Report on "Generalized Białynicki-Birula Decompositions"

by Łukasz Sienkiewicz

The PhD thesis "Generalized Białynicki-Birula Decompositions" by Łukasz Sienkiewicz is of the highest quality. I deem the thesis not only sufficient to grant a PhD but I recommend the PhD be granted "with an honorary distinction."

The PhD revisits the classical result of Białnicki-Birula from 1973 concerning an action of $\mathbb{G}_m$ on a smooth projective variety X. If we let $F_1, \ldots, F_n$ be the connected components of the fixed locus $X^{\mathbb{G}_m}$, then Białnicki-Birula's theorem states that (a) $X_i = \{x \in X \mid \lim_{t\to 0} t \cdot x \in F_i\}$ is a locally closed subvariety of X, (b) F_i is smooth and the map $X_i \to F_i$ defined by $x \mapsto \lim_{t\to 0} t \cdot x$ is an affine fibration, and (c) $X = \coprod X_i$. As explained in the introduction of the thesis, this is a foundational result in algebraic geometry with a variety of applications.

In 2013, Drinfeld offered a functorial approach to Białnicki-Birula decompositions by introducing the functor

$$X^+ = \mathrm{Mor}^{\mathbb{G}_m}(\mathbb{A}^1 \times -, X)\colon \mathrm{Sch}/k \to \mathrm{Sets}$$

assigning a scheme T to the set of $\mathbb{G}_m$-equivariant morphisms $\mathbb{A}^1 \times T \to X$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ via scaling and trivially on T. When X is a smooth projective variety as in Białnicki-Birula's theorem, then X^+ is represented by the scheme $\coprod X_i$. An advantage of Drinfeld's approach is that X^+ can be defined more generally for any (possibly singular) scheme X and in fact for any algebraic space over the ground field k. Drinfeld shows that under very mild hypotheses (namely, X is an algebraic space of finite type over k) that X^+ is representable by an algebraic space of finite type over k and that the morphism $X^+ \to X^{\mathbb{G}_m}$ is affine. If X is smooth, then $X^{\mathbb{G}_m}$ is smooth and $X^+ \to X^{\mathbb{G}_m}$ is affine. On the other hand, if X is proper, then the evaluation at 1 map $X^+ \to X$ is a surjective monomorphism. However, the question of when the connected components of X^+ are locally closed subschemes of X is more nuanced.

This PhD thesis offers a generalization of Białynicki-Birula and Drinfeld's theorems. In some of my own joint work with Hall and Rydh, we generalized Drinfeld's approach to establish a version of Białnicki-Birula decompositions for Deligne–Mumford stacks. Sienkiewicz's PhD thesis offers a generalization in a completely different direction to ours.

The starting point for the generalized Białynicki-Birula decompositions in the PhD thesis is the observation that $\mathbb{G}_m$ is the group of units in the multiplicative monoid $\mathbb{A}^1$ and the functor X^+ is essentially capturing when multiplication maps $\mathbb{G}_m \to X$ extend to the monoid. Working more generally with the group G of units in an algebraic monoid M, the PhD thesis investigated to what extent Białynicki-Birula's theorem holds in this generalized setting. Namely, Sienkiewicz introduces and studies the functor

$$\mathcal{D}_X = \mathrm{Mor}^G(M \times -, X) \colon \mathrm{Sch}/k \to \mathrm{Sets}$$

for a particular class of monoids called Kempf monoids. The defining property of a Kempf monoid is quite simple: in addition to the requirement that M is geometrically integral and affine over k, the monoid M must admit a zero and a central torus $T \subset G$ such that its closure contains the zero.

What I find particular striking about this thesis is how much can be obtained just from the simple defining property of a Kempf monoid. Sienkiewicz consistently leverages this defining property to show that many of the desirable properties of the original Białynicki-Birula decomposition still holds! The class of Kempf monoids is quite broad including reductive monoids with 0.

Given a Kempf monoid M over k with group of units G and a scheme X of finite type over k with an action of G, the main results of the thesis are the following:

- (Theorem A) $\mathcal{D}_X$ is representable by a scheme X^+ of finite type over k and the natural morphism $X^+ \to X^G$ is affine, and
- (Theorem B) the morphism $X^+ \to X^G$ is an affine fibration whenever its smooth.

The exposition in this thesis is fantastic. In Chapters 2 through 5, the author very succinctly and clearly summarizes the necessary technical background of functors, algebraic groups and algebraic monoids often with completely self-contained justifications. The author is successfully able to work with many technical concepts and is careful with the various intricacies involved. The proofs seem all to be correct and there are very few typos (e.g. the conclusion in Corollary 5.4.3 should be that $\bar{T}$ is a linearly reductive monoid and the hypothesis that M admits a zero is missing from Corollary 5.5.4).

The heart of the thesis lies in Chapters 6 and 7. Chapter 6 covers algebraization of formal M-schemes and coherent completeness, subjects of which I am already quite familiar. In my paper "A Luna étale slice theorem for algebraic stacks" joint with Hall and Rydh, coherent completeness played a fundamental role. In our setting, we showed that if G is a linearly reductive group acting on a noetherian affine scheme $X = \operatorname{Spec} A$ and A^G is a complete local noetherian k-algebra, then there is an equivalence of categories

$$\mathrm{Coh}^G(X) = \varprojlim \mathrm{Coh}^G(X_n)$$

where X_n is the nth nilpotent thickening of the unique closed G-orbit $X_0 \subset X$; we then say that X *is coherently complete along* X_0. This result becomes powerful when used in combination with Hall and Rydh's formulation of Tannaka duality for noetherian algebraic stacks: a morphism $X \to Y$ of algebraic stacks can be recovered by its tensor functor $f^*\colon \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$. The consequence is that if X is coherently complete along X_0, then to define a morphism $X \to Y$ it suffices to give compatible maps $X_n \to Y$! This technique featured prominently in the proof of our main theorem.

In Chapter 6, algebraization and coherent completeness is revisited in the context of Kempf monoids. Let M be a Kempf monoid with group of units G. The results of Chapter 6 can be summarized as follows:

- (Coherent completeness) If Z is a locally linear noetherian M-scheme and Z_n denotes the nth nilpotent thickening of $Z^M \subset Z$, then
$$\mathrm{Coh}^G(Z) = \varprojlim \mathrm{Coh}^G(Z_n)$$
is an equivalence of categories. This is Theorem 6.6.1.
- (Algebraization) Any formal M-scheme $\{Z_n\}$ is the completion of a locally linear M-scheme Z. This is Theorem 6.5.1. Moreover, if each Z_n is of finite type over k, then so is Z. This is a consequence of Theorem 6.5.7. Similarly, morphisms are algebraizable: if Z and W are locally linear M-schemes, then
$$\mathrm{Mor}^M(Z, W) = \varprojlim \mathrm{Mor}(Z_n, W_n).$$
The algebraization of morphisms is proved directly in Corollary 6.5.5 but it can also be viewed as a consequence of Tannaka duality and the above result on coherent completeness. Together the two algebraization statements establish an equivalence of categories between locally linear M-schemes and formal M-schemes.

This chapter is very pleasant to read. I find it very impressive that these results are proved essentially from scratch. I'm also very fond with the explicit method of proof. In my collaboration with Hall and Rydh, our first approach at the coherent completeness result resembled the approach taken in this thesis (see Remark 2.5 in "A Luna étale slice theorem for algebraic stacks") by explicitly building the limiting coherent sheaf degree by degree after showing that the isotypic components eventually stabilize. This stabilization is handled by Lemma 6.6.1.1 which is both nicely formulated and proven. In our work, we eventually decided on a different approach that was more adaptable in our setting. In our joint work, we've also considered analogous algebraization results (although in a different context of this thesis), our most powerful statement being Theorem 1.10 in our second paper "The étale local structure of algebraic stacks."

In Chapter 7, Theorems A and B are proven using the algebraization and coherent completeness in Chapter 6 together with Hall and Rydh's Tannaka duality. Again, the exposition is clear and the arguments appear to be correct.

This PhD thesis leaves open some questions. When is the morphism $X^+ \to X^G$ smooth? When are the components of X^+ locally closed in X? Is the result true when X is an algebraic space? These questions however are addressed in joint work of the author with Jelisiejew and are summarized in the last section of Chapter 7.

Theorems A and B represent important and novel contributions to algebraic geometry. The thesis is exceptionally well-written. By introducing the necessary background and highlighting the key concepts, the reader can readily follow the exposition in a self-contained manner. The proofs in addition to being correct are also easy to follow. For these reasons, I recommend that this PhD thesis be approved and be granted "with an honorary distinction."

Yours sincerely,

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