\phi}\big(\frac{w_{k-1}}{T^{1/\beta}}-w_k\big)\widehat{\phi}\big(w_k\big) \\ & \times e^{iw_1x}e^{-u_1\psi_T(w_1)}e^{-u_2\psi(w_2)}e^{-(u_3-D_T^{-1}u_2-u_1)\psi_T(w_3)}\ldots\times \\ & \ldots\times e^{-(u_{k-1}-D_T^{-1}u_{k-2}-u_{k-3})\psi_T(w_{k-1})}e^{-u_k\psi(w_k)} \\ & du_1\ldots du_k dw_1\ldots dw_k, \end{aligned} \tag{4.7.12}$$

where $D_T = T(f(T^{1/\beta}))^{-1}$. For $z, w \in \mathbb{R}$ and $T \geq 1$ let us define

$$a_T(w,z) = \widehat{\phi}(w/T^{1\beta}-z)-\widehat{\phi}(-z), \tag{4.7.13}$$

$$b(z) = \widehat{\phi}(-z). \tag{4.7.14}$$

Then (4.7.12) can be rewritten as

$$\begin{aligned} k!\Big(\frac{1}{2\pi}\Big)^k \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \quad & \mathbf{1}_{\{0<u_1<D_T^{-1}u_2+u_1<u_3<\ldots<u_{k-1}<D_T^{-1}u_k+u_{k-1}<t\}} \\ & \times\big(a_T(w_1,w_2)+b(w_2)\big)\big(\overline{a_T(w_2,w_3)}+\overline{b(w_2)}\big)\ldots \\ & \times\big(a_T(w_{k-1},w_k)+b(w_k)\big)\big(\overline{b(w_k)}\big) \\ & \times e^{iw_1x}e^{-u_1\psi_T(w_1)}e^{-u_2\psi(w_2)}e^{-(u_3-D_T^{-1}u_2-u_1)\psi_T(w_3)}\ldots\times \\ & \ldots\times e^{-(u_{k-1}-D_T^{-1}u_{k-2}-u_{k-3})\psi_T(w_{k-1})}e^{-u_k\psi(w_k)} \\ & \mathrm{d}u_1\ldots\mathrm{d}u_k\,\mathrm{d}w_1\ldots\mathrm{d}w_k. \end{aligned} \tag{4.7.15}$$

We will show that out of all 2^{k-1} expressions that we get by multiplying the parentheses with the terms a_T and b in (4.7.15), the only term that does not converge to zero as $T \to \infty$ is the one in which only b's appear. In fact, we will only prove that the term with

$$a_T(w_1,w_2)\overline{a_T(w_2,w_3)}\ldots a_T(w_{k-1},w_k)\overline{b_T(w_k)}$$

converges to 0, the other cases being very similar as the integral with respect to $w_1,\ldots,w_k$ factorizes. Let us denote this term by M. Since we assume that $\int_{\mathbb{R}}|\phi(y)|dy = 1$ we see that by Assumption (**D**) (we can without loss of generality assume that $C = 1$ in the formulation of the Assumption (**D**)),

$$\begin{aligned} M \quad \leq \quad & \int_{\mathbb{R}^k}\int_0^t\int_0^{D_Tt}\ldots\int_0^t\int_0^{D_Tt} \\ & \big(1\wedge|w_k|^\kappa\big)\big(1\wedge|w_{k-1}T^{-1/\beta}|^{2\kappa}\big)\big(1\wedge|w_{k-3}T^{-1/\beta}|^{2\kappa}\big) \\ & \times\ldots\times\big(1\wedge|w_3T^{-1/\beta}|^{2\kappa}\big)\big(1\wedge|w_1T^{-1/\beta}|^\kappa\big) \\ & \times e^{-u_1\psi_T(w_1)}e^{-u_2\psi(w_2)}e^{-u_3\psi_T(w_3)}\ldots e^{-u_{k-1}\psi_T(w_{k-1})}e^{-u_k\psi(w_k)} \\ & \mathrm{d}u_1\ldots\mathrm{d}u_k\,\mathrm{d}w_1\ldots\mathrm{d}w_k. \end{aligned} \tag{4.7.16}$$

Now, Lemma 4.11.14 in the Appendix implies that

$$M \leq c_1(T^{1/\beta} D_T^{-1} + T^{-\kappa/\beta}) \times (1 + T^{-1/\beta} D_T T^{-\kappa/\beta}), \tag{4.7.17}$$

for some finite constant c_1 independent of T. Since $\kappa > (\beta-1)/2$, M converges to 0 as $T \to \infty$. The only significant term in (4.7.15) is thus given by

$$\begin{aligned} k!\Big(\frac{1}{2\pi}\Big)^k \int_{\mathbb{R}^k}\int_{\mathbb{R}^k} \quad & \mathbf{1}_{\{0<u_1<D_T^{-1}u_2+u_1<u_3<\ldots<u_{k-1}<D_T^{-1}u_k+u_{k-1}<t\}} \\ & \times b_T(w_2)\overline{b_T(w_2)} b_T(w_k)\overline{b_T(w_k)} \\ & \times e^{iw_1x} e^{-u_1\psi_T(w_1)} e^{-u_2\psi(w_2)} e^{-(u_3-D_T^{-1}u_2-u_1)\psi_T(w_3)} \ldots \times \\ & \ldots \times e^{-(u_{k-1}-D_T^{-1}u_{k-2}-u_{k-3})\psi_T(w_{k-1})} e^{-u_k\psi(w_k)} \\ & \mathrm{d}u_1 \ldots \mathrm{d}u_k \, \mathrm{d}w_1 \ldots \mathrm{d}w_k, \end{aligned} \tag{4.7.18}$$

which converges, by dominated convergence theorem, to the right-hand side of (4.7.10) (see Lemma 4.11.1 in the Appendix).

Very similarly one shows that for all odd positive integers k the respective moments converge to 0 and that for any $t_1 \leq \ldots t_k$ and $x \in \mathbb{R}$

$$\mathbb{E}\big(\widetilde{P}_{t_1}^T(x) \ldots \widetilde{P}_{t_k}^T(x)\big) \to c(\phi)^k \mathbb{E}(W^k_{L^\beta_{t_1}(x)}).$$

Proof of Lemma 4.7.3

Assume that $0 \leq s < t < \infty$. Analogously to (4.7.12) and using the fact that $\widehat{\phi}$ is bounded we can estimate

$$\begin{aligned} \mathbb{E}\big(\widetilde{P}_t^T(x) - \widetilde{P}_s^T(x)\big)^k \leq & \\ \int_{\mathbb{R}^{k/2}} \int_{[0,t-s]^{k/2}} & e^{-u_1\psi_T(w_1)} e^{-u_3\psi_T(w_3)} \ldots e^{-u_{k-1}\psi_T(w_{k-1})} \\ & \qquad \mathrm{d}u_1 \, \mathrm{d}u_3 \ldots \mathrm{d}u_{k-1} \, \mathrm{d}w_1 \, \mathrm{d}w_3 \ldots \mathrm{d}w_{k-1}. \end{aligned}$$

This and (4.11.35) imply that

$$\mathbb{E}\big|\widetilde{P}_t^T(x) - \widetilde{P}_s^T(x)\big|^k \leq C_k(t-s)^{k\delta/2} \tag{4.7.19}$$

for some finite constant C_k independent of T. Taking k large enough we may apply the Kolmogorov's tightnes criterion (see Theorem 2.7.2) and infer that the sequence of processes $(P^T(x))$ is tight in $\mathcal{C}[0,\infty)$.

4.8 Proof of Theorem 4.6.1

4.8.1 Outline of the proof

To prove Theorem 4.6.1 we first show the convergence of finite-dimensional distributions, and then establish tightness, which suffices to prove convergence in $\mathcal{C}$ (see [5, Theorem 8.1]). To show the former we will establish the convergence of the characteristic function of (4.6.1) to the characteristic function of (4.1.1).

Let $a_k \in \mathbb{R}, t_k \geq 0$, $k = 1, \ldots, m$, put $\overline{G}^T = \sum_k^m a_k G^T_{t_k}$ and $\overline{P}^T(x) = \sum_k^m a_k P^T_{t_k}(x)$ for $t \geq 0, T \geq 1$ and $x \in \mathbb{R}$, with P^T and G^T defined by (4.7.1) and (4.1.1), respectively. Using the fact that (x_j, z_j) are points of a Poisson random measure we have

$$\mathbb{E}\exp(i\overline{G}^T) = \exp\Bigg(\int_{\mathbb{R}^2} \mathbb{E}\Bigg(\exp\Big(iz\frac{1}{F_T}\sum_{j=1}^m a_j \int_0^{D_T t_j} \phi(C_T x + \eta^1_u)du\Big) - 1\Bigg) |z|^{-1-\alpha} l(z) dz dx\Bigg). \quad (4.8.1)$$

Our aim is to prove that

$$\lim_{T\to\infty} \mathbb{E}\exp(i\overline{G}^T) = \exp\left(C(\alpha,\beta)\int_{\mathbb{R}} \mathbb{E}\left|\sum_{k=1}^m a_k L^\beta_{t_k}(x)\right|^\alpha dx\right), \quad (4.8.2)$$

where $(L^\beta_t(x))$ is the local time of a symmetric β-stable Lévy process at x and $C(\alpha,\beta)$ is a constant depending only on α and β. Here, we have used the properties of stable integrals (see Section (2.3.2)) to get the right-hand side of (4.8.2). This suffices to prove the convergence in the sense of finite-dimensional distributions.

After a change of variables $z := T^{1/\alpha\beta} z$ and $x := -T^{1/\beta} C_T^{-1} x$ (4.8.1) can be written as

$$\mathbb{E}\exp(i\overline{G}^T) \;=\; exp\left(\int_{\mathbb{R}^2}\mathbb{E}\left(e^{iz\overline{P}^T(x)} - 1\right)|z|^{-\alpha-1}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}dzdx\right). \quad (4.8.3)$$

By the symmetry of function l, (4.8.3) can be rewritten as

$$\begin{aligned}\mathbb{E}\exp(i\overline{G}^T) \;&=\; \exp\Big(\int_{\mathbb{R}^2}\mathbb{E}\big(e^{iz\overline{P}^T(x)} - \mathbf{1}_{\{|z|\leq 1\}} iz\overline{P}^T(x) - 1\big) \\ &\quad\times |z|^{-\alpha-1}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}dzdx\Big),\end{aligned} \quad (4.8.4)$$

where l is the slowly varying function from Assumption (**B**). By Proposition 4.5.1 $\overline{P}^T(x)$ converges in law to $\int_{\mathbb{R}} \phi(y)dy \times \sum_{k=1}^m a_k L_{t_k}(x) =: \overline{P}(x)$. Furthermore all the moments of $\overline{P}^T(x)$ converge to the corresponding moments of $\overline{P}(x)$. Since l is slowly varying at infinity $l(T^{1/\alpha\beta}z)/l(T^{1/\alpha\beta})$ converges to 1 as $T \to \infty$. Taking all this into account,

$$\lim_{T\to\infty} \mathbb{E}\big(e^{iz\overline{P}^T(x)} - \mathbf{1}_{\{|z|\le 1\}} iz\overline{P}^T(x) - 1\big)|z|^{-\alpha-1}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}$$
$$= \mathbb{E}\big(e^{iz\overline{P}(x)} - \mathbf{1}_{\{|z|\le 1\}} iz\overline{P}(x) - 1\big)|z|^{-\alpha-1}. \quad (4.8.5)$$

By the formula (4.11.7) we will be able to show that (4.8.2) holds as long as we can justify going to the limit under the integral sign in (4.8.4). For this we need the following lemma.

Lemma 4.8.1. *Let P^T be as in* (4.7.1) *and assume that the conditions of Theorem 4.6.1 are satisfied. Then the following claims are true.*

(i) For each $t > 0$ the functions $x \mapsto \mathbb{E}\big|P_t^T(x)\big|$ and $x \mapsto \mathbb{E}P_t^T(x)^2$ are bounded uniformly in $T \ge 1$.

(ii) For ant $t > 0$

$$\sup_{T\ge 1}\Big(\int_{\mathbb{R}} \mathbb{E}|P_t^T(x)|dx + \int_{\mathbb{R}} \mathbb{E}|P_t^T(x)|^2 dx\Big) < \infty. \quad (4.8.6)$$

(iii) For any $t, \delta > 0$ and there exist $K > 0$ and $T_0 \ge 1$ such that

$$\sup_{T\ge T_0}\Big(\int_{|x|>K} \mathbb{E}|P_t^T(x)|dx + \int_{|x|>K} \mathbb{E}|P_t^T(x)|^2 dx\Big) < \delta. \quad (4.8.7)$$

(iv) For any $\delta \in (0,1)$ there exists $T_0 \ge 1$ and constant $C > 0$, depending only on α, δ, such that for all $r \in (0,1)$

$$\int_{|z|\le r} |z|^{1-\alpha}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}dz \le \delta + Cr^{2-\alpha+\delta}. \quad (4.8.8)$$

Equipped with Lemma 4.8.1 we can show the following, which gives us the desired convergence.

Lemma 4.8.2. *Under the assumptions of Theorem 4.6.1 we have*

$$\lim_{T\to\infty}\int_{\mathbb{R}^2} \mathbb{E}\big(e^{iz\overline{P}^T(x)} - \mathbf{1}_{\{|z|\le 1\}} iz\overline{P}^T(x) - 1\big) \times |z|^{-\alpha-1}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}dzdx$$
$$= \quad C(\alpha,\beta)\int_{\mathbb{R}} \mathbb{E}\left|\sum_{k=1}^m a_k L_{t_k}(x)\right|^{\alpha} dx, \quad (4.8.9)$$

where $(L_t(x))$ is the local time of a symmetric β-stable Lévy process at $x \in \mathbb{R}$ and $C(\alpha,\beta)$ is a constant.

Tightness, under Assumption (**C**), will follow from the following lemma.

Lemma 4.8.3. *Assume that ψ satisfies the conditions of Assumption (**C**) in Section 4.4. Then the family of processes $\{(G_t^T)_{t\geq 0} : T \geq 1\}$ defined by* (4.6.1) *is tight in $\mathcal{C}[0,\tau]$ for any $\tau > 0$.*

4.8.2 Proof of Lemma 4.8.1

Without loss of generality we may assume that $\phi \geq 0$, which implies that $P_t^T(x) \geq 0$. Changing variables and using Plancherel and Fubini's theorems, for $x \in \mathbb{R}, t \geq 0, T > 0$ and $\phi \in L^1(\mathbb{R})$ we have

$$\mathbb{E}P_t^T(x) = \frac{1}{2\pi}\int_0^t \int_{\mathbb{R}} \widehat{\phi}\Big(\frac{w}{T^{1/\beta}}\Big)e^{ixw}e^{-u\psi_T(w)}dw, \tag{4.8.10}$$

where ψ_T equals

$$\psi_T(z) = Tf(T^{1/\beta})^{-1}\psi\big(\frac{z}{T^{1/\beta}}\big). \tag{4.8.11}$$

Hence

$$\mathbb{E}P_t^T(x) \leq \frac{1}{2\pi}\left\|\phi\right\|_1 \int_0^t \int_{\mathbb{R}} e^{-u\psi_T(w)}dwdu, \tag{4.8.12}$$

which is bounded uniformly in $T \geq 1$ by Lemma 4.11.12 in the Section 4.11.2 if the appendix to this chapter. Using similar techniques one can write (recall (4.7.4))

$$\begin{aligned}\mathbb{E}P_t^T(x)^2 \quad &= \quad \frac{2}{(2\pi)^2}\int_0^t\int_{u_1}^t\int_{\mathbb{R}^2} e^{ixw_1}\widehat{\phi}\Big(\frac{w_1-w_2}{T^{1/\beta}}\Big)\widehat{\phi}\Big(\frac{w_2}{T^{1/\beta}}\Big) \\ &\quad\quad e^{-(u_2-u_1)\psi_T(w_2)}e^{-(u_1)\psi_T(w_1)}dw_1dw_2du_1du_2 \\ &\quad \leq \frac{\|\phi\|_1^2}{2\pi^2}\Big(\int_0^t\int_{\mathbb{R}} e^{-u\psi_T(w)}dwdu\Big)^2 \end{aligned} \tag{4.8.13}$$

and argue similarly. This proves (i).

We now turn to (ii). For any $t \geq 0, T \geq 1$ we have

$$\int_{\mathbb{R}} \mathbb{E}P_t^T(x)dx \leq \left\|\phi\right\|_1 t.$$

As for the second part of (ii), using (4.8.13) and obvious substitutions, we may write

$$\int_{\mathbb{R}} \mathbb{E}P_t^T(x)^2dx =$$

$$\int_{\mathbb{R}} \frac{1}{\pi}\int_0^t\int_{u_1}^t\int_{\mathbb{R}} \phi(-x)e^{ixw}\widehat{\phi}(w)T^{1/\beta}e^{-D_T(u_2-u_1)\psi(w)}dwdu_1du_2dx, \tag{4.8.14}$$

where D_T is as in (4.7.1). The above (after substituting $w := w/T^{1/\beta}$) can be bounded by

$$\frac{1}{\pi}\|\phi\|_1 \int_0^t \int_{u_1}^t \int_{\mathbb{R}} \left|\widehat{\phi}\Big(\frac{w}{T^{1/\beta}}\Big)\right| e^{-(u_2-u_1)\psi_T(w)} dw du_1 du_2, \tag{4.8.15}$$

which in turn is no bigger than

$$\frac{1}{\pi}\|\phi\|_1^2 \int_0^t \int_{u_1}^t \int_{\mathbb{R}} e^{-(u_2-u_1)\psi_T(w)} dw du_1 du_2. \tag{4.8.16}$$

By Lemma 4.11.12, the last expression is bounded uniformly in $T \geq 1$. This proves (ii).

Let us now turn to (iii). In order to escape notational complexity we will only consider the integrals over $\{x \in \mathbb{R} : x > K\}$. For $\{x \in \mathbb{R} : x < -K\}$ it is then enough to use the symmetry of η and take $\widetilde{\phi}(x) = \phi(-x)$. We can also assume that $\phi \geq 0$. First notice that after changing variables and using Fubini theorem we get

$$\begin{aligned}\int_K^\infty \mathbb{E}P_t^T(x)dx &= \mathbb{E}\Big(\int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{y+KT^{1/\beta}<\eta_{D_T u}\}}\phi(y)dydu\Big)\\ &= \int_0^t \int_{\mathbb{R}} \mathbb{P}(yT^{-1/\beta}+K < T^{-1/\beta}\eta_{D_T u})\phi(y)dy,\end{aligned}$$

which converges as $T \to \infty$, by dominated convergence and (4.11.21), to

$$\int_{\mathbb{R}} \phi(y)dy \int_0^t \mathbb{P}(\xi_u > K)du = \int_{\mathbb{R}} \phi(y)dy \int_K^\infty \mathbb{E}L_t(x)dx. \tag{4.8.17}$$

By choosing K large enough to begin with and using Lemma 4.11.4 from Section 4.11.2 in the Appendix we see that the first part of (iii) is true. Regarding its second part, write (again after changing variables and using Fubini theorem)

$$\begin{aligned}\int_K^\infty \mathbb{E}P_t^T(x)^2dx &= 2\int_0^t \int_{u_1}^t \int_{\mathbb{R}} \mathbb{E}\Big(\mathbf{1}_{\{x>KT^{1/\beta}-\eta_{D_T u_1}\}}\phi(-x) \qquad (4.8.18)\\ &\quad \times T^{1/\beta}\phi(\eta_{D_T u_2}-\eta_{D_T u_1}-x)\Big)dxdu_1du_2.\end{aligned}$$

Since η is a Lévy process the above equals

$$\begin{aligned}2\int_0^t \int_{u_1}^t \int_{\mathbb{R}} \quad & \mathbb{P}\big(x > KT^{1/\beta}-\eta_{D_T u_1}\big)\phi(-x) \qquad (4.8.19)\\ & \times T^{1/\beta}\mathbb{E}\big(\phi(\eta_{D_T(u_2-u_1)}-x)\big)dxdu_1du_2\\ = \; & 2\int_0^t \int_{u_1}^t \int_{\mathbb{R}} \mathbb{P}\big(xT^{-1/\beta} > K - T^{-1/\beta}\eta_{D_T u_1}\big)\phi(-x) \qquad (4.8.20)\\ & \times \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\phi}\Big(\frac{w}{T^{1/\beta}}\Big)e^{-ixwT^{-1/\beta}}e^{-(u_2-u_1)\psi_T(w)}dwdxdu_1du_2.\end{aligned}$$

The integrand in (4.8.20) the above can be bounded by $\phi(-x)\left\|\phi\right\|_1 e^{-(u_2-u_1)\psi_T(w)}$. Hence

$$\int_K^\infty \mathbb{E}P_t^T(x)^2dx \quad\leq\quad 2t\left\|\phi\right\|_1\int_0^t\Big(\int_{\mathbb{R}^2}\mathbb{P}\big(T^{-1/\beta}\eta_{D_Tu_1}>K-T^{-1/\beta}x\big)\\ \phi(-x)e^{-u\psi_T(w)}dxdw\Big)du. \tag{4.8.21}$$

The integrand in (4.8.21) is non-negative and bounded by an integrable function (again use Lemma 4.11.12 from Section 4.11.2 in the appendix). Thus, the right-hand side of (4.8.21) can be bounded by

$$C(t,\phi,\alpha,\beta)\int_0^t\int_{\mathbb{R}}\phi(-x)\mathbb{P}\big(T^{-1/\beta}\eta_{D_Tu_1}>K-T^{-1/\beta}x\big)dxdu, \tag{4.8.22}$$

where $C(t,\phi,\alpha,\beta)$ is a constant independent of T and K. By dominated convergence and (4.11.21) (4.8.22) converges to

$$C(t,\phi,\alpha,\beta)\left\|\phi\right\|_1\int_0^t\mathbb{P}(\xi_u>K)du, \tag{4.8.23}$$

as $T\to\infty$. Here ξ is a symmetric β-stable Lévy process. Using dominated convergence again we conclude that there exists $T_0>0$ such that

$$\lim_{K\to\infty}\sup_{T\geq T_0}\int_K^\infty \mathbb{E}P_t^T(x)^2dx=0. \tag{4.8.24}$$

The proof of (iv) is relatively straightforward consequences of [42, Theorem 10.5.6] and we skip them.

4.8.3 Proof of Lemma 4.8.2

The proof of Lemma 4.8.2 can be divided into the following steps. First we prove that if we replace integration in (4.8.3) over $\mathbb{R}^2$ with integration over the set $\{(x,z):|x|>K\}$ or $\{(x,z):|z|<r\}$, then the corresponding quantity can be made arbitrarily small for K sufficiently large and r sufficiently small, respectively. Then we show that for any $r\in(0,1)$ and $K>0$ the function

$$(x,z)\ \mapsto\ \mathbb{E}\Big|e^{i\theta z\overline{P^T}(x)}\ -\ \mathbf{1}_{\{|z|\leq 1\}}i\theta z\overline{P^T}(x)\ -\ 1\Big||z|^{-\alpha-1}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})} \tag{4.8.25}$$

can be bounded on $[-K,K]\times(\mathbb{R}\setminus(-r,r))$, uniformly in $T\geq 1$, by an integrable function.

We proceed to show the first fact. Using inequalities $|e^{iw}-1|\leq|w|$ and $|e^{iw}-$

$iw - 1| \le \frac{1}{2}|w|^2$ for $w \in \mathbb{R}$, we see that

$$\begin{aligned}
&\int_{|x|>K}\int_{\mathbb{R}} \mathbb{E}\left|e^{i\theta z\overline{P}^T(x)} - \mathbf{1}_{\{|z|\le 1\}} i\theta z\overline{P}^T(x) - 1\right| \\
&\quad \times |z|^{-\alpha-1}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}dzdx \\
\le &\int_{|z|\le 1}|z|^{1-\alpha}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}dz\int_{|x|>K}\mathbb{E}|\overline{P}^T(x)|^2dx \\
&+\int_{|z|>1}|z|^{-\alpha}dz\int_{|x|>K}\mathbb{E}|\overline{P}^T(x)|dx \qquad (4.8.26)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{|z|\le r} \mathbb{E}\left|e^{i\theta z\overline{P}^T(x)} - \mathbf{1}_{\{|z|\le 1\}} i\theta z\overline{P}^T(x) - 1\right| \\
&\quad \times |z|^{-\alpha-1}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}dzdx \\
\le &\int_{\mathbb{R}}\mathbb{E}|\overline{P}^T(x)|^2dx\int_{|z|\le r}|z|^{1-\alpha}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}dz, \qquad (4.8.27)
\end{aligned}$$

which in view of Lemma 4.8.1 can be made arbitrarily small for all T sufficiently large by first choosing K large enough in (4.8.26) and r small enough in (4.8.27).

We now go on to show the second fact. By Proposition 2.8.5, for each $r \in (0,1)$ fixed there exists $T_0 \ge 1$ such that

$$\left|\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})} - 1\right| \le 1 \qquad (4.8.28)$$

for all $z \in [r,1]$ and $T \ge T_0$. Furthermore, by Proposition 2.8.6, for any $\delta > 0$ there exists $T_1 \ge 1$ such that for all $z \ge 1$ and $T \ge T_1$ we have

$$\left|\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}\right| \le (1+\delta)|z|^\delta. \qquad (4.8.29)$$

This implies that for $|z| > r$ and all T large enough the function

$$(x,z) \;\mapsto\; \mathbb{E}\left|e^{i\theta z\overline{P^T}(x)} - \mathbf{1}_{\{|z|\le 1\}} i\theta z\overline{P^T}(x) - 1\right||z|^{-\alpha-1}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}\mathbf{1}_{\{|z|\le r\}} \qquad (4.8.30)$$

can be bounded by the function

$$\begin{aligned}
(x,z) \mapsto\ &2\mathbb{E}|\overline{P^T}(x)|^2\mathbf{1}_{\{|z|\in(r,1]\}}|\theta|^2|z|^{1-\alpha} \\
&+\mathbb{E}|\overline{P^T}(x)||\theta|\mathbf{1}_{\{|z|>1\}}|z|^{-\alpha}(1+\delta)|z|^\delta \qquad (4.8.31)
\end{aligned}$$

for $x, z \in \mathbb{R}$. Choosing δ small enough and again using Lemma 4.8.1 (part (ii)) we see that the above can be bounded by an integrable function, uniformly for all T large enough.

To conclude, using the fact that $\overline{P^T}(x)$ converges in law to $P = \sum_{k=1}^m a_j L_t(x)$ (with all its moments converging as well) for every $x \in \mathbb{R}$, we have (by dominated convergence)

$$\lim_{T\to\infty} \int_{|z|\geq r} \int_{|x|\leq K} \mathbb{E}\big(e^{iz\overline{P}^T(x)} - \mathbf{1}_{\{|z|\leq 1\}} iz\overline{P}^T(x) - 1\big) \times |z|^{-\alpha-1} \frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})} dz dx$$
$$= \int_{|z|\geq r} \int_{|x|\leq K} \mathbb{E}\big(e^{iz\overline{P}(x)} - \mathbf{1}_{\{|z|\leq 1\}} iz\overline{P}(x) - 1\big)|z|^{-\alpha-1} dz dx. \quad (4.8.32)$$

By dominated convergence again, we can go with r to zero in the integral over dz first, use (4.11.11) and finally go with K to ∞ to get (4.8.9) and finish the proof.

4.8.4 Proof of Lemma 4.8.3

For any $K > 0, T \geq 1$ put $K_T := KT^{1/\alpha\beta}$ and let $G_t^T = G_t^{T,1} + G_t^{T,2}$ for any $t \geq 0$, with

$$G_t^{T,1} := \frac{1}{F_T} \sum_j z_j \mathbf{1}_{\{K_T > |z_j|\}} \int_0^{D_T t} \phi(C_T x^j + \xi_u^j) du, \quad (4.8.33)$$

and

$$G_t^{T,2} := \frac{1}{F_T} \sum_j z_j \mathbf{1}_{\{|z_j| \geq K_T\}} \int_0^{D_T t} \phi(C_T x^j + \xi_u^j) du. \quad (4.8.34)$$

We are going to show that the family of processes $(G_t^{T,1})_{t\geq 0}$ is tight $\mathcal{C}[0,\tau]$ for any $\tau > 0$ and that for any $\delta > 0$

$$\lim_{K\to\infty} \limsup_{T\to\infty} \mathbb{P}(\sup_{t\in[0,\tau]} |G_t^{T,2}| > \delta) = 0, \quad (4.8.35)$$

which suffices to establish tightness.

We now proceed to establish tightness for the family $(G_t^{T,1})_{t\geq 0}$. Notice that, using the formula for variance of an integral with respect to a Poisson random measure,

$$\mathbb{E}\big(G_t^{T,1} - G_s^{T,1}\big)^2 \leq c_1 \int_{\mathbb{R}} \mathbb{E}\big(P_{t-s}^T(x)^2\big) dx \int_{|z|\leq K} |z|^{1-\alpha} \frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})} dz, \quad (4.8.36)$$

for some finite constant c_1. After a change of variables $z := zT^{1/\alpha\beta}$ and an application of [42, Theorem 10.5.6], we conclude that for all T large enough, the integral over $\{|z| \leq K\}$ in (4.8.36) is bounded by $c_2 K^{2-\alpha}$ for some finite constant c_2 depending only on α. Furthermore, by (4.8.15)

$$\begin{aligned}\int_{\mathbb{R}} \mathbb{E}\big(P_{t-s}^T(x)^2\big)dx \quad &\leq \quad \frac{1}{\pi}\,\|\phi\|_1 \int_s^t \int_{u_1}^t \int_{\mathbb{R}} \left|\widehat{\phi}\Big(\frac{w}{T^{1/\beta}}\Big)\right| e^{-(u_2-u_1)\psi_T(w)} dw du_1 du_2 \\ &\leq \frac{1}{\pi}\,\|\phi\|_1^2\,(t-s) \int_0^{t-s} \int_{\mathbb{R}} e^{-u_2\psi_T(w)} dw du_2 \qquad (4.8.37)\end{aligned}$$

Using Lemma 4.11.13, we see that (4.8.37) is bounded by

$$c_3(t-s)^{1+\delta}$$

for some $\delta > 0$ and a constant c_3 independent of s, t and T. An application of Theorem 2.7.2 (not that $G_0^T = 0$) shows that the family $(G_t^{T,1})_{t\geq 0}$ is tight in $\mathcal{C}[0,\tau]$ for any $\tau > 0$.

Proceeding further, notice that for any $\delta, \tau > 0$ (after a change of variables)

$$\begin{aligned}&\mathbb{P}\big(\sup_{t\in[0,\tau]} |G_t^{T,2}| > \delta\big) \\ &\qquad \leq \frac{1}{\delta}\int_{\mathbb{R}}\int_{|z|\geq K} \mathbb{E} \sup_{t\in[0,\tau]} \big|P_t^T(x)\big||z|^{-\alpha}\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})} dz dx, \quad (4.8.38)\end{aligned}$$

which by part (iii) of Lemma 4.8.1 (with ϕ replaced by its absolute value) and [42, Corollary 10.5.8] can, for all T large enough, be bounded by

$$c_4 \int_{|z|\geq K} |z|^{-\alpha}|z|^{\delta} dz, \qquad (4.8.39)$$

with c_4 being a constant independent of T and $\delta > 0$ can be arbitrarily small. This establishes (4.8.35) and finishes the proof of the lemma.

4.9 Proof of Theorem 4.6.2

4.9.1 Outline of the proof

Let F_T, C_T and D_T be as in the formulation of Theorem 4.6.2. Let us define

$$R_t^T(x) := \int_{\mathbb{R}} \phi(y) L_{Tt}^{\beta}(x + T^{-1/\beta}y)dy = \int_0^{Tt} \phi(\eta_s - T^{1/\beta}x)\,\mathrm{d}s, \qquad (4.9.1)$$

for $x \in \mathbb{R}$, $t \geq 0$ and $T > 0$. Let $a_1, \ldots, a_m \in \mathbb{R}$ and $t_1, \ldots, t_m \geq 0$ for some $m \geq 1$. Then, using (4.8.1) with $C_T \equiv 1$ and after the change of variables similar to the one in (4.8.3)

$$\mathbb{E}\exp\big(\sum_{j=1}^m a_j G_{t_j}^T\big) = exp\Bigg(\int_{\mathbb{R}^2} \mathbb{E}\Big(e^{iT^{\frac{\beta-1}{2\beta}} z\sum_{j=1}^m a_j R_{t_j}^T(x)}$$

$$-i\mathbf{1}_{\{|z|\leq 1\}} T^{\frac{\beta-1}{2\beta}} z \sum_{j=1}^m a_j R_{t_j}^T(x) - 1\Big) \frac{l(T^{1/\alpha\beta} z)}{l(T^{1/\alpha\beta})} |z|^{-1-\alpha} dz dx\Bigg). \quad (4.9.2)$$

To show convergence of finite-dimensional distributions we are going to show that for any $a_1, \ldots, a_m \in \mathbb{R}$ and $0 \leq t_1 \leq \ldots \leq t_m < \infty$ with $m \in \mathbb{N}$

$$\lim_{T\to\infty} \mathbb{E}\exp\big(\sum_{j=1}^m a_j G_{t_j}^T\big) = exp\left(-c_1(\alpha)\int_{\mathbb{R}} \mathbb{E}\Big|c_2(\phi,\beta)\sum_{j=1}^m a_j W(L_{t_j}^\beta(x))\Big|^\alpha dx\right), \quad (4.9.3)$$

where $c_2(\phi, \beta)$ is the constant in (4.6.6), $c_1(\alpha)$ is the constant in (4.11.11), $(L_t(x))$ is the local time of a symmetric standard β-stable Lévy process at x and W is a standard one-dimensional Brownian motion independent of $(L_t(x))$. By Proposition 4.5.2 for $x, z \in \mathbb{R}$ fixed the quantity under the integral in (4.9.2) converges to

$$\mathbb{E}\left(e^{iz\sum_{j=1}^m a_j W(L_{t_j}(x))} - i\mathbf{1}_{\{|z|\leq 1\}} z \sum_{j=1}^m a_j W(L_{t_j}^\beta(x)) - 1\right)|z|^{-1-\alpha} \quad (4.9.4)$$

and $\frac{l(T^{1/\alpha\beta}z)}{l(T^{1/\alpha\beta})}$ converges to 1 since l is slowly varying at infinity and symmetric. Thus, analogously as in the proof of Theorem 4.6.1 one must only justify going to the limit under the integral sign. This is made possible by the lemmas below which we prove in the following subsections.

Lemma 4.9.1. *Assume that the conditions of Theorem 4.6.2 are satisfied. Then for every $T \geq 1$ and $\phi \in L^1(\mathbb{R})$ we have*

$$I_1^T := T^{\frac{\beta-1}{2\beta}} \int_{\mathbb{R}} \mathbb{E}|R_t^T(x)| dx < \infty, \quad (4.9.5)$$

and

$$I_2^T := T^{\frac{\beta-1}{\beta}} \int_{\mathbb{R}} \mathbb{E}|R_t^T(x)|^2 dx < \infty. \quad (4.9.6)$$

If, in addition, we assume that $\int_{\mathbb{R}} |\phi(y)||y|^{\frac{\beta-1}{2}} dy < \infty$, then

$$\sup_{T\geq 1} I_1^T + \sup_{T\geq 1} I_2^T < \infty. \quad (4.9.7)$$

Lemma 4.9.2. *Let $t \geq 0$ and assume that an integrable function ϕ satisfies the assumptions in the statement of Theorem 4.6.2. Then, for any $\delta > 0$ there exists $K_0 > 0$ and $T_0 = T_0(K_0)$ such that for all $T \geq T_0, K \geq K_0$ we have*

$$\int_{\{|x|>K\}} T^{\frac{\beta-1}{2\beta}} \mathbb{E}\big|R_t^T(x)\big|dx < \delta. \tag{4.9.8}$$

Moreover, using Hölder inequality, one can easily show that there also holds an inequality analogous to (4.9.8) *when we replace* $T^{\frac{\beta-1}{2\beta}} \mathbb{E}\big|R_t^T(x)\big|$ *by* $T^{\frac{\beta-1}{\beta}} \mathbb{E}\left(R_t^T(x)^2\right)$.

4.9.2 Proof of Lemma 4.9.1

It is not hard to see, using Lemma 4.11.3 and (4.11.4) from the Section 4.11.1 in the appendix, that for any $z, x \in \mathbb{R}$

$$\begin{aligned} \mathbb{E}\big(L_t(x+z) - L_t(x)\big)^2 &\leq \big(\mathbb{E}L_t(x+z) + \mathbb{E}L_t(x)\big)\big(c_3(t,\beta) \wedge c_4(t,\beta)|z|^{\beta-1}\big) \\ &\leq c_3(t,\beta)\Big(1 \wedge |x+z|^{-\beta-1} + 1 \wedge |x|^{-\beta-1}\Big)\big(1 \wedge |z|^{\beta-1}\big) \end{aligned} \tag{4.9.9}$$

for some constants c_1, c_2, c_3 depending only on t and β. By Hölder inequality, (4.9.9) and the fact that $\int_{\mathbb{R}} \phi(y)dy = 0$ we then get

$$\begin{aligned} I_1^T &= T^{\frac{\beta-1}{2\beta}} \int_{\mathbb{R}} \mathbb{E}\Big| \int_{\mathbb{R}} \phi(y)\big(L_t(x+T^{-1/\beta}y) - L_t(x)\big)\Big|dx \\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} T^{\frac{\beta-1}{2\beta}} |\phi(y)| \mathbb{E}\Big|L_t(x+T^{-1/\beta}y) - L_t(x)\Big|dydx \\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} T^{\frac{\beta-1}{2\beta}} |\phi(y)| \Big(\mathbb{E}\Big|L_t(x+T^{-1/\beta}y) - L_t(x)\Big|^2\Big)^{\frac{1}{2}} dydx \\ &\leq c_3(t,\beta) \int_{\mathbb{R}}\int_{\mathbb{R}} T^{\frac{\beta-1}{2\beta}} |\phi(y)| \Big(\big(1 \wedge |x+yT^{-1/\beta}|^{-\beta-1} + 1 \wedge |x|^{-\beta-1}\big) \\ &\quad 1 \wedge |T^{-1/\beta}y|^{\beta-1}\Big)^{\frac{1}{2}} dxdy, \\ &= c_3(t,\beta) \int_{\mathbb{R}}\int_{\mathbb{R}} |\phi(y)| \Big(\big(1 \wedge |x+yT^{-1/\beta}|^{-\beta-1} + 1 \wedge |x|^{-\beta-1}\big) \\ &\quad T^{\frac{\beta-1}{\beta}} \wedge |y|^{\beta-1}\Big)^{\frac{1}{2}} dxdy \end{aligned}$$

Thus, I_1^T is finite since $\beta + 1 > 2$. Note that it is bounded uniformly in $T \geq 1$ for $\phi \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} |\phi(y)||y|^{\frac{\beta-1}{2}} dy < \infty$.

Using very similar manipulations we have

$$\begin{aligned} I_2^T &\leq T^{\frac{\beta-1}{\beta}} \int_{\mathbb{R}} \int_{\mathbb{R}^2} |\phi(y_1)||\phi(y_2)| \Big(\mathbb{E}\big(L_t(x+T^{-1/\beta}y_1) - L_t(x)\big)^2\Big)^{\frac{1}{2}} \\ &\quad \Big(\mathbb{E}\big(L_t(x+T^{-1/\beta}y_2) - L_t(x)\big)^2\Big)^{\frac{1}{2}} dy_1 dy_2 dx \\ &= T^{\frac{\beta-1}{\beta}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\phi(y)| \Big(\mathbb{E}\big(L_t(x+T^{-1/\beta}y) - L_t(x)\big)^2\Big)^{\frac{1}{2}} dy \right)^2 dx \\ &\leq c_3(t,\beta) \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |\phi(y)| \Big(\big(1 + \wedge |x|^{-\beta-1}\big) c_4(T) \wedge |y|^{\beta-1}\Big)^{\frac{1}{2}} dy \right)^2 dx \end{aligned}$$

for some finite constant $c_4(T)$ finite. Again, if $\int_{\mathbb{R}} |\phi(y)||y|^{\frac{\beta-1}{2}} dy < \infty$, then $\sup_{T \geq 1} I_2^T < \infty$.

4.9.3 Proof of Lemma 4.9.2

Choose K_0 so that

$$\int_{\{|x|>K_0\}} c_2(t,\beta) \int_{\mathbb{R}} |\phi(y)||y|^{\frac{\beta-1}{2}} dy dx < \frac{\delta}{2}, \tag{4.9.10}$$

where $c_2(t,\beta)$ is the same as in (4.9.9). Using Hölder inequality and inequality (4.9.9) we have

$$\begin{aligned} &\int_{\{|x|>K\}} T^{\frac{\beta-1}{2\beta}} \mathbb{E}\big|R_t^T(x)\big| dx \leq \\ &\quad \leq \int_{\{|x|>K\}} c_3(t,\beta) \int_{\mathbb{R}} \big(\mathbb{E}L_t(x) + \mathbb{E}L_t(x+T^{-1/\beta})\big)^{\frac{1}{2}}) |\phi(y)||y|^{\frac{\beta-1}{2}} dy dx. \end{aligned} \tag{4.9.11}$$

Using $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for $x, y \geq 0$, the above can be bounded by $A + B$, with

$$\begin{aligned} A &= \int_{\{|x|>K_0\}} 2c_3(t,\beta) \int_{\mathbb{R}} \big(\mathbb{E}L_t(x)\big)^{\frac{1}{2}} |\phi(y)||y|^{\frac{\beta-1}{2}} dy dx, \\ B &= \int_{\{|x|>K_0\}} 2c_3(t,\beta) \int_{\mathbb{R}} \big(\mathbb{E}L_t(x+T^{-1/\beta}y)\big)^{\frac{1}{2}} |\phi(y)||y|^{\frac{\beta-1}{2}} dy dx. \end{aligned}$$

B can be rewritten as

$$2c_3(t,\beta) \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbf{1}_{\{|x-T^{-1/\beta}y|>K\}} |\phi(y)||y|^{\frac{\beta-1}{2}} \big(\mathbb{E}L_t(x)\big)^{\frac{1}{2}} dy dx, \tag{4.9.12}$$

which, by dominated convergence, converges to

$$2c_3(t,\beta)\int_{\{|x|>K_0\}}\int_{\mathbb{R}}|\phi(y)||y|^{\frac{\beta-1}{2}}\left(\mathbb{E}L_t(x)\right)^{\frac{1}{2}}dydx, \tag{4.9.13}$$

as $T\to\infty$. Choosing T_0 sufficiently large, we get the required inequality for all $K\geq K_0$ in view of Lemma 4.11.3 in Section 4.11.1 of the appendix.

The second part of the lemma is proved in a very similar manner and we omit the proof.

4.10 Proof of Theorem 4.6.4

4.10.1 Simple case

To gain some intuition first, let us concentrate on a very concrete choice of ϕ to show what happens when ϕ vanishes relatively slowly at infinity. We will then extend our discussion to the case of functions regularly varying at infinity.

Suppose that

$$\phi(y):=|y|^{-\gamma}\mathbf{1}_{\{y\geq 1\}}-|y|^{-\gamma}\mathbf{1}_{\{y\leq -1\}}, \tag{4.10.1}$$

for $1<\gamma<1+\frac{\beta-1}{2}$. Recall that R^T is defined by (4.9.1). After a change of variables we get

$$R_t^T(x)=T^{\frac{1-\gamma}{\beta}}\int_{T^{-\frac{1}{\beta}}}^{\infty}|y|^{-\gamma}\left(L_t(y+x)-L_t(-y+x)\right)dy. \tag{4.10.2}$$

Put

$$Z_t^{T,\gamma}(x):=\int_{T^{-\frac{1}{\beta}}}^{\infty}|y|^{-\gamma}\left(L_t(y+x)-L_t(-y+x)\right)dy. \tag{4.10.3}$$

For each $x\in\mathbb{R}$ and $t\geq 0$, almost surely, $\lim_{T\to\infty}Z_t^{T,\gamma}(x)=Z_t^{\gamma}(x)$, where

$$Z_t^{\gamma}(x):=\int_0^{\infty}|y|^{-\gamma}\left(L_t(y+x)-L_t(-y+x)\right)dy, \tag{4.10.4}$$

which follows from dominated convergence theorem and Lemma 4.10.1 below.

Lemma 4.10.1. *Let Z be given by* (4.10.4) *and* $1<\gamma<1+\frac{\beta-1}{2}$. *For $\alpha\in[1,2]$ and $t\geq 0$*

$$\int_{\mathbb{R}}\mathbb{E}\left(\int_0^{\infty}|y|^{-\gamma}\left|L_t(y+x)-L_t(-y+x)\right|dy\right)^{\alpha}dx<\infty. \tag{4.10.5}$$

Furthermore,

$$\int_{\mathbb{R}}\mathbb{E}\left(\int_0^{\infty}|y|^{-\gamma}\left|L_t(y+x)-L_t(x)\right|dy\right)^{\alpha}dx<\infty. \tag{4.10.6}$$

Proof of Lemma 4.10.1. Since the proofs of (4.10.5) and (4.10.6) are virtually identical, we will concentrate only on the former. We will show that (4.10.5) holds for $\alpha = 1$ and $\alpha = 2$ which will suffice to prove the lemma. Denote the corresponding integrals (4.10.5) by I_1 and I_2, respectively. Using Jensen's inequality we see that

$$I_1 \leq \int_{\mathbb{R}} \int_0^\infty |y|^{-\gamma} \left(\mathbb{E}\left(L_t(x+y) - L_t(x-y)\right)^2\right)^{\frac{1}{2}} dydx. \tag{4.10.7}$$

Observe that (by Lemma 4.11.1 in the Section 4.11.1)

$$\begin{aligned}\mathbb{E}\left|L_t(x+y) - L_t(x-y)\right|^2 &= 2\int_0^t \int_0^{t-u_1} \Big(p_{u_1}(x+y) + p_{u_1}(x-y)\Big) \\ &\quad \Big(p_{u_2}(0) - p_{u_2}(2y)\Big) du_2 du_1 \\ &\leq 2\Big(\mathbb{E}L_t(x+y) + \mathbb{E}L_t(x-y)\Big)(c_1 \wedge (c_2|y|^{\beta-1})),\end{aligned}$$

where the inequality follows from (4.11.10) and (4.11.4) for $n = 1$, for some constants c_1 and c_2 depending only on β and t. Therefore

$$\begin{aligned}I_1 &\leq \int_{\mathbb{R}} \int_0^\infty |y|^{-\gamma} \left(\Big(c_3 \wedge \big(c_4|x+y|^{-\frac{\beta+1}{2}}\big)\Big) + \Big(c_3 \wedge \big(c_4|x-y|^{-\frac{\beta+1}{2}}\big)\Big)\right) \\ &\quad c_1 \wedge \big(c_2|y|^{\frac{\beta-1}{2}}\big) dydx,\end{aligned}$$

which is finite since $\frac{1+\beta}{2} > 1$ and $1 < \gamma < 1 + \frac{\beta-1}{2}$. As for I_2, notice that by the Hölder inequality

$$\begin{aligned}I_2 &= \int_{\mathbb{R}} \int_0^\infty \int_0^\infty |y_1|^{-\gamma}|y_2|^{-\gamma}\mathbb{E}\Big(\big|L_t(x+y_1) - L_t(x-y_1)\big| \\ &\quad \big|L_t(x+y_2) - L_t(x-y_2)\big|\Big) dy_1 dy_2 dx \\ &\leq \int_{\mathbb{R}} \left(\int_0^\infty |y|^{-\gamma} \left(\mathbb{E}\left(L_t(x+y) - L_t(x-y)\right)^2\right)^{\frac{1}{2}} dy\right)^2 dx\end{aligned}$$

Seeing that, by (4.10.8), for all $x \in \mathbb{R}$

$$\int_0^\infty |y|^{-\gamma} \left(\mathbb{E}\left(L_t(x+y) - L_t(x-y)\right)^2\right)^{\frac{1}{2}} dy \leq \int_0^\infty |y|^{-\gamma} c_3 \wedge \big(c_4|y|^{\frac{\beta-1}{2}}\big) dy, \tag{4.10.8}$$

we conclude I_2 is finite. □

The process $(Z_t^\gamma(x))_{t\geq 0}$ is continuous and has a non-zero mean as long as $x \neq 0$. Using Hölder inequality it is easy to see that the process Z^γ has all moments finite. If we choose

$$F_T = T^{1+1/(\alpha\beta)-\gamma/\beta}, \tag{4.10.9}$$

then, we will see (in the more general setting of Theorem 4.6.4) that the finite dimensional distributions of the process $(G_t^T)_{t\geq 0}$ in (4.1.7) converge to the finite dimensional distributions of the process V given by (4.1.3). The proof resembles closely the proof of Theorem 4.6.2

Now we may prove Proposition 4.6.5.

Proof of Proposition 4.6.5. We will only providfe te proof for V the one for $\widetilde{V}$ is virtually identical. Let $a_1, \ldots, a_m \in \mathbb{R}$ and $0 \leq t_1 \leq \ldots \leq t_m < \infty$. Take $c > 0$ and notice that using Remark 4.11.5 in the Appendix

$$\begin{aligned}
\mathbb{E}\Big(exp\Big(i\sum_{j=1}^m a_j V_{t_j}\Big)\Big) &= exp(-\int_{\mathbb{R}} \mathbb{E}\Big|\sum_{j=1}^m a_j Z_{ct_j}(x)\Big|^\alpha dx) \\
&= exp\Big(-c^{1/\beta}\int_{\mathbb{R}} \mathbb{E}\Big|\sum_{j=1}^m a_j Z_{ct_j}(c^{1/\beta}x)\Big|^\alpha dx\Big) \\
&= exp\Big(-c^{1/\beta}\int_{\mathbb{R}} \mathbb{E}\Big|\sum_{j=1}^m a_j \int_0^\infty |y|^{-\gamma}\Big(L_{ct_j}(c^{1/\beta}x+y) \\
&\qquad -L_{ct_j}(c^{1/\beta}x-y)\Big)\Big|^\alpha dydx\Big) \\
&= exp\Big(-\mathbb{E}\Big|\sum_{j=1}^m a_j \int_0^\infty |y|^{-\gamma}\Big(L_{ct_j}(c^{1/\beta}x+c^{1/\beta}y) \\
&\qquad -L_{ct_j}(c^{1/\beta}x-c^{1/\beta}y)\Big)\Big|^\alpha dxc^{1/\beta}c^{-(\gamma\alpha)/\beta}c^{\alpha/\beta}\Big) \\
&= \mathbb{E}\left(exp(ic^H\sum_{j=1}^m a_j V_{t_j})\right)
\end{aligned}$$

where the last inequality follows from (4.11.5). Hence V is self-similar with Hurst coefficient $H = 1 + \frac{1}{\alpha\beta} - \frac{\gamma}{\beta}$. The stationarity of increments follows immediately once we notice that $L_{t+s}(z) := L_{t+s}^0(z) = L_s(z) + L_t^{\xi_s}(z)$ for $s, t \geq 0$ and $z \in \mathbb{R}$ and use Fubini theorem. □

4.10.2 The main body of proof

Proof of Theorem 4.6.4 in the case $\gamma_1 < \gamma_2$. Recall that by assumption

$$\phi(y) = \mathbf{1}_{\{y<0\}}|y|^{-\gamma_2}g_2(y) + \mathbf{1}_{\{y>0\}}|y|^{-\gamma_1}g_1(y),$$

with g_1 and g_2 being eventually positive. We can always find some positive constants K_1, K_2 such that $g_1(y) > 0$ for $y \geq K_1$, $g_2(y) > 0$ for $y \leq -K_2$ and

$$\int_{[-K_2,K_1]} \phi(y)dy = 0. \tag{4.10.10}$$

Thus, we may write $\phi = \phi_a + \phi_b$ with

$$\phi_a = \phi \mathbf{1}_{[-K_2, K_1]}(\cdot)$$

and $\phi_b := \phi - \phi_a$. If we correspondingly split the functional G^T in (4.1.7) into parts corresponding to ϕ_a and ϕ_b, respectively, then the part $G^{T,a}$ corresponding to ϕ_a satisfies the assumptions of Theorem 4.6.2 and comparing the normalizations

$$F_T = g_1(T^{1/\beta}) T^{1+1/(\alpha\beta)-\gamma/\beta} \tag{4.10.11}$$

and the normalization F_T in the statement of Theorem 4.6.2, we conclude that under normalization (4.10.11) the finite-dimensional distributions of $G^{T,a}$ converge weakly, and hence in probability, to 0. All this means that, without loss of generality we may assume that $\phi = \phi_b$.

Take any $a_1, \dots, a_m \in \mathbb{R}$ and $t_1, \dots, t_m \geq 0$. For G^T as in (4.1.7) we have (after a change of variables, using symmetry and the fact that (x_j, z_j) are points of a Poisson random measure)

$$\begin{aligned} &\mathbb{E} exp\Big(i \sum_{j=1}^m a_j G_{t_j}^T\Big) \\ &= exp\Bigg(\int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}\Big(e^{izM^T(x)} - 1 - i\mathbf{1}_{\{|z|\leq 1\}} z M^T(x)\Big) \mathbf{1}_{\{|z|\geq T^{-1/\alpha\beta}\}} |z|^{-1-\alpha} dz dx \Bigg), \end{aligned} \tag{4.10.12}$$

where

$$\begin{aligned} M^T(x) = \sum_{j=1}^m a_j \Big(& g_1(T^{1/\beta})^{-1} \int_0^\infty g_1(T^{1/\beta} y) |y|^{-\gamma_1} \big(L_{t_j}(x+y) - L_{t_j}(x)\big) dy \\ & - T^{\frac{-\gamma_2+\gamma_1}{\beta}} g_1(T^{1/\beta})^{-1} \int_0^\infty |y|^{-\gamma_2} g_2(T^{1/\beta} y) \big(L_{t_j}(x-y) - L_{t_j}(x)\big) dy \Big). \end{aligned} \tag{4.10.13}$$

Denote

$$f_T(z,x) = \big(e^{izM^T(x)} - 1 - i\mathbf{1}_{\{|z|\leq 1\}} z M^T(x)\big) |z|^{-1-\alpha} \mathbf{1}_{\{|z|\geq T^{-1/\alpha\beta}\}}. \tag{4.10.14}$$

In order to prove the theorem we have to show that

$$\lim_{T\to\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{E}\big(f_T(z,x)\big) dz dx = \int_{\mathbb{R}} \mathbb{E}\Big| \sum_{j}^m a_j Z_{t_j}(x) \Big|^\alpha dx, \tag{4.10.15}$$

where

$$Z_t(x) = \int_0^\infty |y|^{-\gamma_1} \big(L_t(x+y) - L_t(x)\big) \quad t \geq 0. \tag{4.10.16}$$

We can write $M^T(x) = \sum_{j=1}^m a_j\big(M^{T,1}_{t_j}(x) - M^{T,2}_{t_j}(x)\big)$ with

$$M^{T,1}_t(x) = \int_{T^{-1/\beta}K_1}^{\infty} \frac{g(T^{1/\beta}y)}{g(T^{1/\beta})}|y|^{-\gamma_1}\big(L_t(x+y) - L_t(x)\big)dy, \tag{4.10.17}$$

$$M^{T,2}_t(x) = T^{\frac{-\gamma_2+\gamma_1}{\beta}} g_1(T^{1/\beta})^{-1}\int_{T^{-1/\beta}K_2}^{\infty} |y|^{-\gamma_2} g_2(T^{1/\beta}y) \times\big(L_t(x-y) - L_t(x)\big)dy. \tag{4.10.18}$$

We may only consider $M^{T,1}$, since dealing with $M^{T,2}$ is very similar because of symmetry of the local time. In fact $M^{T,2}$ becomes of smaller order than $M^{T,1}$ as $T \to \infty$ since $\gamma_1 < \gamma_2$. Moreover, we will assume that $m = 1$, $a_1 = 1$ and $t_1 = t$. Notice that

$$f_T(z,x) \le \mathbf{1}_{\{|z|\le 1\}} z^{1-\alpha}|M_T(x)|^2 + \mathbf{1}_{\{|z|>1\}}|z|^{-\alpha}|M_T(x)| \tag{4.10.19}$$

The following lemma will be instrumental to showing (4.10.15).

Lemma 4.10.2. *For any $t \ge 0$ there exists $T_0 > 0$ such that*

$$\sup_{T\ge T_0}\int_{\mathbb{R}} \mathbb{E}\big(M^{T,1}_t(x)\big)^2 dx + \sup_{T\ge T_0}\int_{\mathbb{R}} \mathbb{E}\big|M^{T,1}_t(x)|dx < \infty. \tag{4.10.20}$$

Proof of Lemma 4.10.2. By the Hölder inequality, inequality (4.9.9) and the fact that $(a+b)^2 \le 2a^2 + 2b^2$, respectively we have

$$\begin{aligned}\mathbb{E}(M^{T,1}(x))^2 &\le \left(\int_0^{\infty}\left|\frac{g_1(T^{1/\beta}y)}{g_1(T^{1/\beta})}\right||y|^{-\gamma_1}\Big(\mathbb{E}\big(L_t(x+y) - L_t(x)\big)^2\Big)^{1/2}dy\right)^2\\ &\le c_1\big(B_1^T(x) + B_2^T(x)\big).\end{aligned} \tag{4.10.21}$$

where c_1 is some constant independent of T and

$$B_1^T(x) = \left(\int_0^{\infty}\left|\frac{g_1(T^{1/\beta}y)}{g_1(T^{1/\beta})}\right||y|^{-\gamma_1}\Big((1\wedge|x+y|^{-\beta-1})(1\wedge|y|^{\beta-1})\Big)^{\frac{1}{2}}dy\right)^2 \tag{4.10.22}$$

$$B_2^T(x) = \left(\int_0^{\infty}\left|\frac{g_1(T^{1/\beta}y)}{g_1(T^{1/\beta})}\right||y|^{-\gamma_1}\Big((1\wedge|x|^{-\beta-1})(1\wedge|y|^{\beta-1})\Big)^{\frac{1}{2}}dy\right)^2 . \tag{4.10.23}$$

Note that in both cases the quantities under the square power sign on the right-hand sides of both (4.10.22) and (4.10.23) are bounded (uniformly in $x \in \mathbb{R}$) by

$$A^T := \int_0^{\infty}\left|\frac{g_1(T^{1/\beta}y)}{g_1(T^{1/\beta})}\right||y|^{-\gamma_1}\big(1\wedge|y|^{\frac{\beta-1}{2}}\big)dy. \tag{4.10.24}$$

The integral in A^T may be split onto two integrals over $(0,1)$ and $(1,\infty)$, which we will be denoting A_1^T and A_2^T, respectively. In A_1^T we make a change of variables to see that

$$A_1^T = T^{\frac{2\gamma_1-\beta-1}{2\beta}} g_1(T^{1/\beta})^{-1} \int_0^{T^{1/\beta}} |g_1(y)||y|^{-\gamma_1+\frac{\beta-1}{2}} dy, \tag{4.10.25}$$

which converges to a constant by Theorem 2.8.4. Furthermore, using Theorem 2.8.6, one gets

$$A_2^T = \int_1^\infty \left| \frac{g_1(T^{1/\beta} y)}{g_1(T^{1/\beta})} \right| |y|^{-\gamma_1} dy \tag{4.10.26}$$

$$\leq c_2 \int_1^\infty |y|^{-\gamma_1+\epsilon} dy \tag{4.10.27}$$

for some ϵ such that $\gamma_1 - \epsilon > 1$ for all T sufficiently large and some constant c_2 independent of T. Therefore, we may conclude that there is some finite T_0 such that

$$\sup_{T \geq T_0} (A_1^T + A_2^T) < \infty. \tag{4.10.28}$$

Therefore, for some constant c_2, independent of T, we have

$$B_1^T(x) \leq c_2 \int_0^\infty \left| \frac{g_1(T^{1/\beta} y)}{g_1(T^{1/\beta})} \right| |y|^{-\gamma_1} \Big((1 \wedge |x+y|^{-\beta-1})(1 \wedge |y|^{\beta-1}) \Big)^{\frac{1}{2}} dy$$

$$B_2^T(x) \leq c_2 \int_0^\infty \left| \frac{g_1(T^{1/\beta} y)}{g_1(T^{1/\beta})} \right| |y|^{-\gamma_1} \Big((1 \wedge |x|^{-\beta-1})(1 \wedge |y|^{\beta-1}) \Big)^{\frac{1}{2}} dy,$$

so that

$$\max \left(\int_{\mathbb{R}} B_1^T(x) dx, \int_{\mathbb{R}} B_2^T(x) dx \right) \leq (A^T)^2 \int_{\mathbb{R}} \big(1 \wedge |x|^{-\frac{1+\beta}{2}}\big) dx. \tag{4.10.29}$$

The proof that $\sup_{T \geq T_0} \int_{\mathbb{R}} \mathbb{E}\big|M_t^{T,1}(x)| dx < \infty$ is virtually identical and we skip it. □

Notice that the quantity under the integral sign in (4.10.13) converges pointwise to

$$\sum_{j=1}^m a_j \int_0^\infty |y|^{-\gamma_1} \big(L_{t_j}(x+y) - L_{t_j}(x) \big) dy.$$

This and Lemma 4.11.7 in the Appendix, leads us to expect that the limit (up to a multiplicative constant) of (4.10.12) is given by the characteristic function of the corresponding finite-dimensional distribution of (4.1.5). By Lemma 4.10.2 we may forget about the term $\mathbf{1}_{\{|z| \geq T^{-1/\alpha\beta}\}}$ in (4.10.12). We would now like to show (assuming $M^T(x) = M_t^{T,1}$) that (4.10.15) holds. However, we have to justify going with the limit under the integral. In order to do that we fix some $r > 0$ and split $M_t^{T,1}$ as follows:

$$M_t^{T,1,r_-}(x) = \int_0^r \frac{g(T^{1/\beta}y)}{g(T^{1/\beta})}|y|^{-\gamma_1}\big(L_t(x+y) - L_t(x)\big)dy, \tag{4.10.30}$$

and

$$M_t^{T,1,r_+}(x) = \int_r^\infty \frac{g(T^{1/\beta}y)}{g(T^{1/\beta})}|y|^{-\gamma_1}\big(L_t(x+y) - L_t(x)\big)dy. \tag{4.10.31}$$

By Theorem 2.8.6, for r fixed and any $\epsilon > 0$ there exists some T_0 finite such that for all $T \geq T_0$ the integrand in (4.10.31) may be bounded by

$$c_3|y|^{-\gamma_1+\epsilon}\big|L_t(x+y) - L_t(x)\big|dy$$

for some finite constant c_3. In view of inequality (4.10.19) we conclude that by dominated convergence we may go with the limit under the integral in (4.10.15) if we substitute M^T with $M_t^{T,1,r_+}$. It remains to prove that for any $\epsilon > 0$ there exist some $T_0, r > 0$ such that

$$\sup_{T\geq T_0}\int_{\mathbb{R}} \mathbb{E}\big(M^{T,1,r_-}(x)\big)^2 dx + \sup_{T\geq T_0}\int_{\mathbb{R}} \mathbb{E}\big|M^{T,1,r_-}(x)|dx < \epsilon. \tag{4.10.32}$$

Similarly as in the proof of Lemma 4.10.2, to show that (4.10.32) holds it suffices to prove that (with no loss of generality we assume here that $r \in (0,1)$) for any $\epsilon > 0$ there exist some $T_0, r > 0$ such that

$$\sup_{T\geq T_0} A^{T,r} < \epsilon, \tag{4.10.33}$$

where

$$A^{T,r} = \int_0^r \left|\frac{g_1(T^{1/\beta}y)}{g_1(T^{1/\beta})}\right||y|^{-\gamma_1+\frac{\beta-1}{2}}dy \tag{4.10.34}$$

$$= \frac{1}{h(T^{1/\beta})}\int_0^{T^{1/\beta}r} g_1(y)|y|^{-\gamma_1+\frac{\beta-1}{2}}dy, \tag{4.10.35}$$

with

$$h(w) = g_1(w)w^{\frac{\beta+1-2\gamma_1}{2}}.$$

By Proposition 2.8.4, for any $r > 0$ fixed

$$\frac{1}{h(T^{1/\beta}r)}\int_0^{T^{1/\beta}r} g_1(y)|y|^{-\gamma_1+\frac{\beta-1}{2}}dy,$$

converges to a finite constant as $T \to \infty$. Therefore, $A^{T,r}$ may be written as a product of two terms, on of which is bounded uniformly for all T sufficiently large and the other equal to

$$\frac{h(T^{1/\beta}r)}{h(T^{1/\beta})} = \frac{g_1(T^{1/\beta}r)}{g_1(T^{1/\beta})}r^{\frac{\beta+1-2\gamma_1}{2}}.$$

Since g_1 is slowly varying at infinity, the right-hand side of the above can be made arbitrarily small for all $T \geq T_0$ by choosing r sufficiently small and then T_0 sufficiently large. This proves (4.10.33) and finishes the proof of the whole theorem.

□

4.11 Appendices

4.11.1 Properties of stable local limes

By $p_u(x)$ we denote the transition density of a symmetric β-stable Lévy process. In the whole appendix we assume $\beta \in (1,2)$. We use the fact that for any $u > 0$ and $y \in \mathbb{R}$, $p_u(y) \leq p_u(0)$. The transition density satsfies the following scaling property:

$$p_u(y) = u^{-1/\beta} p_1(u^{-1/\beta} y), \quad u > 0, \quad y \in \mathbb{R}. \tag{4.11.1}$$

We use $\Pi(n)$ to denote the set of permutations of the set $\{1, \ldots, n\}$ for $n \in \mathbb{N}$. Finally, we set $\Delta^n_T = \{0 \leq u_1 \leq \ldots u_n < \infty\}$.

The most important facts are given by the following lemma.

Lemma 4.11.1. *Let $(L^\beta_t(x))_{t\geq 0}$ be a local time at $x \in \mathbb{R}$ of a symmetric β-stable process (denoted by ξ) with $\beta \in (1,2)$. Then for any $n \in \mathbb{N}$ and $t > 0$*

$$\mathbb{E}\left(L^\beta_t(x)\right)^n = n! \frac{1}{(2\pi)^n} \int_0^t \cdots \int_{u_{n-1}}^t \int_{\mathbb{R}^n} e^{ixz_1} e^{-(u_n - u_{n-1})|z_n|^\beta} e^{-(u_{n-1}-u_{n-2})|z_{n-1}|^\beta} \cdots e^{-u_1|z_1|^\beta} \, \mathrm{d}z_1 \ldots \mathrm{d}z_n \, \mathrm{d}u_1 \ldots \mathrm{d}u_n. \tag{4.11.2}$$

We also have

$$\mathbb{E}\left(L^\beta_t(x)\right)^n = n! \int_0^t \int_{u_1}^t \cdots \int_{u_{n-1}}^t p_{u_n - u_{n-1}}(0) \ldots p_{u_2-u_1}(0) p_{u_1}(x) \, \mathrm{d}u_n \ldots \mathrm{d}u_1, \tag{4.11.3}$$

and

$$\mathbb{E} L^\beta_t(x_1) \ldots L^\beta_t(x_n) = \sum_{\pi \in \Pi(n)} \int_{\Delta^n_T} p_{u_n - u_{n-1}}(x_{\pi_n} - x_{\pi_{n-1}}) \ldots p_{u_2-u_1}(x_{\pi_2} - x_{\pi_2}) p_{u_1}(x_{\pi_1}) \, \mathrm{d}u_n \ldots \mathrm{d}u_1. \tag{4.11.4}$$

The proof is very similar to the proof of [41, Lemma 1] and we skip it.

Corollary 4.11.2. *From* (4.11.3) *and the fact that* $p_u(x) \leq p_u(0)$ *for* $x \in \mathbb{R}, u > 0$*, it follows that for every* $n \in \mathbb{N}$ *the function*

$$x \mapsto \mathbb{E}\big(L_t^\beta(x)\big)^n$$

is bounded by $\mathbb{E}L_t^\beta(0)$.

We will need a lemma about the asymptotic behavior of $\mathbb{E}L_t^\beta(x)$ as $|x| \to \infty$. The proof is straightforward so we skip it.

Lemma 4.11.3. *For any* $t > 0$ *there exists a constant* C *depending only on* t *and* β *such that*

$$\mathbb{E}L_t^\beta(x) \leq C\big(1 \wedge |x|^{-\beta-1}\big).$$

Proof. By definition $\mathbb{E}L_t^\beta(x)$ is bounded by t for all $x \in \mathbb{R}$. Recall that for all sufficiently large x (in absolute value terms) we have

$$p_1(z) \leq c_1\big(1 \wedge |z|^{-1-\beta}\big),$$

for some constant c_1 (see [33, Theorem 1.12]). Thus, we may write (using the scaling property of the symmetric stable distribution)

$$\begin{aligned}
\mathbb{E}L_t^\beta(x) &= \int_0^t p_u(x)du \\
&= |x|^{\beta-1}\int_0^{t|x|^{-\beta}} p_1(u^{-1/\beta})u^{-1/\beta}du \\
&\leq c_1|x|^{\beta-1}\int_0^{t|x|^{-\beta}} u^{\frac{1}{\beta}+1}u^{-\frac{1}{\beta}}du \\
&= \frac{c_1}{2}t^2|x|^{-\beta-1}.
\end{aligned}$$

□

Lemma 4.11.4, follows easily from Corollary 4.11.2.

Lemma 4.11.4. *The following hold for any* $t > 0$*:*

(i) for any positive $p > 0$

$$\mathbb{E}|L_t^\beta(x)|^p < \infty, \tag{4.11.5}$$

(ii)

$$\mathbb{E}|L_t^\beta(x_1)\dots L_t^\beta(x_m)| < \infty \tag{4.11.6}$$

uniformly in $x_1, \dots, x_m \in \mathbb{R}$,

(iii) for any $p \in [1, \infty)$ *we have*

$$\int_{\mathbb{R}} \mathbb{E}\left|L_t^\beta(x)\right|^p \, \mathrm{d}x < \infty, \tag{4.11.7}$$

uniformly in $x \in \mathbb{R}$.

Proof. (i) and (ii) are easy consequences of Hölder inequality and Corollary 4.11.2. (iii) follows from the fact that $\sup_{x\in\mathbb{R}} L_t^\beta(x) \le t$ almost surely and that

$$\int_{\mathbb{R}} \mathbb{E}L_t^\beta(x)dx = t.$$

□

Remark 4.11.5. *For any* $a > 0$ *and* $z \in \mathbb{R}$ *the process* $(L_{ct}^\beta(z))_{t\ge0}$ *has the same law as* $(c^{1-1/\beta}L_t^\beta(\frac{z}{c^{1/\beta}}))_{t\ge0}$ *(see [42, Proposition 10.4.8]).*

We also have the following lemma.

Lemma 4.11.6. *For* $x \in \mathbb{R}$, $x \neq 0$*:*

$$\int_0^t \left(p_u(0) - p_u(x)\right) \mathrm{d}u = |x|^{\beta-1} \int_0^{t|x|^{-\beta}} \Big(p_1(0) - p_1\big(\frac{1}{u^{1/\beta}}\big)\Big) \frac{1}{u^{1/\beta}} \, \mathrm{d}u. \tag{4.11.8}$$

Here p *is the* β*-stable transition density. Putting*

$$c = \int_0^\infty \Big(p_1(0) - p_1\big(\frac{1}{u^{1/\beta}}\big)\Big) \frac{1}{u^{1/\beta}} \, \mathrm{d}u, \tag{4.11.9}$$

which is finite (see [41]), and noticing that the first integral in (4.11.8) *is bounded by a constant* c_1 *depending only on* t *and* β*, we conclude that*

$$\int_0^t \left(p_u(0) - p_u(x)\right) \mathrm{d}u \le c_1 \wedge c|x|^{\beta-1}, \quad x \neq 0. \tag{4.11.10}$$

Lemma 4.11.7. *For* $\alpha \in (0, 2)$, $x \in \mathbb{R}$ *and any* $R > 0$ *we have (see [13, equation (2.42)]*

$$c(\alpha)|x|^\alpha = \int_{\mathbb{R}} \left(1 - e^{ixu} + ixu\mathbf{1}_{\{|u|\le R\}}\right) \frac{\mathrm{d}u}{|u|^{1+\alpha}}, \tag{4.11.11}$$

where $c(\alpha)$ *is a constant independent of* x *and* R.

4.11.2 Technical results related to regular variation

This section starts with a few technical results and consequences of the Assumption (**A**) in Section 4.4 which will be needed to establish Propositions 4.5.1, 4.5.2 and Theorem 4.6.1 in full generality. Notice that if in the Assumption (**A**) the process η is a symmetric

Throughout this section we let

$$D_T := Tf(T^{1/\beta})^{-1}, \tag{4.11.12}$$

$$\psi_T(z) := Tf(T^{1/\beta})^{-1}\psi\big(\frac{z}{T^{1/\beta}}\big), \tag{4.11.13}$$

for $T \geq 1$ and $z \neq 0$. If not stated otherwise, we always assume that $\int_{\mathbb{R}} |\phi(y)|dy = 1$, which implies that $|\widehat{\phi}|$ is bounded by 1. Note that if $\psi(z) = |z|^\beta$, then $\psi_T(z) = |z|^\beta$ for all $T > 0$ and this means that all of the following lemmas become trivial in this case.

Lemma 4.11.8. *The characteristic exponent from Assumption (**A**) satisfies*

$$\lim_{T\to\infty} \psi_T(z) = |z|^\beta, \tag{4.11.14}$$

for $z \neq 0$.

Proof. After a change of variables we can write

$$\psi_T(z) = c(\beta)^{-1}\int_{\mathbb{R}} \big(1 - e^{iuz} + iuz\mathbf{1}_{\{|u|\leq T^{1/\beta}\}}\big)\frac{f(T^{1/\beta}u)}{f(T^{1/\beta})}|u|^{-1-\beta}du, \tag{4.11.15}$$

with $c(\beta)$ as in Lemma 4.11.7. By the same lemma it only remains to justify going with the limit under the integral sign. Fix some $r \in (0,1)$ and write (using the symmetry of f) $\psi_T(z) = \psi_T^1(z) + \psi_T^2(z)$ with

$$\psi_T^1(z) = c(\beta)^{-1}\int_{|u|\leq r} \big(1 - e^{iuz} + iuz\big)\frac{f(T^{1/\beta}u)}{f(T^{1/\beta})}|u|^{-1-\beta}du, \tag{4.11.16}$$

$$\psi_T^2(z) = c(\beta)^{-1}\int_{|u|>r} \big(1 - e^{iuz} + \mathbf{1}_{\{|u|\leq 1\}}iuz\big)\frac{f(T^{1/\beta}u)}{f(T^{1/\beta})}|u|^{-1-\beta}du. \tag{4.11.17}$$

By [42, Theorem 10.5.6] and inequality $|1 - e^{iz} + iz| \leq 1/2|z|^2, z \in \mathbb{R}$, $\psi_T^1(z)$ can be bounded, for all T large enough, by $c_1(\beta)r^{2-\beta}$, where $c_1(\beta)$ is a finite constant depending only on α. On the other hand, by [42, Theorem 10.5.5 and Corollary 10.5.8], for any $\delta > 0$ there exists $T_0 \geq 1$ such that for all $T \geq T_0$ the integrand in $\psi_T^2(z)$ can be bounded by $c_2|u|^{-\beta}(1+\delta)|u|^\delta$. Thus, by dominated convergence, (4.11.17) converges to

$$c(\beta)^{-1}\int_{|u|>r} \big(1 - e^{iuz} + \mathbf{1}_{\{|u|\leq 1\}}iuz\big)|u|^{-1-\beta}du. \tag{4.11.18}$$

This and Lemma 4.11.7 shows that for any $\delta > 0$ and $z \neq 0$ $|\psi_T(z) - |z|^\beta| < \delta$ for all T large enough. □

It is easy to see that Lemma 4.11.8 implies the following.

Corollary 4.11.9. *We have*

$$\lim_{w \to 0} \frac{\psi(w)}{|w|^\beta f(1/w)} = c_1, \tag{4.11.19}$$

for some finite constant c_1. This in turn means that $\psi \in RV_0(\beta)$ and, since we can always write $\psi(z) = |z|^\beta L_0(z)$ with L_0 slowly varying at 0, we have

$$\lim_{T \to \infty} \frac{L_0(T^{-1})}{f(T)} = 1. \tag{4.11.20}$$

Moreover, for any $u \geq 0$ and $\theta \in \mathbb{R}$ we have

$$\lim_{T \to \infty} \mathbb{E}\Big(exp\big(i\theta T^{-1/\beta}\eta_{D_T u}\big)\Big) = e^{-u|\theta|^\beta}. \tag{4.11.21}$$

Lemma 4.11.10. *Let ψ be a Lévy exponent satisfying*

$$\int_1^\infty \psi(z)^{-1} dz < \infty, \tag{4.11.22}$$

and

$$\psi \in RV_0(\beta), \quad \beta \in (1,2). \tag{4.11.23}$$

Then there exists finite $C_0 > 0$ such that for any $K > 0$ there exists $T_0 \geq 1$ such that for any $T \geq T_0$ we have

$$\int_K^\infty \psi_T(z)^{-1} dz \leq C_0 K^{1-\beta}. \tag{4.11.24}$$

To prove the above lemma we will need the following consequence of [42, Theorem 10.5.6].

Lemma 4.11.11. *Let $h : (0,\infty) \to (0,\infty)$ be in $RV_0(\beta)$, with $\beta \in (1,2)$. Then the function*

$$w \mapsto \int_w^1 h(z)^{-1} dz, \quad w \in (0,1) \tag{4.11.25}$$

is in $RV_0(1-\beta)$ and

$$\lim_{w \to 0} \frac{\int_w^1 h(z)^{-1} dz}{w h(w)^{-1}} = \frac{1}{\beta - 1}. \tag{4.11.26}$$

Proof of Lemma 4.11.11. Changing variables we have

$$\int_w^1 h(z)^{-1}dz = \int_1^{\frac{1}{w}} h(1/z)^{-1}z^{-2}dz \tag{4.11.27}$$

and the function $z \mapsto h(1/z)^{-1}z^{-2}$ is in $RV_\infty(\beta-2)$, so by Theorem 2.8.7 the function $x \mapsto \int_1^x h(1/z)^{-1}z^{-2}$ is in $RV_\infty(\beta-1)$ and

$$\lim_{x\to\infty} \frac{\int_1^x h(1/z)^{-1}z^{-2}dz}{xh(1/x)^{-1}x^{-2}} = \frac{1}{\beta-1}, \tag{4.11.28}$$

which finishes the proof of the lemma. □

Proof of Lemma 4.11.10. Notice that we can write

$$\int_K^\infty \psi_T(z)^{-1}dz = A_T(K) + B_T(K), \tag{4.11.29}$$

where

$$A_T(K) = E_T \int_{KT^{-1/\beta}}^1 \psi(z)^{-1}dz, \tag{4.11.30}$$

$$B_T(K) = E_T \int_1^\infty \psi(z)^{-1}dz, \tag{4.11.31}$$

with $E_T = T^{1/\beta-1}f(T^{1/\beta})$. For T sufficiently large $B_T(K)$ can be made arbitrarily small, irrespective of the value of K. By Lemma 4.11.11

$$\lim_{T\to\infty} \frac{\int_{KT^{-1/\beta}}^1 \psi(z)^{-1}dz}{KT^{-1/\beta}\psi(KT^{-1/\beta})^{-1}} = \frac{1}{\beta-1}, \tag{4.11.32}$$

which means that for T sufficiently large

$$A_T(K) \le c(\beta,K)K^{1-\beta}\frac{f(T^{1/\beta})}{L_0(KT^{-1/\beta})}, \tag{4.11.33}$$

where the fraction on the right-hand side of (4.11.33) converges to 1 as $T \to \infty$ by Corollary 4.11.9 (where L_0 is also defined) and $c(\beta,K)$ is a finite positive constant independent of T. □

Lemma 4.11.12. *There exists $C_2(\psi,t) > 0$ such that for any $t > 0$*

$$\sup_{T\ge1} \int_0^t \int_{\mathbb{R}} e^{-u\psi_T(z)}dzdu \le C_2(\psi,t). \tag{4.11.34}$$

Proof. This is an easy consequence of Lemma 4.11.10. □

If we additionally assume (**C**) then we can rephrase Lemma 4.11.12 to obtain the following.

Lemma 4.11.13. *Suppose that assumptions (**A**) and (**C**) are satisfied. Then there exists a constant c_0, independent of t and T such that for all T large enough*

$$\int_0^t \int_{\mathbb{R}} e^{-u\psi_T(w)} dwdu \leq c_0 t^\delta, \tag{4.11.35}$$

for any $0 < \delta < 1 - 1/\beta$ and all $t \in (0,1)$.

Proof. Notice that, by Lemma 4.11.8

$$\lim_{w \to 0} \frac{\psi(w)}{|w|^\beta f(1/w)} = c_1, \tag{4.11.36}$$

for some finite constant c_1. Thus, there exists $\epsilon_1 > 0$ such that

$$\frac{1}{2} c_1 \leq \frac{\psi(w)}{|w|^\beta f(1/w)} \leq 2c_1, \tag{4.11.37}$$

for $|w| < \epsilon_1$. We may write the left-hand side of (4.11.35) as $I_1 + I_2$, with

$$I_1 = \int_0^t \int_{|wT^{-1/\beta}| > \epsilon_1} e^{-u\psi_T(w)} dwdu, \tag{4.11.38}$$

$$I_2 = \int_0^t \int_{|wT^{-1/\beta}| \leq \epsilon_1} e^{-u\psi_T(w)} dwdu. \tag{4.11.39}$$

(4.11.40)

Let us consider I_1 first. Since for any $t > 0$ and $x > 0$

$$\left| \frac{1 - e^{-tx}}{x} \right| \leq min(t, 1/x),$$

we have that, in particular, for $\kappa \in (0,1)$

$$\left| \frac{1 - e^{-tx}}{x} \right| \leq t^\kappa x^{\kappa - 1}.$$

Therefore, for all T large enough,

$$\begin{aligned} I_1 &\leq \int_{|wT^{-1/\beta}| > \epsilon_1} \left(\frac{1}{\psi_T(w)} \right)^{1-\kappa} t^\kappa dw \\ &= \int_{|w| > \epsilon_1} \left(\frac{1}{D_T \psi(w)} \right)^{1-\kappa} T^{1/\beta} t^\kappa dw \leq c_2 t^\kappa \end{aligned}$$

for some constant c_2 independent of t and T. As for I_2, one can easily deduce from (4.11.37) that for $|wT^{-1/\beta}| \leq \epsilon_1$

$$\psi_T(w) \geq \frac{1}{2} |w|^\beta \frac{f(T^{1/\beta}/w)}{f(T^{1/\beta})}.$$

Fix any $\epsilon_2 > 0$. An application of Karamata's representation theorem (see for example [42, Theorem 10.5.7]) yields the inequality

$$\frac{f(T^{1/\beta}/w)}{f(T^{1/\beta})} \geq c_3|w|^{-\epsilon_2} \tag{4.11.41}$$

for all T large enough, $|w| > 1$ and $|wT^{-1/\beta}| \leq \epsilon_1$, provided we choose ϵ_1 small enough. c_3 is a positive constant independent of T. Using all this we may write

$$\begin{aligned} I_2 &\leq \int_{1<|w|\leq T^{1/\beta}\epsilon_1} \int_0^t e^{-\frac{1}{2}uc_1c_3|w|^{\beta-\epsilon_2}} dudw + t\int_{|w|<1} dw \\ &\leq c_4 t^{1-(\beta-\epsilon_2)^{-1}} + 2t, \end{aligned}$$

provided we choose ϵ_2 small enough. Thus, we can take

$$\delta = 1 - (\beta - \epsilon_2)^{-1} \tag{4.11.42}$$

and the proof is finished since $\beta \in (1, 2)$. □

Lemma 4.11.14. *Assume that (**A**) and (**D**) hold. The for any $t > 0, \kappa > 0$ and all T sufficiently large we have the following inequalities:*

$$\int_{\mathbb{R}} \int_0^t \Big(1 \wedge \Big|\frac{w}{T^{1/\beta}}\Big|^{\kappa}\Big) e^{-u\psi_T(w)} dudw \leq c_1(\psi, \beta, \kappa, t)(T^{1/\beta} D_T^{-1} + T^{-\kappa/\beta}), \tag{4.11.43}$$

$$\int_{\mathbb{R}} \int_0^{D_T t} \Big(1 \wedge |w|^{\kappa}\Big) e^{-u\psi(w)} dudw \leq c_4(\psi, \beta, \kappa, t)(1 + T^{-1/\beta-\kappa/\beta} D_T). \tag{4.11.44}$$

In particular for $\kappa > (\beta - 1)/2$

$$\int_{\mathbb{R}} \int_0^t \Big(1 \wedge \Big|\frac{w}{T^{1/\beta}}\Big|^{2\kappa}\Big) e^{-u\psi_T(w)} dudw \leq c_2(\psi, \beta, \kappa, t) T^{1/\beta} D_T^{-1}. \tag{4.11.45}$$

Furthermore

$$\int_{\mathbb{R}} \int_0^{D_T t} e^{-u\psi(w)} dudw \leq c_3(\psi, \beta, t) T^{-1/\beta} D_T, \tag{4.11.46}$$

for some finite constants c_1, c_2, c_3 and c_4 independent of T.

Proof. After a change of variables $w' = T^{1/\beta}w$, $u' = u/D_T$, the left-hand side of (4.11.43) can be written as

$$T^{1/\beta}D_T^{-1}\int_{\mathbb{R}}\int_0^{D_T t}\Big(1\wedge|w|^\kappa\Big)e^{-u\psi(w)}dudw \tag{4.11.47}$$

$$\leq \quad T^{1/\beta}D_T^{-1}\Big(\int_{|w|>1}\psi(w)^{-1}dw+\int_{|w|\leq 1}\int_0^{D_T t}|w|^\kappa e^{-u\psi(w)}dwdu\Big) \tag{4.11.48}$$

The first integral in (4.11.48) is bounded by Assumption (**A**). We can bound the second by

$$\int_{|w|\leq T^{-1/\beta}}D_T|w|^\kappa dw+\int_{T^{-1/\beta}<|w|\leq 1}|w|^\kappa\psi(w)^{-1}dw.$$

An application of Lemma 4.11.11 gives inequality (4.11.43) and 4.11.44. Inequality (4.11.45) follows immediately once we make a change of variables and use the fact that $\int_{\mathbb{R}}\big(1\wedge|w|^{2\kappa}\big)\psi(w)^{-1}dw<\infty$. The inequality (4.11.46) is just Lemma 4.11.12 after a change of variables. □

Chapter 5

Random walks in doubly random scenery

5.1 Introduction

5.1.1 Motivation

In Chaper 4 we have mentioned and provided a particle picture interpretation for the process (4.3.3) in the special case when $\gamma = 2$. A natural question arises as to how to obtain other members of the class $Y_{\alpha,\widetilde{\beta},\gamma}$. Recall that in [20] the scaling limit obtained by the authors also led only to $Y_{\alpha,\widetilde{\beta},2}$. Thus, providing a setting in which other processes of the form (4.3.3) arise seems natural. In the present chapter we do exactly that, but in a more discrete framework, which is an analogue to the particle system we have considered in Chapters 3 and 4. The content of this chapter is based mainly on the publication [47] by the author of the thesis.

5.1.2 Random walks in random scenery

Our model is based in the framework of *random walks in random scenery models*. They were first considered in [22], where a number of limit theorems regarding the scaling limits of these models were proved. The more specific context in which we will be working was presented in [15]. The model considered therein can be briefly sketched as follows. Assume that there is a *user* moving randomly on the *network* (in this paper the network is just $\mathbb{Z}$) who earns random rewards (governed by the random scenery) associated with the points in the network that they visit. The quantity of interest is then the total amount of rewards collected. To be more precise, assume that that the movement of the user is a random walk on $\mathbb{Z}$ which after suitable scaling converges to the β-stable Lévy process with $\beta \in (1, 2]$. Furthermore, let the random scenery be given by i.i.d. random variables $(\xi_j)_{j\in\mathbb{Z}}$ which belong to the normal domain of attraction of a symmetric strictly stable distribution with index of stability $\alpha \in (0, 2]$. Then the *random walk in random scenery* is given by

$$Z_n = \sum_{k=1}^{n} \xi_{S_k}, \tag{5.1.1}$$

where $S_k = \sum_{j=1}^{k} X_k$ is the random walk determining the movement of the user. If we consider a large number of independent *random walkers* moving in independent random sceneries, then the scaling limit in the corresponding functional limit theorem (see [15, Theorem 1.2]) leads to the process (4.1.1).

5.1.3 The limit process

By [20, Proposition 3.2] the process $Y_{\alpha,\widetilde{\beta},\gamma}$ is H-sssi with Hurst coefficient $H = \widetilde{\beta}/\gamma + (1-\widetilde{\beta})/\alpha$. Here we use $\widetilde{\beta}$ instead of β so as not to confuse it with the notation we have adopted for this thesis. Similarly as in the proof of [35, (3.10)] we can show that for $\widetilde{\beta} \in (0, \frac{1}{2})$

$$(Y_{\alpha,\widetilde{\beta},\gamma}(t))_{t\geq 0} \stackrel{d}{=} c_{\widetilde{\beta}} \left(\int_{\Omega'\times\mathbb{R}} S_\gamma(L_t(x,\omega'),\omega') dZ_\alpha(\omega',x) \right)_{t\geq 0}, \tag{5.1.2}$$

where $c_{\widetilde{\beta}}$ is a constant depending only on $\widetilde{\beta}$, $(L_t(x))_{t\geq 0}$ is the local time of a symmetric β-stable Lévy motion defined independent of the process S_γ (both defined on $(\Omega', \mathcal{F}', \mathbb{P}')$), $\beta = (1-\widetilde{\beta})^{-1}$ and Z_α is a symmetric α-stable random measure on $(\Omega', \mathbb{R})$ with control measure $\mathbb{P}' \otimes \lambda_1$.

5.2 Description of the model and the main result

Imagine that each $x \in \mathbb{Z}$ is associated with a reward (or punishment) given by ξ_x which takes integer values. Now imagine a *random walker* moving on $\mathbb{Z}$ independently of the rewards and starting at 0. Let $S_k = X_1 + \ldots X_k$, $k = 1, 2, \ldots$, denote the consecutive partial sums of the random walk with $X_1, X_2, \ldots$ being i.i.d.. Before the movement the walker generates a sequence $Y_1, Y_2, \ldots$ of i.i.d. random variables which are independent of the ξ_x's and his movement. Now, any time the walker visits a point x he gets a reward (or receives punishment) given by $Y_k \times \xi_x$, where k is number of times that the walker has already stayed at x (including the current visit). One can think of Y_k as per visit weights that do not depend on the location of the visit (x), but only on the number of times we have visited the current location in the past. We will refer to the sequence (Y_k) as *temporal scenery.* Thus the amount by which a potential reward is being multiplied depends only on the number of the visits. The total reward/punishment at time n in this scheme is given by

$$\sum_{x\in\mathbb{Z}} \Big(\sum_{k=1}^{N_n(x)} Y_k \Big) \xi_x, \tag{5.2.1}$$

where

$$N_n(x) := \sum_{k=1}^{n} \mathbf{1}_{\{S_k=x\}} \tag{5.2.2}$$

denotes the number of visits to the point $x \in \mathbb{Z}$ up to time $n \in \mathbb{N}$.

The specific context in which our model is investigated is an extension of the one presented in [15, Section 1.2] and goes as follows. Let $(S_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}$ such that for some positive sequence $(a_n)_{n\geq 1}$ we have

$$\frac{1}{a_n} S_n \Rightarrow Z_\beta, \tag{5.2.3}$$

where Z_β has symmetric β-stable distribution $1 < \beta < 2$. In particular, this choice of β implies that the random walk is recurrent. If we define

$$N_s(x) := N_{[s]}(x) + (s - [s])(N_{[s]+1}(x) - N_{[s]}(x)), \quad s \geq 0, \quad x \in \mathbb{Z},$$

then for any $a < b$ and $s \geq 0$ the distribution of the random variable

$$T_s^n(a,b) := \frac{1}{n} \sum_{x: a \leq n^{-1/\beta} x < b} N_{ns}(x),$$

converges to the distribution of

$$\int_a^b L_s^\beta(y) dy,$$

where for any $y \in \mathbb{R}$, $(L_s^\beta(y))_{s\geq 0}$ is the jointly continuous version of the local time of a symmetric β stable Lévy process (see [22, Section 2]).

In the most general setting $(a_n)_{n\geq 1}$ is regularly varying at infinity with exponent β. We will assume more, i.e., that (S_n) is in the *normal domain of attraction* of Z_β and take $a_n = n^{1/\beta}$. Let $\xi = (\xi_x)_{x\in\mathbb{Z}}$ be a family of i.i.d. random variables such that

$$\frac{1}{n^{1/\alpha}} \sum_{x=0}^{n} \xi_x \Rightarrow Z_\alpha, \tag{5.2.4}$$

where Z_α is a symmetric α-stable random variable with $\alpha \in (0,2)$. What is different from the model considered in [15] is that we introduce more randomness to the model with an i.i.d. sequence $(Y_n)_{n\geq 1}$ such that

$$\frac{1}{n^\gamma}\sum_{j=1}^{n} Y_j \Rightarrow Z_\gamma, \tag{5.2.5}$$

where Z_γ has a symmetric γ-stable distribution with $\alpha < \gamma \leq 2$. In the original formulation of [15] all the Y_n's are equal to one. For technical reasons we will also assume that there exist $\kappa > 1$ such that

$$\sup_{k\in\mathbb{N}} \mathbb{E}\left|\frac{Y_1 + \ldots + Y_k}{k^{1/\gamma}}\right|^{\alpha\kappa} < \infty. \tag{5.2.6}$$

Note that (5.2.5) and (5.2.6) hold trivially for Y_k γ-stable and $\alpha\kappa < \gamma$. The above condition can be viewed as a restriction on the distribution of Y_1. A sufficient condition for (5.2.6) to hold is given in the lemma below. We denote the characteristic function of Y_1 by ϕ.

Lemma 5.2.1. *Assume that* (5.2.5) *holds and* $\alpha > 1$. *Then* (5.2.6) *is satisfied if*

$$\int_r^\infty \frac{|\phi'(\theta)|}{\theta^{\alpha\kappa}} d\theta < \infty \tag{5.2.7}$$

for some $r > 0$ *and there is a finite constant* K *such that* $|\phi'(\theta)| \leq K|\theta|^{\gamma-1}$ *for* θ *in some neighbourhood of zero.*

The base for our study is the behaviour of the process

$$\widetilde{Z}(t) := \sum_{x\in\mathbb{Z}} \Big(\sum_{k=1}^{N_{[t]}(x)} Y_k\Big)\xi_x, \quad t \geq 0. \tag{5.2.8}$$

We also define the rescaled version of (5.2.8) by

$$D_n(t) := r_n^{-1}\widetilde{Z}(nt), \quad n \geq 1, i \geq 1, t \geq 0, \tag{5.2.9}$$

with $r_n = n^{1/\gamma + 1/(\alpha\beta) - 1/(\gamma\beta)}$.

We are interested in the scaling limit in which we consider the aggregate behaviour of a large number of independent walkers with independent strategies and having independent environments from which they collect the rewards. More precisely, consider an i.i.d. sequence of processes $\big((D_n^{(i)}(t))_{t\geq 0}\big)_{i=1}^\infty$, $n \geq 1$, each distributed as (5.2.9) and define for $t \geq 0$

$$G_n(t) := \frac{1}{c_n^{1/\alpha}} \sum_{i=1}^{c_n} D_n^{(i)}(t), \quad n \geq 1, \tag{5.2.10}$$

where c_n is any sequence of positive integers converging to $+\infty$. Now we may state our result concerning the scaling limit of the above process.

Theorem 5.2.2. *For any $0 < \alpha < \gamma \leq 2$ the process $(G_n(t))_{t\geq 0}$ defined by* (5.2.10) *converges (up to a multiplicative constant) as $n \to \infty$, in the sense of finite-dimensional distributions, to the process given by* (5.1.2).

5.3 Proof of Lemma 5.2.1

Take any $\kappa > 1$ such that $\alpha\kappa < \gamma$. In the proof $c_1, c_2, \ldots$ will denote constants independent of k and θ. Since the random variable Y_1 is symmetric we may write (using in [31, Lemma 1.3])

$$m_k(\alpha\kappa) := \mathbb{E}\left|\frac{Y_1 + \ldots + Y_k}{k^{1/\gamma}}\right|^{\alpha\kappa} = c_1 \int_0^\infty \frac{\phi_k'(-\theta)}{\theta^{\alpha\kappa}} d\theta, \tag{5.3.1}$$

for some constant c_1 which depends only on α and κ. Here ϕ_k denotes the characteristic function of $(1/k^{1/\gamma})(Y_1 + \ldots Y_k)$. Since Y_1 in the domain of normal attraction of Z_γ we conclude (see [18] for proofs) that the function

$$\theta \mapsto 1 - \phi(\theta) \tag{5.3.2}$$

is regularly varying at 0 with exponent γ and in particular

$$\lim_{\theta\to 0} \frac{1 - \phi(\theta)}{|\theta|^\gamma} = c_2, \tag{5.3.3}$$

with a finite positive constant c_2 depending only on γ. $m_k(\alpha\kappa)$ can be bounded by

$$c_1 \int_0^\infty \frac{|\phi_k'(\theta)|}{\theta^{\alpha\kappa}} d\theta \tag{5.3.4}$$

which, using $\phi_k(\theta) = \big(\phi(\frac{\theta}{k^{1/\gamma}})\big)^k$ and changing variables, equals

$$c_1 \int_0^\infty \frac{|\phi'(\theta)|\big(1 - (1 - \phi(\theta))\big)^{k-1}}{\theta^{\alpha\kappa}} k^{1-(\alpha\kappa)/\gamma} d\theta. \tag{5.3.5}$$

Fix $c > 0$ such that

$$\begin{aligned} 1 - \phi(\theta) &\geq c_3|\theta|^\gamma, \\ |\phi'(\theta)| &\leq c_4|\theta|^{\gamma-1}, \end{aligned}$$

for $|\theta| \leq c$ and some positive constants c_3, c_4. The integral in (5.3.5) can be written as $I_1 + I_2$, where I_1 and I_2 are integrals over $(0, c)$ and (c, ∞) respectively. First, notice that

$$I_1 \leq c_5 \int_0^c \frac{|\theta|^{\gamma-1}(1 - c_3|\theta|^\gamma)^{k-1}}{\theta^{\alpha\kappa}} k^{1-(\alpha\kappa)/\gamma} d\theta. \tag{5.3.6}$$

Since for z close to zero $1-z \sim \exp(-z)$, I_1 is no bigger than

$$c_6 \int_0^c \theta^{\gamma-1-\alpha\kappa} \exp(-(k-1)\theta^\gamma) k^{1-(\alpha\kappa)/\gamma} d\theta. \tag{5.3.7}$$

Changing variables $\theta = (k-1)^{-1/\gamma}\theta'$ and using the fact that

$$\int_0^\infty \theta^{\gamma-1-\alpha\kappa} e^{-/theta^\gamma} d\theta < \infty,$$

we conclude that $\limsup_{k\to\infty} I_1 < \infty$. The fact that for any $c > 0$, I_2 is bounded uniformly in $k \in \mathbb{N}$ follows directly from the assumptions of Lemma 5.2.1.

5.4 Proof of Theorem 5.2.2

For clarity we divide the proof of Theorem 5.2.2 into a number of lemmas. Basically, we prove the convergence of finite-dimensional distributions by showing the convergence of appropriate characteristic functions. First we will state them and then proceed to their proofs. In order to simplify the notation we put

$$\widetilde{N}_n(x) := \sum_{j=1}^{N_n(x)} Y_j, \tag{5.4.1}$$

for $n \in \mathbb{N}$ and $x \in \mathbb{Z}$. Since we are going to work a lot with the characteristic function of ξ_0 we introduce the following notation. Let

$$\lambda(u) = \mathbb{E}(\exp(iu\xi_0)), \quad u \in \mathbb{R} \tag{5.4.2}$$

and

$$\bar{\lambda}(u) = \exp(-|u|^\alpha), \quad u \in \mathbb{R}. \tag{5.4.3}$$

Assume that $\theta_1, \ldots, \theta_k \in \mathbb{R}$, $t_1, \ldots, t_k \in [0, \infty)$ for $k \geq 1$. We want to show the convergence of the characteristic function of $\sum_{j=1}^k \theta_j G_n(t_j)$ to the corresponding characteristic function of the process given by the right-hand side of (5.1.2).

The first lemma in this section removes the first layer of randomness in our scheme and expresses the characteristic function in question solely in terms of the random walk and the sequence $(Y_k)_{k\geq 1}$. The proof amounts to conditioning on (X_j) and (Y_j) and we skip it.

Lemma 5.4.1. *For the setting as in Section 5.2*

$$\mathbb{E}\Big(\exp\left(i\sum_{j=1}^k \theta_j G_n(t_j)\right)\Big) = \left(\mathbb{E}\Big(\prod_{x\in\mathbb{Z}} \lambda\big(c_n^{-1/\alpha} r_n^{-1} \sum_{j=1}^k \theta_j \sum_{m=1}^{N_{[nt_j]}(x)} Y_m\big)\Big)\right)^{c_n}. \tag{5.4.4}$$

The second lemma says that in the limit only the asymptotic behaviour of λ near zero matters.

Lemma 5.4.2. *For each $x \in \mathbb{Z}$, $k \geq 1$, $0 \leq t_1 \leq \ldots \leq t_k$ and $\theta_j \in \mathbb{R}$, $j = 1, \ldots, k$*

$$\mathbb{E}\left(c_n \Big(\prod_{x\in\mathbb{Z}} \lambda(c_n^{-1/\alpha} r_n^{-1} \sum_{j=1}^{k} \theta_j \widetilde{N}_{[nt_j]}(x)) - \bar{\lambda}(c_n^{-1/\alpha} r_n^{-1} \sum_{j=1}^{k} \theta_j \widetilde{N}_{[nt_j]}(x)) \Big) \right) \tag{5.4.5}$$

converges to 0 *as* $n \to \infty$.

The third lemma is the backbone of the whole proof.

Lemma 5.4.3. *Let*

$$B_n := \sum_{x\in\mathbb{Z}} \left| r_n^{-1} \sum_{j=1}^{k} \theta_j \sum_{m=1}^{N_{[nt_j]}(x)} Y_m \right|^{\alpha}, \quad n \geq 1. \tag{5.4.6}$$

Then,

$$\lim_{n\to\infty} \mathbb{E}(B_n) = c(\alpha)\mathbb{E}\left(\int_{\mathbb{R}} \left| \sum_{j=1}^{k} \theta_j Y(L_{t_j}(x)) \right|^{\alpha} dx \right), \tag{5.4.7}$$

and

$$\mathbb{E}(\exp(-c_n^{-1} B_n)) = 1 - c_n^{-1} c(\alpha)\mathbb{E}(B_n) + o(c_n^{-1}). \tag{5.4.8}$$

Here $c(\alpha)$ is a constant depending only on α.

It is evident that given the lemmas above, Theorem 5.2.2 follows immediately (see the proof of [15, Theorem 1.2]). Lemma 5.4.1 expresses the characteristic function of $\sum_{j=1}^{k} \theta_j G_n(t_j)$ in terms of the random walk and the random variables (Y_j). Lemma 5.4.2 says that we can assume that the scenery has symmetric α-stable distribution. Finally, Lemma 5.4.3 establishes the convergence of finite dimensional distributions to the finite dimensional distributions of the right-hand side of 5.1.2.

First, however, we will show that the random variables B_n, $n \in \mathbb{N}$ introduced in the formulation of Lemma 5.4.3 are uniformly integrable. We do this by showing that $\mathbb{E}|B_n|^{\kappa}$ is bounded uniformly in $n \in \mathbb{N}$ for some $\kappa > 1$.

Lemma 5.4.4. *Assume that* (5.2.6) *holds for some $1 < \kappa < \gamma/(\gamma - \alpha)$. Then, for every $t > 0$ there is a constant C, independent of $n \in \mathbb{N}$ (possibly depending on κ), such that we have*

$$\mathbb{E}(B_n^{\kappa}) \leq C. \tag{5.4.9}$$

Proof of Lemma 5.4.4. It is enough to prove the lemma with $k = 1$ and $\theta_1 = 1$. Fix $n \in \mathbb{N}$ and $t \geq 0$. Let $x_1, \ldots, x_{s_n}$ be the points in the range of the random walk up to time $[nt]$ taken in the increasing order with respect to $N_{[nt]}(x_i)$. We can write

$$B_n = \frac{1}{r_n^\alpha}\left(\left|Y_1 + \ldots + Y_{N_{[nt]}(x_1)}\right|^\alpha + \ldots + \left|Y_1 + \ldots + Y_{N_{[nt]}(x_{s_n})}\right|^\alpha\right). \quad (5.4.10)$$

Notice that by Jensen's inequality, for any $\kappa > 1$ we have

$$B_n^\kappa \leq r_n^{-\kappa\alpha} R_{[nt]}^{\kappa-1}\left(\left|Y_1 + \ldots + Y_{N_{[nt]}(x_1)}\right|^{\alpha\kappa} + \ldots + \left|Y_1 + \ldots + Y_{N_{[nt]}(x_{s_n})}\right|^{\alpha\kappa}\right), \quad (5.4.11)$$

where $R_m = \sum_{x\in\mathbb{Z}} \mathbf{1}_{\{N_m(x)\neq 0\}}$ for $m \in \mathbb{N}$. Since the sequence $(Y_n)_{n\in\mathbb{N}}$ and the random walk are independent, by conditioning on the random walk, we get

$$\begin{aligned}\mathbb{E}(B_n^\kappa) \quad &\leq \quad r_n^{-\kappa\alpha} \sup_{k\in\mathbb{N}} \mathbb{E}\left|\frac{Y_1 + \ldots + Y_k}{k^{1/\gamma}}\right|^{\alpha\kappa} \\ &\qquad \mathbb{E}\Big(R_{[nt]}^{\kappa-1} N_{[nt]}(x_1)^{\frac{\alpha\kappa}{\gamma}} + \ldots + N_{[nt]}(x_{s_n})^{\frac{\alpha\kappa}{\gamma}}\Big), \end{aligned} \quad (5.4.12)$$

We now claim that

$$r_n^{-\kappa\alpha}\mathbb{E}\left(R_{[nt]}^{\kappa-1}\sum_{k=1}^{R_{[nt]}} N_{[nt]}(x_k)^{\frac{\alpha\kappa}{\gamma}}\right) \quad (5.4.13)$$

is bounded uniformly in $n \in \mathbb{N}$ for all $\kappa > 1$ sufficiently close to 0. Recall that $\sum_{x\in\mathbb{Z}} N_{[nt]}(x) = [nt]$. Using Hölder inequality with $p = \frac{\gamma}{\alpha\kappa}$ and $q = \frac{\gamma}{\gamma-\alpha\kappa}$ we see that (5.4.13) is no bigger than

$$\begin{aligned} & r_n^{-\kappa\alpha}\mathbb{E}\Big(R_{[nt]}^{\kappa-1}\Big(\sum_{x\in\mathbb{Z}}\mathbf{1}_{\{N_{[nt]}(x)\neq 0\}}\Big)^{\frac{\gamma-\alpha\kappa}{\gamma}}[nt]^{\frac{\alpha\kappa}{\gamma}}\Big) \\ = \quad & r_n^{-\kappa\alpha}\mathbb{E}\Big(R_{[nt]}^{\frac{(\gamma-\alpha)\kappa}{\gamma}}\Big)[nt]^{\frac{\alpha\kappa}{\gamma}} \\ \leq \quad & r_n^{-\kappa\alpha}\Big(\mathbb{E}(R_{[nt]})\Big)^{\frac{(\gamma-\alpha)\kappa}{\gamma}}\big(\sum_{x\in\mathbb{Z}} N_{[nt]}(x)\big)^{\frac{\alpha\kappa}{\gamma}} \qquad (5.4.14) \\ = \quad & r_n^{-\kappa\alpha}\Big(\mathbb{E}(R_{[nt]})\Big)^{\frac{(\gamma-\alpha)\kappa}{\gamma}}[nt]^{\frac{\alpha\kappa}{\gamma}}, \end{aligned}$$

where the inequality in (5.4.14) follows from Hölder inequality as long as $\kappa \leq \frac{\gamma}{\gamma-\alpha}$. By [22, Lemma 1], $\mathbb{E}(R_{[nt]}) \leq c_1[nt]^{1/\beta}$ for some constant c_1 depending only on β. We thus conclude that (5.4.13) can be bounded by

$$c_1[nt]^{\frac{(\gamma-\alpha)\kappa}{\gamma\beta}}[nt]^{\frac{\alpha\kappa}{\gamma}}n^{-\frac{\kappa\alpha}{\gamma}-\frac{\kappa}{\beta}+\frac{\kappa\alpha}{\gamma\beta}}, \quad (5.4.15)$$

which is bounded uniformly in $n \in \mathbb{N}$. □

Proof of Lemma 5.4.2. The proof presented here is very similar to the proof of Lemma 3.5 in [15]. Let

$$U_n(x) := r_n^{-1} \sum_{j=1}^{k} \theta_j \widetilde{N}_{[nt_j]}(x), \quad n \in \mathbb{N}, x \in \mathbb{Z}. \tag{5.4.16}$$

Using inequality (41) in [15]

$$\left| \prod_{x\in\mathbb{Z}} \lambda(c_n^{-1/\alpha} U_n(x)) - \prod_{x\in\mathbb{Z}} \bar{\lambda}(c_n^{-1/\alpha} U_n(x)) \right| \\ \le \sum_{x\in\mathbb{Z}} \left| \lambda(c_n^{-1/\alpha} U_n(x)) - \bar{\lambda}(c_n^{-1/\alpha} U_n(x)) \right|. \tag{5.4.17}$$

Therefore (5.4.5) can be bounded by

$$c_n \mathbb{E}\left(\sum_{x\in\mathbb{Z}} \left| \lambda(c_n^{-1/\alpha} U_n(x)) - \bar{\lambda}(c_n^{-1/\alpha} U_n(x)) \right| \right). \tag{5.4.18}$$

Define $g(v) = |v|^{-\alpha} |\lambda(v) - \bar{\lambda}(v)|$, for $v \neq 0$ and $g(0) = 0$. Recall that, by assumption,

$$\lambda(u) = \bar{\lambda}(u) + o(|u|^\alpha),$$

as $u \to 0$. Then g is bounded and continuous. With this notation (5.4.18) equals

$$\mathbb{E}\left(\sum_{x\in\mathbb{Z}} |U_n(x)|^\alpha g(c_n^{-1/\alpha} U_n(x)) \right). \tag{5.4.19}$$

Fix any $\epsilon > 0$ and choose $\delta > 0$ such that $|z| < \delta$ implies $|g(z)| < \epsilon$. Then, (5.4.19) can be bounded by

$$\epsilon \mathbb{E}\big(\sum_{x\in\mathbb{Z}} |U_n(x)|^\alpha \big) + \|g\|_\infty \, \mathbb{E}\big(\sum_{x\in\mathbb{Z}} |U_n(x)|^\alpha \mathbf{1}_{\{c_n^{1/\alpha} |U_n(x)| \ge \delta\}} \big), \tag{5.4.20}$$

which in turn is bounded by

$$\epsilon \mathbb{E}\left(\sum_{x\in\mathbb{Z}} |U_n(x)|^\alpha \right) + \|g\|_\infty \, \mathbb{E}\left(\sum_{x\in\mathbb{Z}} |U_n(x)|^\alpha \mathbf{1}_{\{\sum_{x\in\mathbb{Z}} |U_n(x)|^\alpha \ge c_n \delta^\alpha\}} \right). \tag{5.4.21}$$

Since, by Lemma 5.4.4 the sequence of random variables $(\sum_{x\in\mathbb{Z}} |U_n(x)|^\alpha)_{n\in\mathbb{N}}$ is uniformly integrable, the first summand in (5.4.21) is bounded by ϵ times a constant independent of $n \in \mathbb{N}$ and the second converges to 0 as $n \to \infty$. The choice of ϵ was arbitrary and hence the proof is finished. □

Proof of Lemma 5.4.3. First, we are going to show that (5.4.7) holds. Without losing generality we may assume that $0 \le t_1 \le \ldots \le t_k$. For convenience we also put $t_0 = 0$.

We can rewrite $\mathbb{E}(B_n)$ as

$$\int_{\mathbb{R}} \quad \mathbb{E}\Big|(\theta_1 + \dots \theta_k) Z^{(1)}(N_{[nt_1]([a_n x])})\Big(\frac{N_{[nt_1]([a_n x])}}{n a_n^{-1}}\Big)^{1/\gamma}$$
$$+(\theta_2 + \dots \theta_k) Z^{(2)}\big(N_{[nt_2]}([a_n x]) - N_{[nt_1]}([a_n x])\big)$$
$$\times\Big(\frac{N_{[nt_2]}([a_n x]) - N_{[nt_1]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma}$$
$$+ \dots +$$
$$+\theta_k Z^{(k)}\big(N_{[nt_k]}([a_n x]) - N_{[nt_{k-1}]}([a_n x])\big)$$
$$\times\Big(\frac{N_{[nt_2]}([a_n x]) - N_{[nt_1]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma}\Big|^{\alpha} dx,$$

where $Z^{(1)}(\cdot), \dots, Z^{(k)}(\cdot)$ are i.i.d. copies of the sequence

$$Z^{(0)}(m) = \frac{1}{m^{1/\gamma}}\left(Y_1 + \dots Y_m\right), \quad m \in \mathbb{N}, \tag{5.4.22}$$

which are independent of the random walk (S_n) (we put $Z^{(j)}(0) = 0$ for convenience). By Skorochod representation theorem we may assume that for $j = 1, \dots, k$, $Z^{(j)}(m)$ converges almost surely to $Z^{(j)}$, which has symmetric γ-stable distribution and the random variables $Z^{(j)}$ are independent. Let

$$C_n = \int_{\mathbb{R}} \Big|(\theta_1 + \dots \theta_k) Z^{(1)} \times \Big(\frac{N_{[nt_1]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma} \tag{5.4.23}$$
$$+(\theta_2 + \dots \theta_k) Z^{(2)} \times \Big(\frac{N_{[nt_2]}([a_n x]) - N_{[nt_1]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma}$$
$$+ \dots +$$
$$+\theta_k Z^{(k)} \times \Big(\frac{N_{[nt_k]}([a_n x]) - N_{[nt_{k-1}]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma}\Big|^{\alpha}.$$

We are going to show that $\mathbb{E}(B_n) - \mathbb{E}(C_n)$ converges to 0 as $n \to \infty$. For that we will need the inequalities:

$$|a^\alpha - b^\alpha| \le \alpha |a - b|(a^{\alpha-1} + b^{\alpha-1}), \alpha > 1, a, b \ge 0, \tag{5.4.24}$$

and

$$|a^\alpha - b^\alpha| \le |a - b|^\alpha, 0 \le \alpha \le 1, a, b \ge 0. \tag{5.4.25}$$

Assume first that $\alpha > 1$. Put

$$\begin{aligned} A &= \Big|(\theta_1 + \dots \theta_k) Z^{(1)}(N_{[nt_1]([a_n x])})\Big(\frac{N_{[nt_1]([a_n x])}}{n a_n^{-1}}\Big)^{1/\gamma} \\ &\quad +(\theta_2 + \dots \theta_k) Z^{(2)}\big(N_{[nt_2]}([a_n x]) - N_{[nt_1]}([a_n x])\big) \\ &\quad \times \Big(\frac{N_{[nt_2]}([a_n x]) - N_{[nt_1]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma} \\ &\quad + \dots + \\ &\quad +\theta_k Z^{(k)}\big(N_{[nt_k]}([a_n x]) - N_{[nt_{k-1}]}([a_n x])\big) \\ &\quad \times \Big(\frac{N_{[nt_2]}([a_n x]) - N_{[nt_1]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma}\Big|, \end{aligned}$$

and

$$\begin{aligned} B &= \Big|(\theta_1 + \dots \theta_k) Z^{(1)} \times \Big(\frac{N_{[nt_1]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma} \\ &\quad +(\theta_2 + \dots \theta_k) Z^{(2)} \times \Big(\frac{N_{[nt_2]}([a_n x]) - N_{[nt_1]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma} \\ &\quad + \dots + \\ &\quad +\theta_k Z^{(k)} \times \Big(\frac{N_{[nt_k]}([a_n x]) - N_{[nt_{k-1}]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma}\Big|. \end{aligned}$$

Then by (5.4.24) and Hölder inquality

$$\begin{aligned} \mathbb{E}|A^\alpha - B^\alpha| &\leq \alpha \mathbb{E}\big(|A - B|(A^{\alpha-1} + B^{\alpha-1})\big) \\ &\leq \alpha\big(\mathbb{E}|A - B|^\alpha\big)^{1/\alpha}\Big(\big(\mathbb{E}A^\alpha\big)^{(\alpha-1)/\alpha} + \big(\mathbb{E}B^\alpha\big)^{(\alpha-1)/\alpha}\Big). \end{aligned}$$

By triangle inequality

$$\begin{aligned} |A - B| &\leq \sum_{j=1}^{k} \big|\theta_j + \dots + \theta_k\big| \Big|\Big(Z^{(j)}\big(N_{[nt_j]}([a_n x]) - N_{[nt_{j-1}]}([a_n x])\big) - Z^{(j)}\Big) \\ &\quad \times \Big(\frac{N_{[nt_j]}([a_n x]) - N_{[nt_{j-1}]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma}\Big|. \end{aligned} \tag{5.4.26}$$

Notice that by (5.2.6) the sequence of random variables

$$\Big(\Big|\frac{Y_1 + \dots Y_n}{n^{1/\gamma}}\Big|^\alpha\Big)_{n \geq 1} \tag{5.4.27}$$

is uniformly integrable and hence, by conditioning on the random walk and using triangle inequality once again (now for the α-norm of a random variable), we conclude that

$$\begin{aligned}\left(\mathbb{E}|A-B|^{\alpha}\right)^{1/\alpha} \quad \leq \quad & \sum_{j=1}^{k} \left|\theta_j + \ldots + \theta_k\right| && (5.4.28)\\ & \times\Bigg(\mathbb{E}\Big|f\Big(N_{[nt_j]}([a_n x]) - N_{[nt_{j-1}]}([a_n x])\Big) && (5.4.29)\\ & \times\Big(\frac{N_{[nt_j]}([a_n x]) - N_{[nt_{j-1}]}([a_n x])}{n a_n^{-1}}\Big)^{1/\gamma}\Big|^{\alpha}\Bigg)^{1/\alpha} && (5.4.30)\end{aligned}$$

where $f : \mathbb{N} \cup \{0\} \to \mathbb{R}_+$ is a bounded function such that $\lim_{m\to\infty} f(m) = 0$. To be more precise,

$$f(m) := \mathbb{E}|Z^{(j)}(m) - Z^{(j)}|.$$

Using (5.2.6) again one can easily notice that both $\mathbb{E}A^{\alpha}$ and $\mathbb{E}B^{\alpha}$ can be bounded by

$$c_1 \mathbb{E}\Big(\frac{N_{[nt_k]([a_n x])}}{n a_n^{-1}}\Big)^{\alpha/\gamma} \tag{5.4.31}$$

for some finite constant c_1 independent of n. Thus, to show that $|\mathbb{E}(B_n) - \mathbb{E}(C_n)|$ goes to zero as $n \to \infty$ it remains to prove that for any $j = 1, \ldots, k$

$$\begin{aligned}\int_{\mathbb{R}} \quad & \Bigg(\mathbb{E}\Big(f\big(N_{[nt_j]}([a_n x]) - N_{[nt_{j-1}]}([a_n x])\big)^{\alpha}\\ & \times\Big(\frac{N_{[nt_j]}([a_n x]) - N_{[nt_{j-1}]}([a_n x])}{n a_n^{-1}}\Big)^{\alpha/\gamma}\Big)\Bigg)^{1/\alpha}\\ & \times\Bigg(\mathbb{E}\Big(\frac{N_{[nt_k]([a_n x])}}{n a_n^{-1}}\Big)^{\alpha/\gamma}\Bigg)^{(\alpha-1)/\alpha} dx && (5.4.32)\end{aligned}$$

converges to 0 as $n \to \infty$. The integrand in (5.4.32) is bounded by the function

$$x \mapsto c_2 \mathbb{E}\Big(\frac{N_{[nt_k]([a_n x])}}{n a_n^{-1}}\Big)^{\alpha/\gamma}, \tag{5.4.33}$$

for some constant c_2 independent of n. It follows from the proof of Lemma 6 in [22] (see also Section 5.2) that for any $K > 0$ and $t > 0$

$$\int_{|x|>K} \Big(\frac{N_{[nt]([a_n x])}}{n a_n^{-1}}\Big)^{\alpha/\gamma} dx \tag{5.4.34}$$

converges in distribution to

$$\int_{|x|>K} L_t(x)^{\alpha/\gamma} dx \tag{5.4.35}$$

were $(L_t(x))_{t\geq 0, \in\mathbb{R}}$ is a jointly continuous version of local time of a symmetric β-stable Lévy process. By [15, Lemma 3.3] the convergence holds also in $L^1(\Omega)$.

Since the expected value of (5.4.35) converges to 0 as $K \to \infty$ (see [15, Lemma 2.1]), we see that by choosing K large enough,

$$\int_{|x|>K} \mathbb{E}\Big(\frac{N_{[nt_k]([a_n x])}}{na_n^{-1}}\Big)^{\alpha/\gamma} dx \tag{5.4.36}$$

can be made arbitrarily small for all n large enough. Thus it remains to show that for any $K > 0$

$$\begin{aligned}\int_{|x|\le K} \quad & \bigg(\mathbb{E}\Big(f\big(N_{[nt_j]}([a_n x]) - N_{[nt_{j-1}]}([a_n x])\big)^{\alpha} \\ & \times\Big(\frac{N_{[nt_j]}([a_n x]) - N_{[nt_{j-1}]}([a_n x])}{na_n^{-1}}\Big)^{\alpha/\gamma}\Big)\bigg)^{1/\alpha} \\ & \times\bigg(\mathbb{E}\Big(\frac{N_{[nt_k]([a_n x])}}{na_n^{-1}}\Big)^{\alpha/\gamma}\bigg)^{(\alpha-1)/\alpha} dx\end{aligned} \tag{5.4.37}$$

converges to zero as $n \to \infty$. This is relatively easy and we will only sketch the idea. Fix any $r > 0$ and $j = 1, \ldots, k$. The integral in (5.4.37) can be written as a sum of two integrals I_1, I_2 depending on whether

$$\frac{N_{[nt_j]([a_n x])} - N_{[nt_{j-1}]([a_n x])}}{na_n^{-1}} \tag{5.4.38}$$

is greater than r or not. In the first case, taking n sufficiently large, the integrand can be bounded by an arbitrarily small constant (in this case $N_{[nt_j]([a_n x])} - N_{[nt_{j-1}]([a_n x])}$ must be large since $na_n^{-1} \to \infty$). In the second case we simply bound the integrand by

$$r^{1/\gamma}\Big(\frac{N_{[nt_k]([a_n x])}}{na_n^{-1}}\Big)^{(\alpha-1)/\gamma} \tag{5.4.39}$$

and the corresponding integral (again by [15, Lemma 3.3]) can be bounded from above by a constant independent of n times $c^{1/\gamma}$. Choosing r small in the first place gives us what was needed. The case $0 \le \alpha \le 1$ is very similar and we skip the proof.

Now, by the stability and independence of $Z^{(1)}, \ldots, Z^{(k)}$, $\mathbb{E}(C_n)$ is equal to

$$\begin{aligned}\sum_{x\in\mathbb{Z}} \quad & r_n^{-\alpha}\bigg(\mathbb{E}\Big||\theta_1 + \ldots \theta_k|^{\gamma} N_{[nt_1]}(x) \\ & +|\theta_2 + \ldots \theta_k|^{\gamma}\big(N_{[nt_2]}(x) - N_{[nt_1]}(x)\big) \\ & + \ldots + \\ & +|\theta_k|^{\gamma}\big(N_{[nt_k]}(x) - N_{[nt_{k-1}]}(x)\big)\Big|^{\alpha/\gamma}\bigg)\mathbb{E}\big(|Y_1|^{\alpha}\big).\end{aligned} \tag{5.4.40}$$

By Lemmas 3.2 and 3.3 in [15], (5.4.40) converges as $n \to \infty$, to

$$\int_{\mathbb{R}} \mathbb{E}\left|\sum_{j=1}^{k} \big(|\theta_j + \ldots + \theta_k|^\gamma - |\theta_{j+1} + \ldots + \theta_k|^\gamma\big) L_{t_j}(x)\right|^\alpha dx, \tag{5.4.41}$$

which finishes the proof of (5.4.7). Now let us turn to (5.4.8). Define $f_n(x) := c_n(1 - \exp(-c_n^{-1}(x)))$ for $x \in \mathbb{R}, n \in \mathbb{N}$. Then, (5.4.8) is equivalent to

$$\lim_{n\to\infty} \mathbb{E} f_n(B_n) = \mathbb{E}B. \tag{5.4.42}$$

We can write, for $\delta > 0$

$$\begin{aligned} \mathbb{E} f_n(B_n) &= \mathbb{E}\Big(f_n(B_n)\mathbf{1}_{\{|B_n| > c_n^\delta\}}\Big) + \mathbb{E}\Big(f_n(B_n)\mathbf{1}_{\{|B_n| \le c_n^\delta\}}\Big) \qquad (5.4.43) \\ &= I_1 + \mathbb{E}\left(c_n\Big(1 - \big(1 - B_n/c_n + O((B_n/c_n)^2)\big)\Big)\mathbf{1}_{\{|B_n| \le c_n^\delta\}}\right), \end{aligned}$$

where (using $|f_n(x)| \le |x|$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$)

$$|I_2(x)| \le \mathbb{E}\left(|B_n|\mathbf{1}_{\{|B_n| > c_n^\delta\}}\right), \tag{5.4.44}$$

which converges to 0 as $n \to \infty$ by the uniform integrability of $(B_n)_{n\ge1}$. Using this, and taking $\delta < \frac{1}{2}$ we see that (again by the uniform integrability of $(B_n)_{n\ge1}$) (5.4.42) holds. □

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