# Silne kliki w grafach

Strong cliques in graphs

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**Summary**

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# 1 Introduction

## 1.1 Strong chromatic index of graphs

A **strong edge coloring** of a graph $G$ is an edge coloring of $G$ such that every two edges which are at distance at most 2 receive different colors (a distance between two edges $e, f$ of $G$ is the distance in the line graph of $G$ between vertices corresponding to $e$ and $f$). A **strong chromatic index** of a graph $G$, denoted by $\chi'_s(G)$, is the minimum number of colors in a strong edge coloring of $G$.

A strong edge coloring is a model of a channel assignment in a wireless radio network consisting of a group of transceivers which can send and receive data. Each transceiver has a range in which it can communicate with others. Communication between a pair of nodes is possible if both are in range of each other. We can construct a graph that models this network. Vertices of our graph correspond to transceivers, two vertices are adjacent if and only if they can communicate with each other (see Figure 1). We say that a primary interference occurs if one node communicates with two other nodes by the same channel; a secondary interference occurs if two adjacent nodes $n_1, n_2$ use the same channel to communicate with some of their neighbour (different from $n_1, n_2$). Assigning channels to all pairs of communicating nodes to avoid primary and secondary interference corresponds to finding a strong edge coloring of the graph that models the network ([1],

[15]). Each used color corresponds to a channel, so it is important to find a strong edge coloring of the network graph which uses a minimum possible number of colors – i.e. to find the strong chromatic index of this graph.

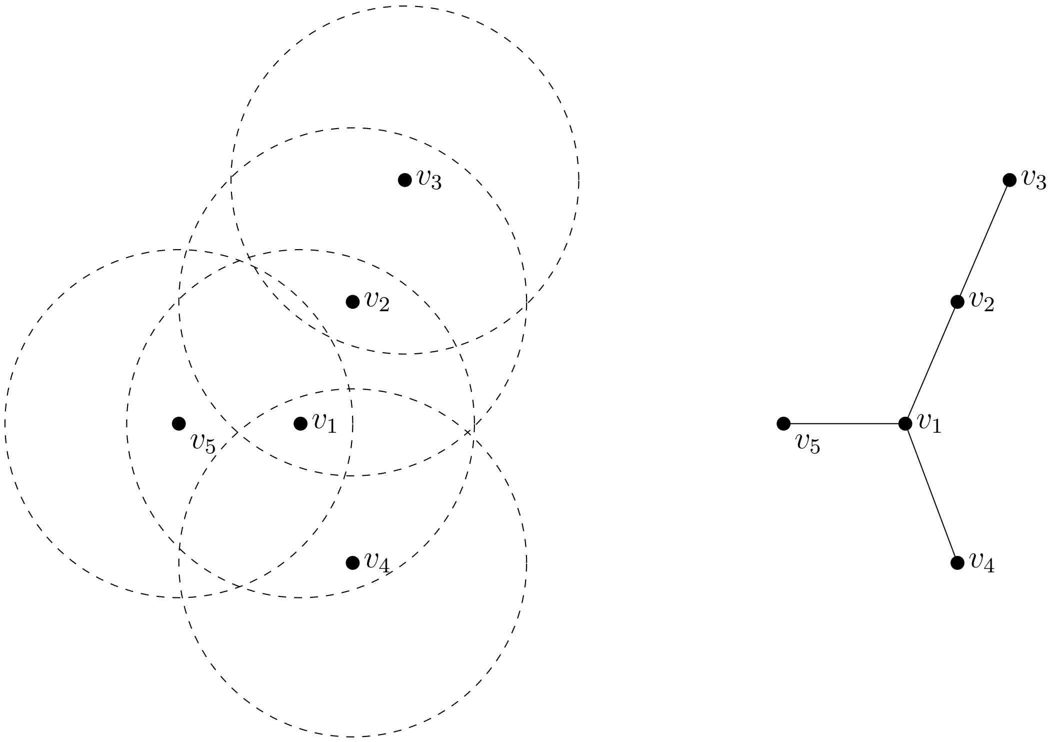



Figure 1: Example of a wireless radio network and a graph that models it.

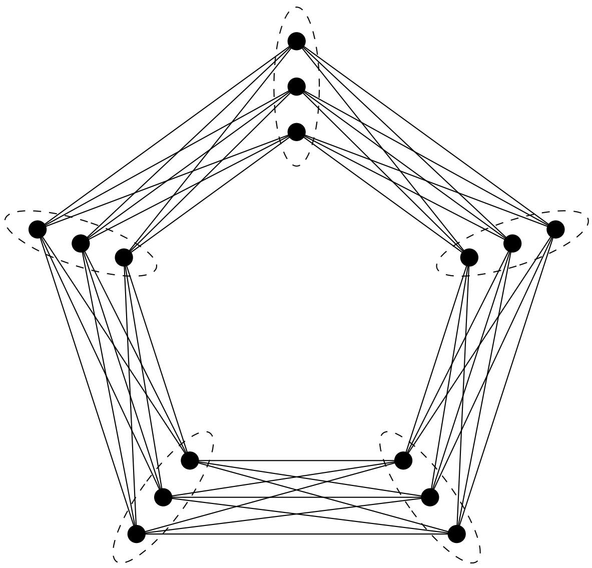
Our main goal is to know how many colors we need to have a strong edge coloring of a graph with maximum degree $\Delta$. In 1985 Erdős and Nešetřil conjectured that $\frac{5}{4}\Delta^2$ colors are enough.

**Conjecture 1.1** (P. Erdős, J. Nešetřil, 1985 [9])**.** *Let $G$ be a graph with maximum degree $\Delta$. Then*

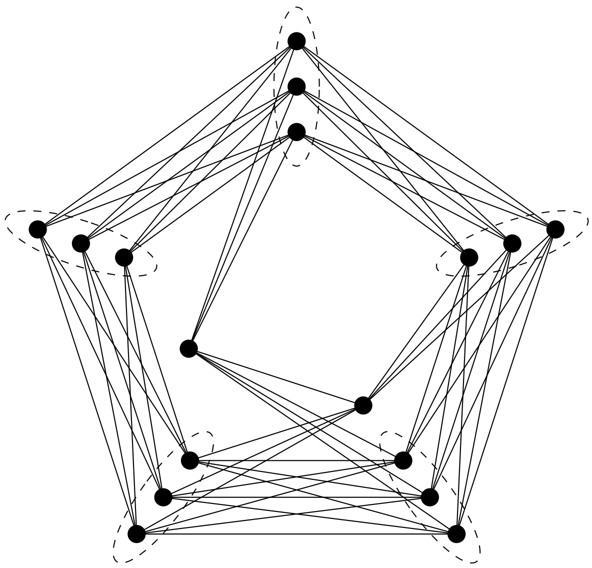
$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \textit{for even } \Delta, \\ \frac{5}{4}\Delta^2 - \frac{2\Delta-1}{4}, & \textit{for odd } \Delta. \end{cases}$$

The bound $\frac{5}{4}\Delta^2$, if true, would be tight, as witnessed by the graph obtained from cycle $C_5$ by blowing each vertex up to an independent set of $\frac{\Delta}{2}$ vertices (see Picture 2).

The conjecture of Erdős and Nešetřil is still open. A simple greedy algorithm shows that for a graph $G$ with maximum degree $\Delta$ the strong chromatic index of $G$ is at most



(a) Even case, $\Delta = 6$.



(b) Odd case, $\Delta = 7$.

Figure 2: Blowing up of $C_5$.

$2\Delta^2 - 2\Delta + 1$. Even a small improvement over the constant 2 next to $\Delta^2$ is nontrivial. In 1997 Molloy and Reed proved that $\chi'_s(G) \leq 1.998\Delta^2$ for graphs with sufficiently large maximum degree $\Delta$ (see [14]). For many years it was the best known bound. In 2015 Bruhn and Joos improved this result to $1.93\Delta^2$ (see [3] and [4]). In 2018 Bonamy, Perrett and Postle proved that $\chi'_s(G) \leq 1.835\Delta^2$ for graphs with sufficiently large $\Delta$ (see [2]). This is the best known result – note that it is still far from the conjectured value.

## 1.2 Strong cliques in graphs

The main topic of this dissertation is strong cliques in graphs. A **strong clique** in a graph $G$ is a subset of edges of $G$ such that every two edges are at distance at most 2. The maximum size of a strong clique in $G$ is denoted by $\omega_s(G)$. There is a clear relation between strong cliques and strong chromatic index. Every edge of a strong clique in a graph $G$ must receive a different color in a strong edge coloring of $G$, so $\omega_s(G) \leq \chi'_s(G)$.

Note that edges of the aforementioned graph, which needs $\frac{5}{4}\Delta^2$ colors in a strong edge coloring, form a strong clique. Therefore, there exist graphs for which $\omega_s(G) = \frac{5}{4}\Delta^2$. Is it an optimal upper bound on $\omega_s(G)$ for all graphs with maximum degree $\Delta$? Gyárfás, Schelp and Tuza stated a weaker version of the conjecture of Erdős and Nešetřil.

**Conjecture 1.2** (R. J. Faudree, A. Gyárfás, R. H. Schelp, Zs. Tuza, 1990 [11]). *Let $G$ be a graph with maximum degree $\Delta$. Then*

$$\omega_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \textit{for even } \Delta, \\ \frac{5}{4}\Delta^2 - \frac{2\Delta-1}{4}, & \textit{for odd } \Delta. \end{cases}$$

We hope that improving an upper bound on $\omega_s(G)$ will provide us a supporting evidence for the conjecture of Erdős and Nešetřil. It is possible that an upper bound on the size of a strong clique will yield a bound on the strong chromatic index. In 1998 Reed conjectured that if $H$ is a graph with maximum degree $\Delta_H$ and a clique number $\omega_H$ then $\chi(H) \leq \left\lceil \frac{\Delta_H+1+\omega_H}{2} \right\rceil$ (see [16]). Consider $L^2(G)$ (the square of the line graph of some graph $G$ with maximum degree $\Delta$) as $H$. The maximum degree of $H$ is at most $2\Delta^2 - 2\Delta$ and $\omega_H = \omega_s(G)$. It follows that if the Reed's conjecture is true, then $\chi_s'(G) \leq \left\lceil \frac{2\Delta^2+\omega_s(G)}{2} \right\rceil$.

What do we know about the maximum size of a strong clique in a graph with maximum degree $\Delta$? In 1990 Faudree, Gyárfás, Schelp, and Tuza proved that $\omega_s(G) \leq (2-\epsilon)\Delta^2$, for sufficiently large $\Delta$ (see [11]). For many years it was the only known result. In 2015 Bruhn and Joos proved that $\omega_s(G) \leq 1.74\Delta^2$, for $\Delta > 400$ (see [3] and [4]). In 2016 we improved this bound to $1.5\Delta^2$ (see [17]). Nowadays, the best known result is $\frac{4}{3}\Delta^2$; it was proved by Faron and Postle in 2019 (see [10]). If the conjecture of Reed is true, then the theorem of Faron and Postle would imply that $\chi_s'(G) \leq \left\lceil \frac{5}{3}\Delta^2 \right\rceil$.

## 1.3 $t-$strong cliques in graphs

We also investigate more general problems: $t-$strong edge coloring and $t-$strong cliques. For integer $t \geq 2$, a $t-$**strong edge coloring** of a graph $G$ is an edge coloring of $G$ such that every two edges which are at distance at most $t$ receive different colors. A $t-$**strong chromatic index** of a graph $G$ is the minimum number of colors in a $t-$strong edge coloring of $G$. A $t-$**strong clique** in a graph $G$ is a subset of edges of $G$ such that every two edges are at distance at most $t$.

A simple greedy algorithm gives an upper bound on a $t-$strong chromatic index of a graph with maximum degree $\Delta$ equals $2\sum_{i=1}^{t}(\Delta-i)^i + 1$. In 2014 Kaiser and Kang improved this result to $(2-\epsilon)\Delta^t$ for $\epsilon = 0.00008$ (see [12]). It is the best known upper bound.

# 2 Results

## 2.1 Strong chromatic index of graphs

We confirm the conjecture of Erdős and Nešetřil (Conjecture 1.1) for claw-free graphs with maximum degree at least 12.

**Theorem 2.1** (M. Dębski, K. Junosza-Szaniawski, M. Ś-N, 2020 [6]). *Let $G$ be a claw-free graph with maximum degree $\Delta$. Then $\chi'_s(G) \leq 1.125\Delta^2 + \Delta$.*

We also show an upper bound on $\chi'_s(G)$ for $K_{1,r}-$free graphs, where $r \geq 4$.

**Theorem 2.2** (M. Dębski, K. Junosza-Szaniawski, M. Ś-N, 2020 [6]). *Let $G$ be a $K_{1,r}-$free graph with maximum degree $\Delta$. Then for all $r \geq 4$*

$$\chi'_s(G) \leq \left(2 - \frac{1}{r-2}\right)\Delta^2 - \frac{r-4}{r-2}\Delta - \frac{1}{r-2}.$$

Our bound is better than the result of Bonamy, Perret and Postle ($1.835\Delta^2$) for $r \leq 8$.

A family of unit disk graphs are a subfamily of $K_{1,6}-$free graphs, so from Theorem 2.2 we have the bound on strong chromatic index of these graphs equals $1.75\Delta^2$. We improve this result.

**Theorem 2.3** (M. Dębski, K. Junosza-Szaniawski, M. Ś-N, 2020 [6]). *Let $G$ be a unit disk graph with maximum degree $\Delta$. Then $\chi'_s(G) \leq 1.625\Delta^2$.*

Our proofs give algorithms to find strong edge colorings of graphs from aforementioned classes. We do not know if our theorems are tight. The highest strong chromatic index of graphs that we are able to construct is $\frac{9}{16}(\Delta+1)^2$ for claw-free and unit disk graphs, and $\frac{3}{4}(\Delta+1)^2$ for $K_{1,4}-$free graphs.

The conjecture of Erdős and Nešetřil is confirmed for chordless graphs. An upper bound on strong chromatic index of these graphs is linear in $\Delta$. We improve the result from $7\Delta - 9$ (see [1]) to $4\Delta - 3$.

**Theorem 2.4** (M. Dębski, J. Grytczuk, M. Ś-N., 2015 [5]). *Let $G$ be a chordless graph with maximum degree $\Delta$. Then $\chi'(G) \leq 4\Delta - 3$.*

## 2.2 Strong cliques in graphs

Our main theorem says that strong cliques in graph with maximum degree $\Delta$ have at most $1.5\Delta^2$ edges. It was the best known upper bound for 3 years (improved by Faron and Postle to $\frac{4}{3}\Delta^2$).

**Theorem 2.5** (M. Ś-N., 2016 [17]). *Let $G$ be a graph with maximum degree $\Delta$. Then*

$$\omega_s(G) \leq 1.5\Delta^2.$$

We also show that $\omega_s(G) \leq \Delta^2 + \frac{1}{2}\Delta$ for claw-free graphs; thereby we improve the bound $1.125\Delta^2 + \Delta$ resulting from Theorem 2.1.

**Theorem 2.6** (M. Dębski, M. Ś-N., 2020+ [7]). *Let $G$ be a claw-free graph with maximum degree $\Delta$. Then*

$$\omega\left(L^2(G)\right) \leq \Delta^2 + \frac{1}{2}\Delta.$$

## 2.3 $t-$strong cliques in graphs

We prove that $t-$strong cliques have at most $1.75\Delta^t + O(\Delta^{t-1})$ edges.

**Theorem 2.7** (M. Dębski, M. Ś-N., 2020+ [8]). *Let $G$ be a graph with maximum degree $\Delta$. For all $t \geq 2$, $t-$strong cliques in $G$ have at most $1.75\Delta^t + O\left(\Delta^{t-1}\right)$ edges.*

For bipartite graphs we show the bound $\Delta^t + O\left(\Delta^{t-1}\right)$.

**Theorem 2.8** (M. Dębski, M. Ś-N., 2020+ [8]). *Let $G$ be a bipartite graph with maximum degree $\Delta$. For all $t \geq 2$, $t-$strong cliques in $G$ have at most $\Delta^t + O\left(\Delta^{t-1}\right)$ edges.*

We also present results for some special classes of graphs.

**Theorem 2.9** (M. Dębski, M. Ś-N., 2020+ [8]). *Let $G$ be a $K_{1,r}-$free graph with maximum degree $\Delta$. For all $t \geq 2$, $t-$strong cliques in $G$ have at most $2\left(\frac{r-2}{r-1}\right)^{t-2}\Delta^t + O\left(\Delta^{t-1}\right)$ edges.*

**Theorem 2.10** (M. Dębski, M. Ś-N., 2020+ [8]). *Let $G$ be a graph with maximum degree $\Delta$ and girth at least $2t + 2x + 1$, for $t \geq 2$ and $0 \leq x \leq \left\lfloor \frac{t}{2} \right\rfloor - 1$. Then $t-$strong cliques in $G$ have at most $2^{t+2}\Delta^{t-x-1}$ edges.*

By noticing a connection between $t-$strong cliques and a degree-diameter problem, we show that upper bounds on $t-$strong chromatic index and $t-$strong cliques cannot be smaller than $\frac{1}{2}\left(\frac{1}{1{,}59}\right)^{t-1}\Delta^t$ (see [8]); thereby improving previous result $\frac{1}{2(t-1)^{t-1}}\Delta^t$, which was proved by Kang and Manggala (see [13]).

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