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Generalized Białynicki-Birula Decompositions

PhD dissertation

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I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

The classical Białynicki-Birula decomposition states that a smooth algebraic variety X with an action of $\mathbb{G}_m$ can be decomposed into a disjoint sum of locally closed subvarieties, Moreover, each locally closed subvariety in the decomposition admits a locally trivial affine space fibration over certain connected component of $X^{\mathbb{G}_m}$. In particular, this result simplifies various cohomological considerations about such varieties.
This dissertation concerns generalizations of the classical Białynicki-Birula decomposition. We consider an algebraic group $\mathbf{G}$ and a dense embedding $\mathbf{G} \hookrightarrow \mathbf{M}$ of $\mathbf{G}$ into an algebraic monoid $\mathbf{M}$. Then for any algebraic space X equipped with an action of $\mathbf{G}$ we define a functor $\mathcal{D}_X$ that parametrizes $\mathbf{G}$-schemes over X for which the action of $\mathbf{G}$ extends to an action $\mathbf{M}$. We can rephrase the classical Białynicki-Birula decomposition in this language by setting $\mathbf{G} = \mathbb{G}_m$ and $\mathbf{M} = \mathbb{A}^1$ and taking for X a smooth algebraic variety with $\mathbb{G}_m$-action.
The functorial approach we propose enables two orthogonal ways of generalizing the original Białynicki-Birula result. The first generalizes the embedding $\mathbb{G}_m \hookrightarrow \mathbb{A}^1$ and the other concerns replacement of smooth algebraic varieties with more general schemes or algebraic spaces (not necessarily smooth or normal). To address the first generalization we introduce the class of Kempf's monoids. In particular, every monoid having reductive group of units is a Kempf's monoid. Under this assumptions we obtain the representability of $\mathcal{D}_X$ and prove that certain canonical morphism $\mathcal{D}_X \to X^{\mathbf{G}}$ is affine. For the smooth case we also prove that the latter morphism is an affine fibration precisely as in the classical case. In particular, this gives an independent proof of the original Białynicki-Birula decomposition.

Keywords: algebraic groups, algebraic monoids, formal schemes, Białynicki-Birula decomposition

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List of Symbols

$\mathcal{C}^{\mathrm{op}}$	opposite category of $\mathcal{C}$
$\mathbf{Fun}(\mathcal{C},\mathcal{D})$	category of functors $\mathcal{C} \to \mathcal{D}$
$\langle f,g\rangle$	the unique morphism $X \to Y \times Z$ for two morphisms $f : X \to Y$ and $g : X \to Z$ in some category
$F \dashv G$	$F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors and F is left adjoint to G
$\mathbf{Set}$	category of sets
$\widehat{\mathcal{C}}$	category $\mathbf{Fun}(\mathcal{C}^{\mathrm{op}},\mathbf{Set})$ (presheaves on $\mathcal{C}$)
$\mathbf{Mon}$	category of monoids
$\mathbf{Grp}$	category of groups
$\mathbf{Ab}$	category of commutative groups
$\mathbf{Top}$	category of topological spaces
M^*	group of units of an abstract monoid M
M^{op}	opposite monoid of an abstract monoid M
R_+	additive group of a ring R
$R^\times$	multiplicative monoid of a ring R
R^*	group of multiplicative invertible elements in a ring R
$\mathbf{Mod}(R)$	category of (left) modules over a ring R
$\mathrm{M}_{n\times n}(R)$	the ring of $n \times n$ matrices over a ring R
k	commutative ring
$\mathbf{Alg}_k$	category of commutative k-algebras
$\mathrm{Mor}_k(A,B)$	set of morphism $A \to B$ of k-algebras
$\mathbf{Sch}_k$	category of schemes over k
$\mathrm{Mor}_k(X,Y)$	set of morphism $X \to Y$ of k-schemes
$f^\#$	morphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ associated with morphism of $f : X \to Y$ of schemes or a local morphism of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induced by f
$X \times_k Y$	fiber-product of k-schemes X and Y over $\mathrm{Spec}\,k$
$\mathbf{Aff}_k$	category of affine schemes over k
$\widehat{A}$	$\mathfrak{m}$-adic completion of a local noetherian ring $(A,\mathfrak{m})$
$\Omega_{X/Y}$	sheaf of Kähler differentials of a morphism $f : X \to Y$ of schemes
$\Omega_{B/A}$	B-module of Kähler differentials of a morphism $A \to B$ of commutative rings
$\Gamma(U,\mathcal{F})$	space of sections over an open set U of a sheaf $\mathcal{F}$

$k[\mathbf{M}]$	bialgebra of global regular functions on an affine monoid k-scheme $\mathbf{M}$
$k[\mathbf{G}]$	Hopf algebra of global regular functions on an affine group k-scheme $\mathbf{G}$
$\mathrm{Hom}_k(V, W)$	k-module of homomorphisms $V \to W$ of k-modules
$V^\vee$	$\mathrm{Hom}_k(V, k)$ where V is a k-module
$\mathrm{Sym}(V)$	symmetric algebra of a k-module V
$\mathbf{Vect}_k$	category of vector spaces over a field k

Chapter 1

Introduction

1.1 Historical background and main results

1.1.1 Classical Białynicki-Birula decomposition

This work is concerned with the generalization of the celebrated result of Białynicki-Birula ([BB73, Theorem 4.3]). We explain this classical result over complex number field $\mathbb{C}$ for simplicity (Białynicki-Birula proved his result for varieties defined over arbitrary algebraically closed field). Consider a complex smooth projective variety X with an action of $\mathbb{C}^*$. We may view X as a projective manifold and for each x in X we define

$$x_0 = \lim_{t \to 0} t \cdot x$$

Note that this limit exists for every point x in X according to the fact that X is projective. Moreover, x_0 is a fixed point of the $\mathbb{C}^*$-action. Classically the fixed point locus $X^{\mathbb{C}^*}$ of X is a disjoint union $F_1, F_2, ..., F_n$ of smooth, closed subvarieties of X. For each i we define

$$X_i^+ = \left\{ x \in X \,\middle|\, \lim_{t \to 0} t \cdot x \in F_i \right\}$$

Białynicki-Birula proved the following result.

Theorem. *In the situation described above the following assertions hold.*

(1) $X_i^+ \cap X_j^+ = \varnothing$ *for* $i \neq j$.

(2) *The map* $X_i^+ \hookrightarrow X$ *is a locally closed immersion of algebraic varieties for every* i.

(3) *The canonical map*

$$X_i^+ \ni x \mapsto \lim_{t \to 0} t \cdot x \in F_i$$

is a morphism of algebraic varieties and moreover, it is a Zariski locally trivial fibration with fiber $\mathbb{C}^{n_i}$ *for some* $n_i \in \mathbb{N}$. *This holds for every* i.

The theorem above (and its generalizations to singular varieties) has profound applications in algebraic geometry. [BBCM13, II, 4.2] contains a survey of classical applications to Betti numbers and homology. Here we give a sample of recent developments which were based on this result. Brosnan ([Bro05]) applied Białynicki-Birula decomposition to obtain decomposition of motives of isotropic smooth homogeneous projective varieties. Results due to Jelisiejew on Hilbert schemes ([Jel19a], [Jel19b]) used generalized version of the decomposition as their main tool. There are applications to cell decompositions of quiver varieties ([RW19], [Sau17]), localization formulas in equivariant cohomology ([Web17]) and mirror theorem for toric varieties ([Iri17]).

1.1.2 Drinfeld's result

In [Dri13] Drinfeld proposed the following functorial generalization of the classical Białynicki-Birula result. Let k be a field and let X be an arbitrary algebraic space over k with an action of $\mathbb{G}_m$. Consider the functor $\mathcal{D}_X$ on the category of k-schemes defined by the formula

$$\mathbf{Sch}_k \ni Y \mapsto \left\{\gamma : \mathbb{A}^1_k \times_k Y \to X \,\middle|\, \gamma \text{ is } \mathbb{G}_m\text{-equivariant}\right\} \in \mathbf{Set}$$

There are canonical morphisms of functors

$$\begin{tikzcd} \mathcal{D}_X \arrow[r, "i_X"] \arrow[d, "r_X"] & X \\ X^{\mathbb{G}_m} \arrow[u, bend left, "s_X"] & \end{tikzcd}$$

which we define now. For this let $\gamma \in \mathcal{D}_X(Y)$ for some k-scheme Y. We define

$$i_X(\gamma) = \gamma_{|\{1\}\times_k Y}, \qquad r_X(\gamma) = \gamma_{|\{0\}\times_k Y}$$

where $1 : \operatorname{Spec} k \to \mathbb{A}^1_k$ is the inclusion of 1 and $0 : \operatorname{Spec} k \to \mathbb{A}^1_k$ is the inclusion of the zero. Next if $f : Y \to X$ is a morphism which factors through $X^{\mathbb{G}_m}$, then we define

$$s_X(f) = f \cdot pr_Y$$

where $pr_Y : \mathbb{A}^1_k \times_k Y \to Y$ is the projection. The definition of $\mathcal{D}_X$ is a functorial reformulation of the limiting procedure discussed above. In order to provide intuitive justification of this claim let us make some observations.

- Consider a k-scheme Y and let $f : Y \to X$ be a morphism. Then f is a Y-point of X and the morphism
$$\mathbb{G}_m \times_k Y \ni (t, y) \mapsto t \cdot f(y) \in X$$
is the orbit of Y-point f with respect to the $\mathbb{G}_m$-action. A limiting procedure may be interpreted as the existence of the extension of the morphism above to a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1_k \times_k Y \to X$. This is the motivation for the definition of $\mathcal{D}_X$.
- Under this interpretation one may view r_X as the morphism sending each Y-point to its limit Y-point provided that the latter exists.
- Similarly i_X can be considered as the inclusion of the space of points that admit limit into X and s_X can be considered as the inclusion of fixed points into the space of points that have limits.

The following theorem is one of main results of Drinfeld's article [Dri13, Theorem 1.4.3].

Theorem. *Let X be an algebraic space of finite type over k with an action of $\mathbb{G}_m$. Then*

(1) *$\mathcal{D}_X$ is representable by an algebraic space of finite type over k.*

(2) *The morphism r_X is affine.*

1.1.3 The research questions

Note that the scheme $\mathbb{A}^1_k$ is a monoid k-scheme with respect to the canonical operation that makes the set of its k-points into the abstract monoid $k^\times$. Then $0 \in k$ defines the zero of the monoid k-scheme $\mathbb{A}^1_k$. Moreover, the group of units of this monoid k-scheme can be identified with $\mathbb{G}_m$ via canonical open immersion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1_k$. This suggests that one can generalize Drinfeld's functorial formulation as follows. Consider a monoid k-scheme $\mathbf{M}$ with zero $\mathbf{o}$. Let $\mathbf{G}$ be its group of units. Then $\mathbf{G}$ is a group k-scheme. For every k-scheme (or algebraic space) X with an action of $\mathbf{G}$ define the functor $\mathcal{D}_X$ by the formula

$$\mathbf{Sch}_k \ni Y \mapsto \left\{\gamma : \mathbf{M} \times_k Y \to X \,\middle|\, \gamma \text{ is } \mathbf{G}\text{-equivariant}\right\} \in \mathbf{Set}$$

on the category of k-schemes. Clearly one can define morphisms r_X, s_X and i_X of functors as above. The goal of this work is to provide answers to the following questions.

Question. *Is $\mathcal{D}_X$ representable?*

Question. *Suppose that $\mathcal{D}_X$ is representable and smooth over $X^{\mathbf{G}}$. Is r_X locally trivial fibration with affine spaces as fibers?*

1.1.4 The results

Originally Jelisiejew and the author were interested in answering these questions for *(linearly) reductive monoids*. It turns out that both our questions have affirmative answers if X is a scheme locally of finite type over k and $\mathbf{M}$ is a reductive monoid over k. There is even wider class of *Kempf monoids* for which this is the case. Precisely the following two theorems are main results of this thesis.

Theorem A (Corollary 7.8.5)**.** *Let $\mathbf{G}$ be a group k-scheme and let $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a scheme locally of finite type over k with an action of $\mathbf{G}$. Then $\mathcal{D}_X$ is representable by a scheme X^+ and $r_X : X^+ \to X^{\mathbf{G}}$ is affine and of finite type.*

Theorem B (Theorem 7.9.3)**.** *Let $\mathbf{G}$ be a group k-scheme and let $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Let X be a scheme locally of finite type over k with an action of $\mathbf{G}$. Suppose that x is a point of $X^{\mathbf{G}}$ such that the morphism $r_X : X^+ \to X^{\mathbf{G}}$ is smooth at $s_X(x)$. Then there exist an open neighborhood V of x in $X^{\mathbf{G}}$ and an isomorphism $\phi : r_X^{-1}(V) \to \mathbb{A}^n_V$ of k-schemes such that the triangle*

$$\begin{tikzcd} r_X^{-1}(V) \arrow[rr, "\phi"] \arrow[dr, "\text{the restriction of } r_X"'] & & \mathbb{A}^n_V \arrow[dl, "\mathrm{pr}_V"] \\ & V & \end{tikzcd}$$

is commutative, where pr_V *is the projection. Moreover, if* $\mathbf{G}$ *is linearly reductive, then one can choose* ϕ *to be* $\mathbf{M}$*-equivariant with respect to some action of* $\mathbf{M}$ *on* $\mathbb{A}^n_V$.

Note that Theorem A is a generalization of the Drinfeld's result mentioned above (Subsection 1.1.2). Theorem B shows that the essential feature of the classical Białynicki-Birula decomposition – that is the fact that the canonical morphism $X^+ \to X^{\mathbf{G}}$ is a Zariski locally trivial fibration with affine spaces as fibers – holds also for this much more general setup.
In Example 7.10.8 we present an application of Theorems A and B for actions of $\mathbb{G}_m \times_k \mathbb{G}_m$.

1.2 Main ideas involved in the proof of Theorem A

In order to facilitate the readability of this work and make our main results described above more intuitive in this and next section we give sketches of proofs of Theorems A and B.
In this section we start by discussing algebraic monoids. Next we introduce formal version of the Białynicki-Birula functor $\widehat{\mathcal{D}}_X$ and explain briefly the proof of its representability. In the last subsection we outline, how representability of this functor combined with *coherent completeness* and *tannakian formalism* imply that the canonically defined morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism. Theorem A (i.e. representability of $\mathcal{D}_X$) is a consequence of the fact that $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism and representability of $\widehat{\mathcal{D}}_X$.

1.2.1 Kempf monoids

Let us first delve a little into the theory of algebraic monoids. The category of algebraic monoids is a rich and beautiful extension of the category of algebraic groups. There are whole monographs ([Ren06], [CLSW14]) devoted to this subject. In particular, (similarly to the case of algebraic groups) researchers and pioneers in the field of algebraic monoids concentrate they efforts on studying *reductive monoids*. An algebraic monoid $\mathbf{M}$ over k is *reductive* if the group $\mathbf{G}$ of units of $\mathbf{M}$ is a reductive algebraic group. Renner in [Ren06, Theorem 5.4] classifies normal reductive monoids over algebraically closed fields in terms of pairs $(\mathbf{G}, \overline{T}_{\max})$ consisting of a reductive group $\mathbf{G}$ and a normal toric monoid $\overline{T}_{\max}$ with maximal torus $T_{\max}$ of $\mathbf{G}$ as the group of units. He proves that if the action of the Weyl group of $T_{\max} \hookrightarrow \mathbf{G}$ extends to $\overline{T}_{\max}$, then there exists a unique (up to an isomorphism) normal reductive monoid $\mathbf{M}$ with $\mathbf{G}$ as the group of units such that the closure of $T_{\max}$ in $\mathbf{M}$ is $\overline{T}_{\max}$. Moreover, if $\overline{T}_{\max}$ is a monoid with zero, then also $\mathbf{M}$ is a monoid with zero.
It turns out, and this is the result due to Rittatore in [Rit98], that the class of reductive monoids with zero is contained in a larger class of *Kempf monoids*. By definition a geometrically integral algebraic monoid $\mathbf{M}$ with zero $\mathbf{o}$ is *a Kempf monoid* if there exists a central torus T inside the group of units of $\mathbf{M}$ such that its closure $\mathbf{cl}(T)$ in $\mathbf{M}$ contains $\mathbf{o}$. Representations of $\mathbf{M}$ are more tractable due to existence of the central torus T, which is linearly reductive

and hence admits semisimple category of representations. Moreover, $\mathbf{M}$ is determined by the formal neighborhood of its zero (Theorem 5.5.8).
In the remaining part of this section and in the next section we fix a Kempf monoid $\mathbf{M}$ and its group of units $\mathbf{G}$. For every $n \in \mathbb{N}$ let $\mathbf{M}_n \hookrightarrow \mathbf{M}$ be an n-th infinitesimal neighborhood of the zero $\mathbf{o}$ in $\mathbf{M}$.

1.2.2 Formal Białynicki-Birula functor

Let X be a k-scheme equipped with an action of $\mathbf{G}$. For every k-scheme Y we define

$$\widehat{\mathcal{D}}_X(Y) = \left\{ \{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \,\middle|\, \forall_{n\in\mathbb{N}}\, \gamma_n \text{ is } \mathbf{G}\text{-equivariant and } \gamma_{n+1|\mathbf{M}_n\times_k Y} = \gamma_n \right\}$$

This gives rise to a functor $\widehat{\mathcal{D}}_X$, which may be intuitively viewed as a formal-geometric version of $\mathcal{D}_X$. It turns out that the representability of $\widehat{\mathcal{D}}_X$ reduces easily to the algebraization in formal $\mathbf{M}$-equivariant geometry. Namely we consider formal $\mathbf{M}$-schemes, i.e., formal schemes ([FGI05, 8.1.3.2])

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \ldots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \ldots$$

such that each Z_n is equipped with action of a monoid k-scheme $\mathbf{M}$, all closed immersions $Z_n \hookrightarrow Z_{n+1}$ are $\mathbf{M}$-equivariant and $Z_n^{\mathbf{M}} = Z_0$ for every $n \in \mathbb{N}$. For every k-scheme Z with an action of $\mathbf{M}$ the sequence of infinitesimal neighborhoods $\widehat{Z}$ of fixed points $Z^{\mathbf{M}}$ in Z is an example of a formal $\mathbf{M}$-scheme. It turns out that every formal $\mathbf{M}$-scheme is of this form. This result takes form of an equivalence of categories (Corollary 6.5.6) and is a consequence of the fact mentioned above that $\mathbf{M}$ is determined by $\{\mathbf{M}_n\}_{n\in\mathbb{N}}$. As a consequence we obtain that $\widehat{\mathcal{D}}_X$ is representable and affine over $X^{\mathbf{G}}$.

1.2.3 Coherent completeness and tannakian formalism

Functors $\mathcal{D}_X$ and $\widehat{\mathcal{D}}_X$ are related by the canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$. It is not difficult to prove that this map is a monomorphism of functors (Theorem 7.7.6). However, its surjectivity turn out to be a more subtle problem, since it is not clear how to recover topologically an element $\mathcal{D}_X$ out of a given element of $\widehat{\mathcal{D}}_X$. In order to explain this let us inspect the surjectivity of $\mathcal{D}_X(\operatorname{Spec} k) \to \widehat{\mathcal{D}}_X(\operatorname{Spec} k)$. A k-point of $\widehat{\mathcal{D}}_X$ is a sequence of morphisms $\{\mathbf{M}_n \to X\}_{n\in\mathbb{N}}$. All these morphisms have their images contained in the infinitesimal neighborhood of $X^{\mathbf{G}}$ and hence they contain information on the infinitesimal neighborhood of $X^{\mathbf{G}}$. If the map $\mathcal{D}_X(\operatorname{Spec} k) \to \widehat{\mathcal{D}}_X(\operatorname{Spec} k)$ is surjective, then the family $\{\mathbf{M}_n \to X\}_{n\in\mathbb{N}}$ can be lifted to a morphism $\mathbf{M} \to X$, which existence depends on the topology of X and this (at least in general) is not encapsulated by the infinitesimal neighborhood of $X^{\mathbf{G}}$. We prove that $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is surjective by the two step argument. Let Z be a scheme representing $\widehat{\mathcal{D}}_X$. Then Z is a locally noetherian scheme with an action of $\mathbf{M}$ such that Z can be covered by open affine $\mathbf{M}$-stable subschemes. It follows that for such Z the category of coherent $\mathbf{G}$-sheaves on Z is canonically equivalent with appropriately defined category of coherent $\mathbf{G}$-sheaves on a formal $\mathbf{M}$-scheme $\widehat{Z}$ consisting of the sequence of formal neighborhoods of fixed points $Z^{\mathbf{M}}$ of Z (Theorem 6.6.1). This type of phenomenon is called *coherent completeness* in [AHR20] and it resembles the celebrated Grothendieck's existence theorem ([FGI05, Theorem 8.4.2]). We derive from it that there exists a functor

$$\mathfrak{Coh}_{\mathbf{G}}(X) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$$

Secondly, according to the result due to Hall and Rydh (Theorem 7.5.1) or by preprint by Jelisiejew and the author (Theorem 7.10.2) there exists a canonical $\mathbf{G}$-equivariant morphism $Z \to X$ which induces the functor discussed above on categories of coherent $\mathbf{G}$-sheaves. Results of this type, which reconstruct a morphism of schemes $f : X \to Y$ (stacks, algebraic spaces) out of a certain monoidal functors $F : \mathfrak{Coh}(Y) \to \mathfrak{Coh}(X)$ in such a way that $f^* \simeq F$ in the category of functors, are called *tannakian formalisms* in this work. This is justified by the fact that classical Tannaka duality ([Mil17, Note 9.4]) can be interpreted as the reconstruction of an algebraic group $\mathbf{G}$ from its category of linear representations considered as a monoidal category over vector spaces. From the existence of this $\mathbf{G}$-equivariant morphism $Z \to X$ (or in other words the morphism $\widehat{\mathcal{D}}_X \to X$) one can deduce that each family $\{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}}$ of compatible $\mathbf{G}$-equivariant morphisms can be extended to a $\mathbf{G}$-equivariant morphism $\mathbf{M} \times_k Y \to X$ (Theorem 7.8.1). This is equivalent with the fact that the natural transformation $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is surjective on every level and from this Theorem A is inferred.

1.3 Main ideas involved in the proof of Theorem B

Theorem B is less demanding and its proof can be explained by referring to the notion of tubular neighborhoods. A tubular neighborhood in differential topology ([BJ82, Definition 12.10]) is a certain differentiable map from the normal bundle of a submanifold to the ambient manifold, which induces a diffeomorphism of the normal bundle with the neighborhood of the submanifold. For differentiable manifolds tubular neighborhoods always exist ([BJ82, Theorem 12.11]). In the world of schemes they exists affine locally under some additional smoothness assumptions. Now if $r_X : X^+ \to X^{\mathbf{G}}$ is a smooth morphism at $s_X(x)$, then in some affine neighborhood of $s_X(x)$ there exists a morphism from the normal bundle of the closed subscheme $s_X : X^{\mathbf{G}} \hookrightarrow X^+$ to X^+. This morphism is étale (it is an analogon of a tubular neighborhood). Moreover, one can construct this morphism as equivariant with respect to some toric submonoid of $\mathbf{M}$. Then by some result from formal $\mathbf{M}$-geometry (Theorem 7.9.1) one can prove that this morphism is an isomorphism and hence r_X is locally isomorphic to vector bundle, which is what Theorem B asserts.

1.4 Relation of this thesis to joint works of Jelisiejew and the author

Theorems A and B are fruits of the collaboration of Jelisiejew and the author ([JS19], [JS20]). Let us now explain how approach presented in this thesis deviates from the content of these two papers.

In [JS19] there is some stress on the notion of the formal $\mathbf{M}$-scheme, but formal geometry is not studied (due to the usual brevity of research papers) in a systematic way. In particular, that work does not contain coherent completeness. This lack is filled in the second paper [JS20], but coherent completeness is studied there somewhat out of the context of formal geometry. Chapter 6 of this work and its main results (Corollary 6.5.6 and Theorems 6.5.7, 6.6.1) are thought as an exhaustive and unified exposition of the theory of the formal $\mathbf{M}$-schemes for a Kempf monoid $\mathbf{M}$.

There is also a minor technical difference between coherent completeness studied in [JS20] and in this thesis. Here we get rid of the notion and usage of *Serre subcategories*. The reader

may judge, if this makes our presentation clearer than that of [JS20].
Moreover, there is a key difference between [JS19] and this work. The first relies on affine étale $\mathbf{G}$-equivariant neighborhoods obtained via the result of Alper, Hall and Rydh ([AHR20, Theorem 2.6]. This restricts the scope of generality of that paper to linearly reductive monoids. Here this was eliminated thanks to coherent completeness, tannakian formalism and properties of Kempf monoids. This makes Corollary 7.8.5 more general with respect to the class of algebraic monoids for which it holds than its counterpart [JS19, Theorem 6.17].
Thanks to an additional observation in the proof of Corollary 7.8.5 we were able to obtain a slightly stronger result than [JS20, Theorem 1.1]. Namely Theorem A is derived for schemes locally of finite type over k and this can be further refined to locally noetherian case if one accepts unpublished result ([JS20, Theorem A.1]). In contrast [JS20, Theorem 1.1] is restricted to the quasi-compact case. Here representability is formulated as the isomorphism between $\widehat{\mathcal{D}}_X$ and $\mathcal{D}_X$, which is the original approach of [JS19] and seems natural, but is not expressed explicitly in [JS20] (again due to brevity).
The proof (in the present thesis) of Theorem B relies on Theorem 7.9.1 and the concept of the tubular neighborhood known from differential geometry. This is significantly different from the original approach of [JS19], which was based on affine étale $\mathbf{G}$-equivariant neighborhoods, and [JS20], which does not refer to any results in formal geometry.
Let us also indicate that there are some interesting results of [JS19] and [JS20], which are not proved in this thesis, but can be obtained by means that are either introduced here or are based on [AHR20, Theorem 2.6]. In Section 7.10 we give an overview of these other results.

1.5 Overview

Let us now briefly introduce the reader to contents of the thesis. Since this work heavily relies on functorial language in the sense of Gabriel and Demazure [DG70], we decide to devote Chapter 2 entirely to introducing this linguistic framework. Next chapter is a short course on basic results concerning monoid and group k-schemes and their linear representations. All the results contained there are classical or of auxiliary character with respect to latter parts of the work. Chapters 4 and 5 are devoted to algebraic groups and monoids. We proved there or give references to proofs of all results concerning these vast subjects that we are going to use in the remaining part of this work. In particular, Kempf monoids are introduced and studied in the last section of Chapter 5. The last two chapters contain our main results and original contributions. In Chapter 6 we study formal $\mathbf{M}$-schemes and prove all the results concerning their algebraization. Technically it is the heart of this thesis. Then in Chapter 7, after introducing tannakian formalism, we use results of Chapter 6 to prove our main theorems according to the plan described in Sections 1.2 and 1.3.
For the readers convenience we included list of symbols at the beginning of this work. Moreover, each chapter starts with short introductory section in which we comment on its contents. We hope that these make the process of reading significantly easier.

Chapter 2

k-Functors And Presheaves

2.1 Introduction

This chapter is devoted to study k-functors and presheaves on $\mathbf{Sch}_k$. We introduce elements of the functorial language that will be used significantly in the following chapters. Mostly we follow first part of [DG70], but in order to make our presentation self-contained we introduce all notions and add some results from other sources. The reader may find this chapter tedious and formal. Since it is somewhat obvious that the category of k-functors i.e. the category of copresheaves on $\mathbf{Alg}_k$ and the category of presheaves on $\mathbf{Sch}_k$ have equivalent subcategories of Zariski sheaves, she may even find a bit irritating the fact that we express each notion in these two linguistically different, but geometrically equivalent, settings. Nevertheless this was the only route that we had found to present this material in a clear and complete way. On the other hand the fact that subcategories of sheaves with respect to Zariski topology in these two categories are equivalent is not entirely obvious result in the theory of sheaves.
Throughout this chapter we assume that k is a commutative ring.

2.2 k-functors

Definition 2.2.1. The category $\mathbf{Fun}(\mathbf{Alg}_k, \mathbf{Set})$ of copresheaves on $\mathbf{Alg}_k$ is called *the category of k-functors*.

Since $\mathrm{Spec} : \mathbf{Alg}_k^{\mathrm{op}} \to \mathbf{Aff}_k$ is an equivalence of categories, the category of k-functors is equivalent with the category of presheaves $\widehat{\mathbf{Aff}_k}$.
If $\mathfrak{X}$ and $\mathfrak{Y}$ are k-functors, then we denote by $\mathrm{Mor}_k(\mathfrak{X}, \mathfrak{Y})$ the class of morphisms $\mathfrak{X} \to \mathfrak{Y}$ of k-functors. If $\sigma : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of k-functors, then for every k-algebra A we denote by $\sigma^A : \mathfrak{X}(A) \to \mathfrak{Y}(A)$ the corresponding component of σ.
Let $\mathfrak{X}$ and $\mathfrak{Y}$ be A-functors for some k-algebra A. Then we denote by $\mathrm{Mor}_A(\mathfrak{X}, \mathfrak{Y})$ the class of morphisms of A-functors $\mathfrak{X} \to \mathfrak{Y}$. For every A-algebra B and a morphism $\sigma : \mathfrak{X} \to \mathfrak{Y}$ of A-functors we denote by $\mathfrak{X}_B$, $\mathfrak{Y}_B$, σ_B the restrictions $\mathfrak{X}_{|\mathbf{Alg}_B}$, $\mathfrak{Y}_{|\mathbf{Alg}_B}$, $\sigma_{|\mathbf{Alg}_B}$ of these entities to the category of B-algebras. We note the following result.

Fact 2.2.2. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be k-functors. Assume that A is a k-algebra, B is an A-algebra, C is an B-algebra. Then the composition of maps of classes*

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_B} \mathrm{Mor}_B(\mathfrak{X}_B, \mathfrak{Y}_B) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C, \mathfrak{Y}_C)$$

equals

$$\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A) \xrightarrow{\sigma \mapsto \sigma_C} \mathrm{Mor}_C(\mathfrak{X}_C, \mathfrak{Y}_C)$$

Definition 2.2.3. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be k-functors and suppose that for every k-algebra A the class $\mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$ is a set. We define

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) = \mathrm{Mor}_A(\mathfrak{X}_A, \mathfrak{Y}_A)$$

for every k-algebra A. This is a k-functor. Indeed, for every k-algebra A and A-algebra B we have a map

$$\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A) \ni \sigma \mapsto \sigma_B \in \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(B)$$

and according to Fact 2.2.2 these maps make $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ into a k-functor. The k-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ is called *a hom k-functor of $\mathfrak{X}$ and $\mathfrak{Y}$*.

Definition 2.2.4. Let $\mathfrak{X}$ be a k-functor and let A be a k-algebra. Then elements of $\mathfrak{X}(A)$ are called *A-points of $\mathfrak{X}$*.

We denote by $\mathbf{1}$ a k-functor that assigns to every k-algebra a set with one element. Then for every k-algebra A the restriction $\mathbf{1}_A$ is a terminal object in the category of A-functors.
Let $\mathfrak{X}$ be a k-functor. Suppose that A is a k-algebra and $x \in \mathfrak{X}(A)$. Then x determines a morphism $\mathbf{1}_A \to \mathfrak{X}_A$ that for every A-algebra B with structural morphism $f : A \to B$ sends the unique element of $\mathbf{1}_A(B)$ to $\mathfrak{X}(f)(x) \in \mathfrak{X}_A(B)$. This gives rise to a bijection

$$\mathfrak{X}(A) \simeq \mathrm{Mor}_A(\mathbf{1}_A, \mathfrak{X}_A)$$

natural in k-algebra A.

Definition 2.2.5. Let $\mathfrak{Z}, \mathfrak{X}, \mathfrak{Y}$ be k-functors and let $\sigma : \mathfrak{Z} \times \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Fix $z \in \mathfrak{Z}(A)$ for some k-algebra A. We denote by $i_z : \mathbf{1}_A \to \mathfrak{Z}_A$ the morphism of A-functors corresponding to z. Since $\mathbf{1}_A$ is a terminal A-functor, $\sigma_A \cdot (i_z \times 1_{\mathfrak{X}_A})$ is isomorphic to a morphism $\sigma_z : \mathfrak{X}_A \to \mathfrak{Y}_A$ of A-functors. We call σ_z *the slice of σ along z*.

Consider now k-functors $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ and assume that the internal hom $\mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ exists. Let $\sigma : \mathfrak{Z} \times \mathfrak{X} \to \mathfrak{Y}$ be a morphism. Then the family of maps

$$\mathfrak{Z}(A) \ni z \mapsto \sigma_z \in \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})(A)$$

give rise to a morphism $\tau : \mathfrak{Z} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X}, \mathfrak{Y})$ of k-functors. Indeed, for a morphism $f : A \to B$ of k-algebras and $z \in \mathfrak{Z}(A)$ we have

$$\sigma_B \cdot \left(i_{\mathfrak{Z}(f)(z)} \times 1_{\mathfrak{X}_B}\right) = \left(\sigma_A \cdot \left(i_z \times 1_{\mathfrak{X}_A}\right)\right)_B$$

and hence $\sigma_{\mathfrak{Z}(f)(z)} = (\sigma_z)_B$. This gives rise to a map Φ of classes

$$\mathrm{Mor}_k\left(\mathfrak{Z}\times\mathfrak{X},\mathfrak{Y}\right) \ni \sigma \mapsto \tau \in \mathrm{Mor}_k\big(\mathfrak{Z},\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})\big)$$

Consider next a morphism $\tau : \mathfrak{Z} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})$ of k-functors and define $\sigma : \mathfrak{Z}\times\mathfrak{X} \to \mathfrak{Y}$ by formula $\sigma^A(z,x) = \left(\tau^A(z)\right)^A(x)$ for every k-algebra A and points $z \in \mathfrak{Z}(A)$, $x \in \mathfrak{X}(A)$. Let $f : A \to B$ be a morphism of k-algebras. Then

$$\sigma^B\left(\mathfrak{Z}(f)(z),\mathfrak{X}(f)(x)\right) = \left(\tau^B\left(\mathfrak{Z}(f)(z)\right)\right)^B\left(\mathfrak{X}(f)(x)\right) = \left(\left(\tau^A(z)\right)_B\right)^B\left(\mathfrak{X}(f)(x)\right) =$$

$$= \left(\tau^A(z)\right)^B\left(\mathfrak{X}(f)(x)\right) = \mathfrak{Y}(f)\left(\left(\tau^A(z)\right)^A(x)\right) = \mathfrak{Y}(f)\left(\sigma^A(z,x)\right)$$

Thus $\sigma : \mathfrak{Z}\times\mathfrak{X} \to \mathfrak{Y}$ is a well defined morphism of k-functors. This gives rise to a map Ψ of classes

$$\mathrm{Mor}_k\big(\mathfrak{Z},\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})\big) \ni \tau \mapsto \sigma \in \mathrm{Mor}_k\left(\mathfrak{Z}\times\mathfrak{X},\mathfrak{Y}\right)$$

Theorem 2.2.6. *Let $\mathfrak{Z},\mathfrak{X},\mathfrak{Y}$ be k-functors and assume that the k-functor $\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})$ exists. Then maps Φ and Ψ are mutually inverse bijections and hence they induce a bijection*

$$\mathrm{Mor}_k\left(\mathfrak{Z}\times\mathfrak{X},\mathfrak{Y}\right) \simeq \mathrm{Mor}_k\big(\mathfrak{Z},\mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})\big)$$

Proof. Pick a morphism $\tau : \mathfrak{Z} \to \mathcal{M}\mathrm{or}_k(\mathfrak{X},\mathfrak{Y})$ of k-functors. Let A be a k-algebra and $z \in \mathfrak{Z}(A)$. Let us first prove that $\Psi(\tau)_z = \tau^A(z)$. Indeed, let $f : A \to B$ be a morphism of k-algebras and x be an element in $\mathfrak{X}(B)$. Then we have

$$\left(\Psi(\tau)_z\right)^B(x) = \Psi(\tau)^B\left(\mathfrak{Z}(f)(z),x\right) = \left(\tau^B\left(\mathfrak{Z}(f)(z)\right)\right)^B(x) = \left(\left(\tau^A(z)\right)_B\right)^B(x) = \left(\tau^A(z)\right)^B(x)$$

Hence $\Psi(\tau)_z = \tau^A(z)$ because B and a B-point x are arbitrary. Now we use this fact and obtain

$$\left(\Phi(\Psi(\tau))\right)^A(z) = \Psi(\tau)_z = \tau^A(z)$$

and hence $\Phi\cdot\Psi$ is the identity. On the other hand fix a morphism $\sigma : \mathfrak{Z}\times\mathfrak{X} \to \mathfrak{Y}$. Let A be a k-algebra and let $z \in \mathfrak{Z}(A)$, $x \in \mathfrak{X}(A)$ be points. Then

$$\left(\Psi\left(\Phi(\sigma)\right)\right)^A(z,x) = \left(\Phi(\sigma)^A(z)\right)^A(x) = \sigma_z^A(x) = \sigma^A(z,x)$$

Thus $\Psi\cdot\Phi$ is the identity map. Therefore, Φ and Ψ are mutually inverse bijections. □

2.3 Zariski local k-functors and Zariski sheaves

In this part we use the notion of a Grothendieck topology on a category. For this notion we refer the reader to [MM94, Chapter III, Section 2, Definition 1].

Definition 2.3.1. Let $\left\{f_i : X_i \to X\right\}_{i\in I}$ be a family of morphisms of k-schemes. We say that $\{f_i\}_{i\in I}$ is *a Zariski covering of* X if the following conditions are satisfied.

(1) For every $i \in I$ the morphism f_i is an open immersion of schemes.

(2) The morphism $\coprod_{i\in I} X_i \to X$ induced by $\left\{f_i\right\}_{i\in I}$ is surjective.

The collection of all Zariski coverings on $\mathbf{Sch}_k$ is a Grothendieck pretopology.

Definition 2.3.2. We call the Grothendieck topology generated by the pretopology consisting of Zariski coverings on $\mathbf{Sch}_k$ *the Zariski topology on* $\mathbf{Sch}_k$. A presheaf on $\mathbf{Sch}_k$ that is a sheaf with respect to Zariski topology on $\mathbf{Sch}_k$ is called *a Zariski sheaf*.

Let $\mathfrak{X}$ be a presheaf on the category of k-schemes. By [MM94, Chapter III, Section 4, Proposition 1] $\mathfrak{X}$ is a Zariski sheaf if and only if for every k-scheme X and for every Zariski covering $\{f_i : X_i \to X\}$ of X the diagram

$$\begin{tikzcd} \mathfrak{X}(X) \arrow[r, "\langle \mathfrak{X}(f_i)\rangle_{i\in I}"] & \prod_{i\in I} \mathfrak{X}(X_i) \arrow[r, shift left, "\langle \mathfrak{X}(f'_{ij})\cdot pr_i\rangle_{(i,j)}"] \arrow[r, shift right, "\langle \mathfrak{X}(f''_{ij})\cdot pr_j\rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I} \mathfrak{X}(X_i \times_X X_j) \end{tikzcd}$$

is a kernel of a pair of arrows, where for every $(i,j) \in I \times I$ morphisms f'_{ij} and f''_{ij} form a cartesian square

$$\begin{tikzcd} X_i \times_X X_j \arrow[r, "f''_{ij}"] \arrow[d, "f'_{ij}"'] & X_j \arrow[d, "f_j"] \\ X_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

Now we repeat these definitions for k-algebras and k-functors.

Definition 2.3.3. Let $\{f_i : A \to A_i\}_{i\in I}$ be a family of morphisms of k-algebras. We say that $\{f_i\}_{i\in I}$ is *a Zariski covering of A* if the following conditions are satisfied.

- **(1)** For every $i \in I$ the morphism $\operatorname{Spec} f_i$ is an open immersion of schemes.
- **(2)** The morphism $\coprod_{i\in I} \operatorname{Spec} A_i \to \operatorname{Spec} A$ induced by $\{\operatorname{Spec} f_i\}_{i\in I}$ is surjective.

The collection of all Zariski coverings on $\mathbf{Alg}_k$ induces on its opposite category $\mathbf{Aff}_k$ of affine k-schemes a Grothendieck pretopology.

Definition 2.3.4. We call the Grothendieck topology generated by the pretopology consisting of Zariski coverings on $\mathbf{Aff}_k$ *the Zariski topology on* $\mathbf{Aff}_k$. A k-functor that is a sheaf with respect to Zariski topology on $\mathbf{Aff}_k$ is called *a Zariski local k-functor*.

Let $\mathfrak{X}$ be a k-functor. Again by [MM94, Chapter III, Section 4, Proposition 1] $\mathfrak{X}$ is a Zariski local k-functor if and only if for every k-algebra A and for every Zariski covering $\{f_i : A \to A_i\}$ of A the diagram

$$\begin{tikzcd} \mathfrak{X}(A) \arrow[r, "\langle \mathfrak{X}(f_i)\rangle_{i\in I}"] & \prod_{i\in I} \mathfrak{X}(A_i) \arrow[r, shift left, "\langle \mathfrak{X}(f'_{ij})\cdot pr_i\rangle_{(i,j)}"] \arrow[r, shift right, "\langle \mathfrak{X}(f''_{ij})\cdot pr_j\rangle_{(i,j)}"'] & \prod_{(i,j)\in I\times I} \mathfrak{X}(A_i \otimes_A A_j) \end{tikzcd}$$

is a kernel of a pair of arrows, where for every $(i,j) \in I \times I$ morphisms f'_{ij} and f'_{ij} form a cocartesian square

$$\begin{tikzcd} A \arrow[r, "f_j"] \arrow[d, "f_i"'] & A_j \arrow[d, "f'_{ji}"] \\ A_i \arrow[r, "f'_{ij}"'] & A_i \otimes_A A_j \end{tikzcd}$$

Now we state the main result of this section.

Theorem 2.3.5. *Let*

$$\widehat{\mathbf{Sch}_k} \longrightarrow \text{the category of } k\text{-functors}$$

be the restriction of presheaves on $\mathbf{Sch}_k$ *to presheaves on* $\mathbf{Aff}_k$ *(k-functors) induced by the inclusion* $\mathbf{Aff}_k \hookrightarrow \mathbf{Sch}_k$*. Then it induces an equivalence of categories between Zariski sheaves on* $\mathbf{Sch}_k$ *and Zariski local k-functors.*

Proof. According to [GW10, Proposition 8.8] every representable functor $\widehat{\mathbf{Sch}_k}$ is a Zariski sheaf. This means that Zariski topology on $\mathbf{Sch}_k$ is *subcanonical*. Note that $\mathbf{Aff}_k$ is a full subcategory of $\mathbf{Sch}_k$ and if we consider $\mathbf{Sch}_k$ as a category equipped with Zariski topology, then $\mathbf{Aff}_k$ satisfies the assumptions of [MM94, Appendix, Section 4, Corollary 3] and the induced topology on $\mathbf{Aff}_k$ is the Zariski topology. Hence the assertion follows from [MM94, Appendix, Section 4, Corollary 3]. □

The notion of creation of limits and colimits ([ML98, Definition on page 112]) is essential to our discussion below. Recall that Yoneda embedding

$$\mathbf{Sch}_k \hookrightarrow \widehat{\mathbf{Sch}_k}$$

is full and faithful. Moreover, it creates limits. [GW10, Proposition 8.8] states that every representable functor $\widehat{\mathbf{Sch}_k}$ is a Zariski sheaf. Let

$$\mathbf{Sch}_k \xhookrightarrow{\mathfrak{P}} k\text{-functors}$$

be the functor defined by the composition of the Yoneda embedding and the restriction $\widehat{\mathbf{Sch}_k} \to \widehat{\mathbf{Aff}_k}$. This functor is full, faithful and creates limits and its image consists of Zariski local k-functors. Thus Theorem 2.3.5 and the discussion above imply that we have the following result.

Theorem 2.3.6. *There exists the commutative triangle of functors and categories*

$$\begin{tikzcd} \text{Zariski sheaves on } \mathbf{Sch}_k \arrow[rr, "\simeq"] & & \text{Zariski local } k\text{-functors} \\ & \mathbf{Sch}_k \arrow[ul, hook] \arrow[ur, hook] & \end{tikzcd}$$

where the horizontal functor is an equivalence, the left hand side functor is the Yoneda embedding and the right hand side functor is the restriction of $\mathfrak{P}$ to the category of Zariski local k-functors, which contains its essential image. In particular, both nonhorizontal functors in the diagram are full, faithful and create limits.

Definition 2.3.7. Let X be a k-scheme. Then the image of X under $\mathfrak{P}$ is a k-functor given by formula

$$\mathbf{Alg}_k \ni A \mapsto \mathrm{Mor}_k(\operatorname{Spec} A, X) \in \mathbf{Set}$$

We call this k-functor *the functor of points of X*.

Remark 2.3.8. By means of identifications in Theorem 2.3.6 we do not make any formal and notational distinction between k-scheme X and its functor of points. In particular, we denote by X the functor of points of a k-scheme X. According to the same result we also do not distinguish between functor of points as a Zariski local k-functor and as a Zariski sheaf on $\mathbf{Sch}_k$.

Definition 2.3.9. Let $\mathfrak{X}$ be a k-functor (or presheaf on $\mathbf{Sch}_k$). We say that $\mathfrak{X}$ is *representable* or is *a scheme* if it is a functor of points of some k-scheme.

Finally let us observe that:

Fact 2.3.10. *Let X, Y be k-schemes. Then $\mathcal{M}\mathrm{or}_k(X, Y)$ exists.*

Proof. Fix a k-algebra A and observe that the class $\mathrm{Mor}_A(X_A, Y_A)$ of natural transformations (morphisms of A-functors) is in bijective correspondence (via Yoneda lemma) with the set of morphisms $\mathrm{Mor}_A(\operatorname{Spec} A \times_k X, \operatorname{Spec} A \times_k Y)$ of A-schemes. □

2.4 Closed, open k-subfunctors and criterion for representability

Suppose now that A is a k-algebra and $\mathfrak{a} \subseteq A$ is an ideal. Then we define $V(\mathfrak{a}) = \operatorname{Spec} A/\mathfrak{a}$ as a closed subscheme $\operatorname{Spec} A$ induced by the quotient morphism $A \to A/\mathfrak{a}$. We define an open subscheme $D(\mathfrak{a}) = \operatorname{Spec} A \setminus V(\mathfrak{a})$ of $\operatorname{Spec} A$.

Definition 2.4.1. Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Assume that for every k-algebra A and every morphism $\tau : \operatorname{Spec} A \to \mathfrak{Y}$ of k-functors there exists an ideal $\mathfrak{a}$ in A and a morphism $\tau' : D(\mathfrak{a}) \to \mathfrak{X}$ of k-functors such that the square

$$\begin{tikzcd} D(\mathfrak{a}) \arrow[r, "\tau'"] \arrow[d, hook] & \mathfrak{X} \arrow[d, "\sigma"] \\ \operatorname{Spec} A \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

is cartesian. Then σ is *an open immersion of k-functors*.

Definition 2.4.2. Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of k-functors. Assume that for every k-algebra A and every morphism $\tau : \operatorname{Spec} A \to \mathfrak{Y}$ of k-functors there exist an ideal $\mathfrak{a}$ in A and morphism $\tau' : V(\mathfrak{a}) \to \mathfrak{X}$ such that the square

$$\begin{tikzcd} V(\mathfrak{a}) \arrow[r, "\tau'"] \arrow[d, hook] & \mathfrak{X} \arrow[d, "\sigma"] \\ \operatorname{Spec} A \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

is cartesian. Then σ is *a closed immersion of k-functors*.

Now for completeness we state the analogical definitions for presheaves on $\mathbf{Sch}_k$.

Definition 2.4.3. Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of presheaves on $\mathbf{Sch}_k$. Assume that for every k-scheme Y and every morphism $\tau : Y \to \mathfrak{Y}$ of presheaves on $\mathbf{Sch}_k$ there exist an open subscheme $X \hookrightarrow Y$ and a morphism $\tau' : X \to \mathfrak{X}$ of presheaves such that the square

$$\begin{tikzcd} X \arrow[r, "\tau'"] \arrow[d, hook] & \mathfrak{X} \arrow[d, "\sigma"] \\ Y \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

is cartesian. Then σ is *an open immersion of presheaves on* $\mathbf{Sch}_k$.

Definition 2.4.4. Let $\sigma : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of presheaves on $\mathbf{Sch}_k$. Assume that for every k-scheme Y and every morphism $\tau : Y \to \mathfrak{Y}$ of presheaves on $\mathbf{Sch}_k$ there exist a closed subscheme $X \hookrightarrow Y$ and a morphism $\tau' : X \to \mathfrak{X}$ of presheaves such that the square

$$\begin{tikzcd} X \arrow[r, "\tau'"] \arrow[d, hook] & \mathfrak{X} \arrow[d, "\sigma"] \\ Y \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

is cartesian. Then σ is *a closed immersion of presheaves on* $\mathbf{Sch}_k$.

We make an easy observation.

Fact 2.4.5. *The class of open (closed) immersions of k-functors (presheaves on* $\mathbf{Sch}_k$*) is closed under base change and composition.*

Now we define open covers.

Definition 2.4.6. Let $\mathfrak{X}$ be a k-functor and $\{\sigma_i : \mathfrak{X}_i \to \mathfrak{X}\}_{i\in I}$ be a family of open immersions. Then for every k-algebra A and $x \in \mathfrak{X}(A)$ we have a family of ideals $\{\mathfrak{a}_i\}_{i\in I}$ defined by cartesian squares

$$\begin{tikzcd} D(\mathfrak{a}_i) \arrow[r, "\tau'"] \arrow[d, hook] & \mathfrak{X}_i \arrow[d, "\sigma_i"] \\ \operatorname{Spec} A \arrow[r, "\tau"'] & \mathfrak{X} \end{tikzcd}$$

in which bottom vertical morphism $\tau : \operatorname{Spec} A \to \mathfrak{X}$ corresponds to x. We say that $\{\sigma_i\}_{i\in I}$ is *an open cover of* $\mathfrak{X}$ if for every k-algebra A and $x \in \mathfrak{X}(A)$ we have

$$\operatorname{Spec} A = \bigcup_{i\in I} D(\mathfrak{a}_i)$$

or in other words $A = \sum_{i\in I} \mathfrak{a}_i$.

Definition 2.4.7. Let $\mathfrak{X}$ be a presheaf on $\mathbf{Sch}_k$ and $\left\{\sigma_i : \mathfrak{X}_i \to \mathfrak{X}\right\}_{i\in I}$ be a family of open immersions of presheaves. Then for every k-scheme X and $x \in \mathfrak{X}(X)$ we have a family of open subschemes $\{X_i\}_{i\in I}$ of X defined by cartesian squares

$$\begin{tikzcd} X_i \arrow[r, "\tau'"] \arrow[d, hook] & \mathfrak{X}_i \arrow[d, "\sigma_i"] \\ X \arrow[r, "\tau"'] & \mathfrak{X} \end{tikzcd}$$

in which bottom vertical morphism $\tau : X \to \mathfrak{X}$ corresponds to x. We say that $\{\sigma_i\}_{i\in I}$ is *an open cover of* $\mathfrak{X}$ if for every k-scheme X and $x \in \mathfrak{X}(X)$ we have

$$X = \bigcup_{i\in I} X_i$$

Remark 2.4.8. Equivalence described in Theorem 2.3.5 identifies the class of open (closed) immersions of Zariski local k-functors on the one hand and the class of open (closed) immersions of Zariski sheaves on $\mathbf{Sch}_k$ on the other. Moreover, the equivalence preserves and reflects open covers. We will not need this result, but it is worth noting.

These notions are intertwined in the following elementary yet beautiful result.

Theorem 2.4.9. *Let $\mathfrak{X}$ be (a k-functor) a presheaf on $\mathbf{Sch}_k$. Then the following are equivalent.*

- **(i)** *$\mathfrak{X}$ is representable.*
- **(ii)** *$\mathfrak{X}$ is (a Zariski local k-functor) a Zariski sheaf on $\mathbf{Sch}_k$ and there exists an open cover*

  $$\left\{\sigma_i : X_i \to \mathfrak{X}\right\}_{i\in I}$$

  by k-schemes.

Proof. The case for presheaves on $\mathbf{Sch}_k$ is [GW10, Theorem 8.9]. The case for k-functors is [DG70, page 18, Théoréme de comparaison, part b] (note that the authors define scheme as a k-functor satisfying **(ii)**). Moreover, according to Remark 2.4.8 the two cases considered above is really a single theorem expressed in different languages. □

Proposition 2.4.10. *Let $\sigma : \mathfrak{X} \hookrightarrow \mathfrak{Y}$ be a monomorphism of k-functors and let $\mathfrak{Y}$ be a Zariski local k-functor. Assume that for every k-algebra A and every morphism $\tau : \operatorname{Spec} A \to \mathfrak{Y}$ of k-functors there exist a Zariski local k-functor $\mathfrak{Z}$ that fits into a cartesian square*

$$\begin{tikzcd} \mathfrak{Z} \arrow[r] \arrow[d, hook] & \mathfrak{X} \arrow[d, hook, "\sigma"] \\ \operatorname{Spec} A \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

Then $\mathfrak{X}$ is a Zariski local k-functor.

Proof. Let $\{f_i : A \to A_i\}_{i \in I}$ be a Zariski covering of a k-algebra A. For every pair i, j of elements of I consider a cocartesian square

$$\begin{tikzcd} A \arrow[r, "f_j"] \arrow[d, "f_i"'] & A_j \arrow[d, "f'_{ji}"] \\ A_i \arrow[r, "f'_{ij}"'] & A_i \otimes_A A_j \end{tikzcd}$$

Now assume that for every i there we are given an element $x_i \in \mathfrak{X}(A_i)$ such that for every pair $i, j \in I$ we have

$$\mathfrak{X}(f'_{ij})(x_i) = \mathfrak{X}(f'_{ji})(x_j)$$

Our goal is to show that there exists a unique A-point $x \in \mathfrak{X}(A)$ such that $\mathfrak{X}(f_i)(x) = x_i$. For this observe first that the family $y_i = \sigma^{A_i}(x_i)$ for $i \in I$ satisfies

$$\mathfrak{Y}(f'_{ij})(y_i) = \mathfrak{Y}(f'_{ji})(y_j)$$

and since $\mathfrak{Y}$ is a Zariski local k-functor, there exists a unique $y \in \mathfrak{Y}(A)$ such that $\mathfrak{Y}(f_i)(y) = y_i$. Let $\tau : \operatorname{Spec} A \to \mathfrak{Y}$ be a morphism of k-functors determined by $y \in \mathfrak{Y}(A)$. There exists a Zariski local k-functor $\mathfrak{Z}$ that fits into a cartesian square

$$\begin{tikzcd} \mathfrak{Z} \arrow[r, "\tau'"] \arrow[d, hook, "\sigma'"'] & \mathfrak{X} \arrow[d, hook, "\sigma"] \\ \operatorname{Spec} A \arrow[r, "\tau"'] & \mathfrak{Y} \end{tikzcd}$$

of k-functors. Since the square is cartesian, we derive that for each $i \in I$ there exists $z_i \in \mathfrak{Z}(A_i)$ such that $\tau'^{A_i}(z_i) = x_i$ and $\sigma'^{A_i}(z_i) = \operatorname{Spec} f_i$. Moreover, we have

$$\mathfrak{Z}(f'_{ij})(z_i) = \mathfrak{Z}(f'_{ji})(z_j)$$

for every pair $i, j \in I$. Since $\mathfrak{Z}$ is a Zariski local k-functor, there exists a unique $z \in \mathfrak{Z}(A)$ such that $\mathfrak{Z}(f_i)(z) = z_i$. Then $x = \tau'^{A}(z)$ is a unique element such that $\mathfrak{X}(f_i)(x) = x_i$. $\square$

Corollary 2.4.11. *Let X be a k-scheme and let $j : \mathfrak{X} \hookrightarrow X$ be a closed (open) immersion of k-functors (respectively). Then $\mathfrak{X}$ is a k-scheme and $j : \mathfrak{X} \hookrightarrow X$ is a closed (open) immersion of k-schemes (respectively).*

Proof. Proposition 2.4.10 implies that $\mathfrak{X}$ is a Zariski local k-functor. Consider a Zariski covering $\{f_i : \operatorname{Spec} A_i \to X\}_{i \in I}$ of X. For each $i \in I$ consider the cartesian square

$$\begin{tikzcd} \mathfrak{X}_i \arrow[r, "f_i'"] \arrow[d, hook, "j_i"'] & \mathfrak{X} \arrow[d, hook, "j"] \\ \operatorname{Spec} A_i \arrow[r, "f_i"'] & X \end{tikzcd}$$

of k-functors. Then each $\mathfrak{X}_i$ is a closed (open) subscheme of $\operatorname{Spec} A_i$ (this follows by definition) and $\{f_i' : \mathfrak{X}_i \to \operatorname{Spec} A_i\}_{i \in I}$ is an open cover of $\mathfrak{X}$ by k-schemes. Since $\mathfrak{X}$ is a Zariski local k-functor, Theorem 2.4.9 implies that $\mathfrak{X}$ is representable. The fact that $j : \mathfrak{X} \hookrightarrow X$ is a closed (open) immersion of k-schemes follows from the fact that closed (open) immersions of k-schemes are local on the base and j is a morphism that after base change to an affine open k-scheme is a closed (open) immersion. □

2.5 Closed immersions and hom k-functors

We close this chapter by stating important and nontrivial theorem relating hom k-functors and closed immersions. For this we need to introduce the following notion.

Definition 2.5.1. Let X be a k-scheme. Suppose that there exists an open affine cover $X = \bigcup_{i \in I} X_i$ such that k-algebra $\Gamma(X_i, \mathcal{O}_{X_i})$ is free as a k-module. Then we say that X is *a locally free k-scheme*.

Theorem 2.5.2. *Let $j : \mathfrak{Y}' \hookrightarrow \mathfrak{Y}$ be a closed immersion of k-functors and let X be a locally free k-scheme. Suppose that the classes $\operatorname{Mor}_A(X_A, \mathfrak{Y}_A)$ are sets for every k-algebra A. Then the classes $\operatorname{Mor}_A(X_A, \mathfrak{Y}'_A)$ are sets for every k-algebra A and the morphism*

$$\mathcal{M}\mathrm{or}_k(1_X, j) : \mathcal{M}\mathrm{or}_k(X, \mathfrak{Y}') \to \mathcal{M}\mathrm{or}_k(X, \mathfrak{Y})$$

is a closed immersion of k-functors.

Proof. Note that j is a monomorphism of k-functors and hence for every k-algebra j_A is a monomorphism of A-functors. Thus for every k-algebra A the map of classes

$$\operatorname{Mor}_A(1_{X_A}, j_A) : \operatorname{Mor}_A(X_A, \mathfrak{Y}'_A) \to \operatorname{Mor}_A(X_A, \mathfrak{Y}_A)$$

is injective. This proves that $\operatorname{Mor}_A(X_A, \mathfrak{Y}'_A)$ is a set provided that $\operatorname{Mor}_A(X_A, \mathfrak{Y}_A)$ is a set. The main result that

$$\mathcal{M}\mathrm{or}_k(1_X, j) : \mathcal{M}\mathrm{or}_k(X, \mathfrak{Y}') \to \mathcal{M}\mathrm{or}_k(X, \mathfrak{Y})$$

is a closed immersion of k-functors is [DG70, page 64, Proposition 7.5]. □

Remark 2.5.3. If k is a field, then every k-scheme is locally free.

Chapter 3

Monoid k-Schemes And Their Linear Representations

3.1 Introduction

In this chapter we introduce monoid k-schemes and their linear representations. We discuss basic properties of the category of linear representations of affine monoid k-schemes. In our presentation we closely follow the first chapters of the second part of [DG70]. We add some material concerning the category of comodules over coalgebras and comparison between linear representations of an affine monoid k-scheme and its group of units provided that the canonical inclusion of units into a monoid is a schematically dense open immersion. As we shall see later this comparison plays a fundamental role in the proof of representability of Białynicki-Birula functors in the affine case (Theorem 7.8.1). For reader's convenience we also recall with complete proofs facts concerning completely reducible representations and isotypic components. In the last section we prove that quasi-coherent $\mathbf{G}$-sheaves on locally linear $\mathbf{G}$-schemes can be described in terms of representations of $\mathbf{G}$. In Remark 3.12.4 we include (what we conceive to be) an intuitive explanation of the notion of $\mathbf{G}$-sheaf.

3.2 Monoid k-functors and monoid k-schemes

We assume that the reader is familiar with notions of a monoid, group etc. in an arbitrary category with finite products. For definitions and some discussion related to these notions cf. [ML98, pages 2-5].

Definition 3.2.1. *A monoid (group, commutative group) k-functor* is a monoid (group, commutative group) object in the category of k-functors.

It is useful to note that monoid (group, commutative group) k-functor structures on a given k-functor $\mathfrak{X}$ are in bijective correspondence with lifts of $\mathfrak{X}$ to the category **Mon** (**Grp**, **Ab**) of monoids (groups, commutative groups). That is a structure of a monoid k-functor on $\mathfrak{X}$ is the same as the following commutative triangle of categories and functors.

$$\begin{tikzcd} & \mathbf{Mon} \arrow[d, "|-|"] \\ \mathbf{Alg}_k \arrow[ur] \arrow[r, "\mathfrak{X}"'] & \mathbf{Set} \end{tikzcd}$$

Here $|-|: \mathbf{Mon} \to \mathbf{Set}$ denotes the forgetful functor which sends each monoid to its underlying set. Analogical descriptions holds for groups and commutative groups.

Definition 3.2.2. Let $\mathfrak{M}$ be a monoid k-functor. A morphism $\mathbf{o}: \mathbf{1} \to \mathfrak{M}$ of k-functors is called *the zero of* $\mathfrak{M}$ if for every k-algebra A the unique element $\mathbf{o}_A$ in the image of $\mathbf{o}^A: \mathbf{1}(A) \to \mathfrak{M}(A)$ satisfies

$$\mathbf{o}_A \cdot m = \mathbf{o}_A = m \cdot \mathbf{o}_A$$

for every $m \in \mathfrak{M}(A)$.

Definition 3.2.3. Let $\mathfrak{M}$ be a monoid k-functor. Then we denote by $\mathfrak{M}^*$ the k-subfunctor of $\mathfrak{M}$ defined by

$$\mathfrak{M}^*(A) = \mathfrak{M}(A)^*$$

for every k-algebra A. This is a group k-functor. We call $\mathfrak{M}^*$ *the unit group k-functor of* $\mathfrak{M}$.

Definition 3.2.4. *A monoid (group) k-scheme* $\mathbf{M}$ is a monoid (group) object in the category of k-schemes. If $\mathbf{M}$ is affine, then we say that $\mathbf{M}$ is *an affine monoid (group) k-scheme*.

Corollary 3.2.5. *The functor*

$$\mathbf{Sch}_k \xrightarrow{\mathfrak{P}} k\text{-functors}$$

induces an equivalence of categories

$$\text{monoid } k\text{-schemes} \simeq \text{monoid } k\text{-functors representable by } k\text{-schemes}$$

Analogically for categories of groups and commutative groups.

Proof. This follows from the fact that $\mathfrak{P}$ is full, faithful and creates limits (in particular, it preserves and creates products). □

Proposition 3.2.6. *Let* $\mathbf{M}$ *be a monoid k-scheme. Then the group k-functor of units* $\mathbf{M}^*$ *of* $\mathbf{M}$ *is representable. If* $\mathbf{M}$ *is affine, then* $\mathbf{M}^*$ *is representable by an affine k-scheme.*

Proof. Note that $\mathbf{M}^*$ fits into a cartesian square of k-functors

$$\begin{tikzcd} \mathbf{M}^* \arrow[r] \arrow[d] & \mathbf{1} \arrow[d, "e"] \\ \mathbf{M} \times \mathbf{M} \arrow[r, "\mu"'] & \mathbf{M} \end{tikzcd}$$

where $\mu : \mathbf{M} \times \mathbf{M} \to \mathbf{M}$ is the multiplication and $e : \mathbf{1} \to \mathbf{M}$ is the unit. The functor $\mathfrak{P}$ creates limits and hence it creates fiber-products. This implies that $\mathbf{M}^*$ is represented by a unique (up to an isomorphism) k-scheme $\mathbf{M}^*$ that fit into a cartesian square of k-schemes below.

$$\begin{tikzcd} \mathbf{M}^* \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "e"] \\ \mathbf{M} \times_k \mathbf{M} \arrow[r, "\mu"'] & \mathbf{M} \end{tikzcd}$$

Note that if $\mathbf{M}$ is affine, then, since the diagram of k-schemes above is cartesian and affine k-schemes are closed under fiber-products, also $\mathbf{M}^*$ is affine. □

Definition 3.2.7. Let $\mathbf{M}$ be a monoid k-scheme. Then the group k-scheme $\mathbf{M}^*$ is called *the group of units of* $\mathbf{M}$.

Finally we employ the opposite monoid functor.

Definition 3.2.8. Let $(-)^{\mathrm{op}} : \mathbf{Mon} \to \mathbf{Mon}$ be the opposite monoid functor and let $\mathfrak{M}$ be a monoid k-functor. Then the composition $\mathfrak{M}^{\mathrm{op}} = (-)^{\mathrm{op}} \cdot \mathfrak{M}$ is called *the opposite monoid k-functor of* $\mathfrak{M}$.

Let us note the following elementary result.

Fact 3.2.9. *Let $\mathfrak{G}$ be a group k-functor. Then a morphism $\mathfrak{G} \to \mathfrak{G}^{\mathrm{op}}$ given by formula*

$$\mathfrak{G}(A) \ni g \mapsto g^{-1} \in \mathfrak{G}(A)$$

for k-algebra A is an isomorphism of group k-functors.

3.3 Bialgebras and affine monoid k-schemes

We start here with the general notion of k-coalgebras.

Definition 3.3.1. Let (C, Δ, ξ) be a triple consisting of a module C over k and morphisms

$$\Delta : C \to C \otimes_k C, \xi : C \to k$$

of k-modules such that the following diagrams are commutative.

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes_k C \arrow[d, "1_C \otimes_k \Delta"] \\ C \otimes_k C \arrow[r, "\Delta \otimes_k 1_C"'] & C \otimes_k C \otimes_k C \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "\simeq"'] & C \otimes_k C \arrow[d, "1 \otimes_k \xi"] \\ & C \otimes_k k \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\Delta"] \arrow[dr, "\simeq"'] & C \otimes_k C \arrow[d, "\xi \otimes_k 1_C"] \\ & k \otimes_k C \end{tikzcd}$$

Then (C, Δ, ξ) is called *a k-coalgebra*. Morphisms Δ, ξ are called *a comultiplication* and *a counit*, respectively.

Definition 3.3.2. Let (C_1, Δ_1, ξ_1) and (C_2, Δ_2, ξ_2) are k-coalgebras. Then a morphism $f : C_1 \to C_2$ of k-modules is *a morphism of k-coalgebras* if the following diagrams are commutative.

$$\begin{tikzcd} C_1 \otimes_k C_1 \arrow[r, "f \otimes_k f"] & C_2 \otimes_k C_2 \\ C_1 \arrow[u, "\Delta_1"] \arrow[r, "f"'] & C_2 \arrow[u, "\Delta_2"'] \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[rr, "f"] \arrow[dr, "\xi_1"'] & & C_2 \arrow[dl, "\xi_2"] \\ & k & \end{tikzcd}$$

Definition 3.3.3. Let B be a k-module with structures of both k-algebra and k-coalgebra. Assume that the comultiplication $B \to B \otimes_k B$ and the counit $B \to k$ of B are morphisms of k-algebras. Then we say that B with these structures is *a k-bialgebra*.

Definition 3.3.4. Let B_1, B_2 be k-bialgebras and let $f : B_1 \to B_2$ be a morphism of k-modules. We say that f is *a morphism of k-bialgebras* if it is simultaneously morphism of k-algebras and k-coalgebras.

Theorem 3.3.5. *The functor* Spec : $\mathbf{Alg}_k \to \mathbf{Sch}_k$ *induces an equivalence of categories*

$$k\text{-bialgebras} \simeq \text{the category of affine monoid } k\text{-schemes}$$

Proof. This is an exercise in translation. For details see [DG70, Proposition on page 146]. □

3.4 Examples of monoid k-functors

In this section we introduce several examples of monoid and group k-functors.

Example 3.4.1. Consider the monoid k-functor that sends each k-algebra A into its additive group A_+. This defines a commutative group k-functor. Since this k-functor is representable by an affine line $\operatorname{Spec} k[x]$, we derive by Corollary 3.2.5 that $\operatorname{Spec} k[x]$ carries a structure of a commutative group k-scheme. We denote this commutative group k-scheme by $\mathbb{G}_a$ and call it *the additive group scheme over k*.

Example 3.4.2. Consider the monoid k-functor that sends each k-algebra A into its multiplicative monoid $A^{\times}$. This defines a commutative monoid k-functor. Again as in Example 3.4.1 this k-functor is representable by $\operatorname{Spec} k[x]$ and hence by Corollary 3.2.5 $\operatorname{Spec} k[x]$ carries a structure of a commutative monoid k-scheme. We denote this commutative monoid k-scheme by $\mathbb{A}^1_k$ and call it *the affine line over k*. Note that $\mathbb{A}^1_k$ is the monoid k-scheme with zero.

Example 3.4.3. Consider the monoid k-functor that sends each k-algebra A into its multiplicative group of invertible elements A^*. This defines a commutative group k-functor. This k-functor is representable by $\operatorname{Spec} k[x, x^{-1}]$. As above we derive by Corollary 3.2.5 that this k-scheme carries a structure of a commutative group k-scheme. We denote this commutative group k-scheme by $\mathbb{G}_m$ and call it *the multiplicative group scheme over k*.

Definition 3.4.4. Let $\mathfrak{M}$ be a monoid k-functor. Then a morphism $\mathfrak{M} \to \mathbb{A}^1_k$ of monoid k-functors is called *a character of* $\mathfrak{M}$.

Remark 3.4.5. Note that $\mathbb{G}_m$ is the group of units of $\mathbb{A}^1_k$ and the canonical monomorphism $\mathbb{G}_m \hookrightarrow \mathbb{A}^1_k$ is an open immersion. In particular, it follows that if $\mathfrak{G}$ is a group k-functor, then a character of $\mathfrak{G}$ is a morphism $\mathfrak{G} \to \mathbb{G}_m$ of group k-functors.

The next example plays a fundamental role in theory of linear representations of monoid k-functors.

Example 3.4.6. Let V be a k-module. We define a k-functor $\mathcal{L}_V$. We set

$$\mathcal{L}_V(A) = \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V) = \mathrm{Hom}_k(V, A \otimes_k V)$$

for every k-algebra A. Next for every morphism $f : A \to B$ of k-algebras and every morphism $\phi : A \otimes_k V \to A \otimes_k V$ of A-modules we define $\mathcal{L}_V(f)(\phi)$ as a unique morphism of B-modules such that the diagram

$$\begin{tikzcd} A \otimes_k V \arrow[r, "\phi"] \arrow[d, "f \otimes_k 1_V"'] & A \otimes_k V \arrow[d, "f \otimes_k 1_V"] \\ B \otimes_k V \arrow[r, "\mathcal{L}_V(\phi)"'] & B \otimes_k V \end{tikzcd}$$

is commutative. Note also that for fixed k-algebra A the set of endomorphisms $\mathcal{L}_V(A)$ of A-module $A \otimes_k V$ is a monoid with respect to the usual composition of endomorphisms. Moreover, if $f : A \to B$ is a morphism of k-algebras, then $\mathcal{L}_V(f)$ is a morphism of such defined monoids. Thus $\mathcal{L}_V$ admits a structure of a monoid k-functor. We call it *the general linear monoid of* V. Note that $\mathcal{L}_V$ admits the zero given by the zero morphism $A \otimes_k V \to A \otimes_k V$ for every k-algebra A.

Remark 3.4.7. Suppose that V is a finitely generated, projective k-module. Then for each k-algebra A we have a chain of isomorphisms

$$\mathcal{L}_V(A) = \mathrm{Hom}_k(V, A \otimes_k V) \simeq A \otimes_k V^\vee \otimes_k V \simeq \mathrm{Hom}_A(A \otimes_k V \otimes_k V^\vee, A) \simeq$$

$$\simeq \mathrm{Hom}_k(V \otimes_k V^\vee, A) \simeq \mathrm{Mor}_k(\mathrm{Sym}(V \otimes_k V^\vee), A) \simeq \mathrm{Mor}_k(\mathrm{Spec}\, A, \mathrm{Spec}\,\mathrm{Sym}(V \otimes_k V^\vee))$$

Clearly these isomorphisms are natural in A. Hence $\mathcal{L}_V$ is representable and we denote the corresponding affine monoid k-scheme by $\mathbf{L}(V)$.

Example 3.4.8. For k-module V we define a group k-functor $\mathcal{GL}_V$ as the group of units of the general linear monoid $\mathcal{L}_V^*$ of V. We call this group k-functor *the general linear group of* V. Note that

$$\mathcal{GL}_V(A) = \mathrm{Aut}_A(A \otimes_k V, A \otimes_k V)$$

for every k-algebra A.

Remark 3.4.9. Suppose that V is a finitely generated, projective k-module. According to Proposition 3.2.6 and Remark 3.4.7 the group k-functor $\mathcal{GL}_V$ is representable. The corresponding affine group k-scheme is denoted by $\mathbf{GL}(V)$.

Now we give an important example of a morphism of monoid k-functors.

Example 3.4.10. Suppose that V is a finitely generated, projective k-module of constant rank n on $\mathrm{Spec}\, k$. Then the exterior product $\bigwedge^n V$ is a projective module of rank one. Thus for every k-algebra A each element of $\mathrm{Hom}_A\left(A \otimes_k \bigwedge^n V, A \otimes_k \bigwedge^n V\right)$ is the multiplication by some fixed element of A. This defines a morphism of monoids

$$\mathcal{L}_V(A) = \mathrm{Hom}_A(A \otimes_k V, A \otimes_k V) \xrightarrow{\phi \mapsto \bigwedge^n \phi} \mathrm{Hom}_A(A \otimes_k \bigwedge^n V, A \otimes_k \bigwedge^n V) \xrightarrow{\simeq} A^{\times}$$

which is natural in A. Thus we obtain the morphism of monoid k-functors

$$\mathcal{L}_V \xrightarrow{\det} \mathbb{A}^1_k$$

We call it *the determinant of* $\mathcal{L}_V$.

Remark 3.4.11. Suppose that V is a finitely generated, projective k-module of constant rank n on $\mathrm{Spec}\, k$. According to Example 3.4.10 there is the determinant $\det : \mathcal{L}_V \to \mathbb{A}^1_k$. Pick k-algebra A and recall that for $\phi \in \mathcal{L}_V(A)$ we have $\phi \in \mathcal{GL}_V(A)$ if and only if $\det(\phi) \in A^*$. Thus we have a cartesian square

$$\begin{tikzcd} \mathcal{GL}_V \arrow[r] \arrow[d, hook] & \mathbb{G}_m \arrow[d, hook] \\ \mathcal{L}_V \arrow[r, "\det"'] & \mathbb{A}^1_k \end{tikzcd}$$

where vertical morphisms are inclusion of units. Since the inclusion of units $\mathbb{G}_m \hookrightarrow \mathbb{A}^1_k$ is an open immersion (Remark 3.4.5), we derive by Fact 2.4.5 that $\mathcal{GL}(V)$ is an open subgroup k-functor of $\mathcal{L}_V$ or in the language of k-schemes $\mathbf{GL}(V) \hookrightarrow \mathbf{L}(V)$ is an open immersion.

Our last example is also related to determinants.

Example 3.4.12. The identity of $\mathbb{A}^1_k$ in terms of morphisms of k-functors is a closed immersion $\mathbf{1} \hookrightarrow \mathbb{A}^1_k$, where $\mathbf{1}$ is a terminal k-functor, that for each k-algebra A sends the unique element of $\mathbf{1}(A)$ to $1 \in A^{\times}$. Suppose that V is a finitely generated, projective k-module of constant rank n on $\mathrm{Spec}\, k$. Then a cartesian diagram

$$\begin{tikzcd} \mathcal{SL}_V \arrow[r] \arrow[d, hook] & \mathbf{1} \arrow[d, hook] \\ \mathcal{L}_V \arrow[r, "\det"'] & \mathbb{A}^1_k \end{tikzcd}$$

defines by Fact 2.4.5 a closed subgroup k-functor $\mathcal{SL}_V$ of $\mathcal{L}_V$. Since $\mathcal{L}_V$ is representable by $\mathbf{L}(V)$ (Remark 3.4.7), we derive that $\mathcal{SL}_V$ is representable by an affine group k-scheme $\mathbf{SL}(V)$ which is closed in $\mathbf{L}(V)$. We call it *the special linear group of* V.

The closing part of this section is devoted to opposite monoids of general linear monoids.

Fact 3.4.13. *Let V be a finitely generated, projective k-module. Then we have an identification of monoid k-functors*

$$\mathcal{L}_V^{\mathrm{op}} = \mathcal{L}_{V^\vee}$$

Proof. Since V is finitely generated and projective, there exists an anti-isomorphism of abstract monoids

$$\mathrm{Hom}_A\left(A \otimes_k V, A \otimes_k V\right) \ni \phi \mapsto \phi^\vee \in \mathrm{Hom}_A\left(\left(A \otimes_k V\right)^\vee, \left(A \otimes_k V\right)^\vee\right)$$

natural in k-algebra A. This means that $\mathcal{L}_V^{\mathrm{op}}$ can be identified with $\mathcal{L}_{V^\vee}$. □

3.5 Linear representations of monoid k-functors

Definition 3.5.1. Let $\mathfrak{M}$ be a monoid k-functor. A pair (V, ρ) consisting of a k-module V and a morphism $\rho : \mathfrak{M} \to \mathcal{L}_V$ of monoid k-functors is called *a linear representation of* $\mathfrak{M}$.

Remark 3.5.2. Observe that $\mathbf{L}(k) = \mathbb{A}_k^1$. Thus for a monoid functor $\mathfrak{M}$ the characters of $\mathfrak{M}$ are in bijective correspondence with the class of its representations having k as the underlying k-module.

Definition 3.5.3. Let $\mathfrak{M}$ be a monoid k-functor and let (V, ρ), (W, δ) be its linear representations. A morphism $\phi : V \to W$ of k-modules such that

$$(1_A \otimes_k \phi) \cdot \rho(m) = \delta(m) \cdot (1_A \otimes_k \phi)$$

for every k-algebra A and every $m \in \mathfrak{M}(A)$ is called *a morphism of linear representations of* $\mathfrak{M}$.

Let $\mathfrak{M}$ be a monoid k-functor. We denote by $\mathbf{Rep}(\mathfrak{M})$ the category of linear representations of $\mathfrak{M}$. This is an additive category.

Definition 3.5.4. Let (V, ρ) be a linear representation of a monoid k-functor $\mathfrak{M}$. Then the k-submodule

$$V^{\mathfrak{M}} = \left\{v \in V \,\middle|\, \rho(m)(1 \otimes v) = 1 \otimes v \text{ for every } k\text{-algebra } A \text{ and every } m \in \mathfrak{M}(A)\right\}$$

of V is called *the module of invariants of* (V, ρ).

If $\mathfrak{M}$ is a monoid k-functor, then the assignment $(V, \rho) \mapsto V^{\mathfrak{M}}$ gives rise to an additive functor

$$(-)^{\mathfrak{M}} : \mathbf{Rep}(\mathfrak{M}) \to \mathbf{Mod}(k)$$

Now we describe certain constructions concerning linear representations of monoid k-functors.

Example 3.5.5. Let (V_1, ρ_1) and (V_2, ρ_2) be linear representations of monoid k-functors $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively. Then we define a linear representation of $\mathfrak{M}_1 \times \mathfrak{M}_2$ with $V_1 \otimes_k V_2$ as the underlying k-module that corresponds to a morphism $\rho : \mathfrak{M}_1 \times \mathfrak{M}_2 \to \mathcal{L}_{V_1 \otimes_k V_2}$ of monoid k-functors given by

$$\rho\left(m_1, m_2\right) = \left(\rho_1(m_1) \otimes_A \rho_2(m_2) : A \otimes_k V_1 \otimes_k V_2 \to A \otimes_k V_1 \otimes_k V_2\right)$$

for $(m_1, m_2) \in \mathfrak{M}_1(A) \times \mathfrak{M}_2(A)$, where A is a k-algebra. This linear representation is called *the outer tensor product of* (V_1, ρ_1) *and* (V_2, ρ_2).

Example 3.5.6. Let (V_1, ρ_1) and (V_2, ρ_2) be linear representations of monoid k-functor $\mathfrak{M}$. Then we define a linear representation of $\mathfrak{M}$ with $V_1 \otimes_k V_2$ as the underlying k-module given as the composition of the outer tensor product of (V_1, ρ_1) and (V_2, ρ_2) with the diagonal $\mathfrak{M} \hookrightarrow \mathfrak{M} \times \mathfrak{M}$. Explicitly the corresponding morphism $\rho : \mathfrak{M} \to \mathcal{L}_{V_1 \otimes_k V_2}$ of monoid k-functors is given by

$$\rho(m) = \Big(\rho_1(m) \otimes_A \rho_2(m) : A \otimes_k V_1 \otimes_k V_2 \to A \otimes_k V_1 \otimes_k V_2\Big)$$

for $m \in \mathfrak{M}(A)$, where A is a k-algebra. This linear representation is called *the tensor product of* (V_1, ρ_1) *and* (V_2, ρ_2).

Remark 3.5.7. Since $k \otimes_k k \ni 1 \otimes 1 \mapsto 1 \in k$ is a canonical isomorphism, we derive by Remark 3.5.2 that for every monoid k-functor $\mathfrak{M}$ tensor product of two linear representation of $\mathfrak{M}$ both having k as the underlying module induces the structure of an abstract monoid on the class of characters of $\mathfrak{M}$.

Example 3.5.8. Let $\mathfrak{M}$ be a monoid k-functor, let V be k-module and let $\rho : \mathfrak{M} \to \mathcal{L}_V$ be a morphism of monoid k-functors. Suppose that V is a projective and finitely generated k-module. Fact 3.4.13 implies that morphism of a monoid k-functors $\rho^{\mathrm{op}} : \mathfrak{M}^{\mathrm{op}} \to \mathcal{L}_V^{\mathrm{op}}$ can be identified with some morphism $\rho^{\vee} : \mathfrak{M}^{\mathrm{op}} \to \mathcal{L}_{V^{\vee}}$. Hence a pair $(V^{\vee}, \rho^{\vee})$ is a linear representation of $\mathfrak{M}^{\mathrm{op}}$. We call it *the dual representation of* (V, ρ).

Example 3.5.9. Let (V_1, ρ_1) and (V_2, ρ_2) be linear representations of a monoid k-functor $\mathfrak{M}$. Suppose that V_1 is a finitely generated, projective k-module. Then we have an identification

$$\mathrm{Hom}_k(V_1, V_2) = V_1^{\vee} \otimes_k V_2$$

of k-modules. By Examples 3.5.5 and 3.5.8 this isomorphism makes $\mathrm{Hom}_k(V_1, V_2)$ into a linear representation of $\mathfrak{M}^{\mathrm{op}} \times \mathfrak{M}$.

Example 3.5.10. Let (V_1, ρ_1) and (V_2, ρ_2) be linear representations of a monoid k-functor $\mathfrak{M}$. Suppose that V_1 is a finitely generated, projective k-module. By Example 3.5.9 *$Hom_k(V_1, V_2)$* carries natural structure of a linear representation of $\mathfrak{G}^{\mathrm{op}} \times \mathfrak{G}$. According to Fact 3.2.9 we deduce that $\mathfrak{G}$ and $\mathfrak{G}^{\mathrm{op}}$ are canonically isomorphic. Hence group k-functors $\mathfrak{G}^{\mathrm{op}} \times \mathfrak{G}$ and $\mathfrak{G} \times \mathfrak{G}$ are canonically isomorphic. Thus $\mathrm{Hom}_k(V_1, V_2)$ has a natural structure of a linear representation of $\mathfrak{G} \times \mathfrak{G}$. By means of the diagonal $\mathfrak{G} \to \mathfrak{G} \times \mathfrak{G}$ this induces a structure of linear representation of $\mathfrak{G}$ on $\mathrm{Hom}_k(V_1, V_2)$. We call it *the hom representation of* (V_1, ρ_1) *and* (V_2, ρ_2).

Now we prove elementary yet important result.

Proposition 3.5.11. *Let (V_1, ρ_1) and (V_2, ρ_2) be linear representations of a group k-functor $\mathfrak{G}$. Suppose that V_1, V_2 are a finitely generated, projective k-modules. Then*

$$\mathrm{Hom}_k(V_1, V_2)^{\mathfrak{G}} = \text{morphisms of } \mathfrak{G}\text{-representations } (V_1, \rho_1) \to (V_2, \rho_2)$$

as k-submodules of $\mathrm{Hom}_k(V_1, V_2)$.

Proof. Let ρ be a morphism from $\mathfrak{G}$ to the general linear monoid of $\mathrm{Hom}_k(V_1, V_2)$ determining the structure of the hom representation of (V_1, ρ_1) and (V_2, ρ_2). Pick $f \in V_1^{\vee} = \mathrm{Hom}_k(V_1, k)$

and $w \in V_2$. Since $f : V_1 \to k$ and $w \in V_2$, we denote by fw and element of $\mathrm{Hom}_k(V_1, V_2)$ given by formula $(fw)(v) = f(v)w$ for every $v \in V_1$. Now fix k-algebra A and $g \in \mathfrak{G}(A)$. Since V_1 is projective and finitely generated, for every k-algebra we have a canonical isomorphism

$$A \otimes_k \mathrm{Hom}_k(V_1, V_2) \ni 1 \otimes \phi \mapsto 1_A \otimes_k \phi \in \mathrm{Hom}_A(A \otimes_k V_1, A \otimes_k V_2)$$

Examining Example 3.5.10 we deduce that the image of $\rho(g)\big(1 \otimes (fw)\big)$ under this isomorphism is

$$\rho_2(g) \cdot \big(1_A \otimes_k (fw)\big) \cdot \rho_1(g)^{-1}$$

Since every morphism $\phi \in \mathrm{Hom}_k(V_1, V_2)$ is expressible as a sum of morphisms of the form fw, we infer that the image of $\rho(g)(1 \otimes \phi)$ under this isomorphism is

$$\rho_2(g) \cdot \big(1_A \otimes_k \phi\big) \cdot \rho_1(g)^{-1}$$

Thus $\phi \in \mathrm{Hom}_k(V_1, V_2)^{\mathfrak{G}}$ if and only if $1_A \otimes_k \phi = \rho_2(g) \cdot \big(1_A \otimes_k \phi\big) \cdot \rho_1(g)^{-1}$ for every k-algebra A and $g \in \mathfrak{G}(A)$. Thus ϕ is in $\mathrm{Hom}_k(V_1, V_2)^{\mathfrak{G}}$ if and only if it is a morphism of linear representations $(V_1, \rho_1) \to (V_2, \rho_2)$. $\square$

3.6 Comodules over k-coalgebras

For an affine monoid k-scheme $\mathbf{M}$ its linear representations can be described in terms of vector spaces with some additional structure related to k-coalgebra $k[\mathbf{M}]$. In this section we introduce the relevant notions.

Definition 3.6.1. Let C be a k-coalgebra with the comultiplication Δ and the counit ξ. A pair (V, d) consisting of a k-module V and a morphism $d : V \to C \otimes_k V$ of k-modules such that the following diagrams are commutative

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, "d"'] & C \otimes_k V \arrow[d, "1_C \otimes_k d"] \\ C \otimes_k V \arrow[r, "\Delta \otimes_k 1_V"'] & C \otimes_k C \otimes_k V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "d"] \arrow[dr, "\simeq"'] & C \otimes_k V \arrow[d, "\xi \otimes_k 1_V"] \\ & k \otimes_k V \end{tikzcd}$$

is called *a C-comodule*. Morphism d is called *a coaction of C on V*.

Definition 3.6.2. Let C be a k-coalgebra and let $(V_1, d_1), (V_2, d_2)$ be two comodules over C. A morphism of k-modules $f : V_1 \to V_2$ is *a morphism of C-comodules* if the diagram

$$\begin{tikzcd} C \otimes_k V_1 \arrow[r, "1_C \otimes_k f"] & C \otimes_k V_2 \\ V_1 \arrow[u, "d_1"] \arrow[r, "f"'] & V_2 \arrow[u, "d_2"'] \end{tikzcd}$$

is commutative.

We denote by $\mathbf{coMod}(C)$ the category of C-comodules for a k-coalgebra C.
Recall the notion [ML98, Definition on page 112] of a functor creating (co)limits.

Theorem 3.6.3. *Let C be a k-coalgebra. Then the forgetful functor $\mathbf{coMod}(C) \to \mathbf{Mod}(k)$ creates colimits.*

Proof. Let Δ, ξ be the comultiplication and the counit of C, respectively. Suppose that $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$ is a diagram of C-comodules indexed by some category I. Let V together with $u_i : V_i \to V$ for $i \in I$ be a colimit of the diagram $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$. By the universal property of colimits we deduce that there exists a unique morphism $d : V \to C \otimes_k V$ such that diagrams

$$\begin{tikzcd} C \otimes_k V_i \arrow[r, "1_C \otimes_k u_i"] & C \otimes_k V \\ V_i \arrow[u, "d_i"] \arrow[r, "u_i"'] & V \arrow[u, "d"'] \end{tikzcd}$$

are commutative for every $i \in I$. In order to verify that diagrams

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, "d"'] & C \otimes_k V \arrow[d, "1_C \otimes_k d"] \\ C \otimes_k V \arrow[r, "\Delta \otimes_k 1_V"'] & C \otimes_k C \otimes_k V \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "d"] \arrow[dr, "\simeq"'] & C \otimes_k V \arrow[d, "\xi \otimes_k 1_V"] \\ & k \otimes_k V \end{tikzcd}$$

are commutative it suffices to note that for every $i \in I$ we have chains of equalities

$$(1_C \otimes_k d) \cdot d \cdot u_i = (1_C \otimes_k 1_C \otimes_k u_i) \cdot (1_C \otimes_k d_i) \cdot d_i =$$

$$= (1_C \otimes 1_C \otimes_k u_i) \cdot \left(\Delta \otimes_k 1_{V_i}\right) \cdot d_i = (\Delta \otimes_k 1_V) \cdot d \cdot u_i$$

and

$$(\xi \otimes_k 1_V) \cdot d \cdot u_i = (1_k \otimes_k u_i) \cdot \left(\xi \otimes_k 1_{V_i}\right) \cdot d_i = (1_k \otimes_k u_i) \cdot j_{V_i} = j_V \cdot u_i$$

where $j_W : W \to k \otimes_k W$ is the natural isomorphism for every k-module W. Hence (V, d) is a C-comodule. Suppose now that (W, e) is a C-comodule and $w_i : V_i \to W$ for $i \in I$ is a family of C-comodule morphisms compatible with the diagram $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$. Since $\{u_i : V_i \to V\}_{i \in I}$ form a colimiting cocone for $I \ni i \mapsto V_i \in \mathbf{Mod}(k)$, there exists a unique morphism of k-modules $f : V \to W$ such that $f \cdot u_i = w_i$. Note that

$$e \cdot f \cdot u_i = e \cdot w_i = (1_C \otimes_k w_i) \cdot d_i = (1_C \otimes_k f) \cdot (1_C \otimes_k u_i) \cdot d_i = (1_C \otimes_k f) \cdot d \cdot u_i$$

for every $i \in I$. Hence $e \cdot f = (1_C \otimes_k f) \cdot d$. Thus f is a morphism of C-comodules. Thus (V, d) together with a family $\left\{u_i : (V_i, d_i) \to (V, d)\right\}_{i \in I}$ is a colimit of the diagram $I \ni i \mapsto (V_i, d_i) \in \mathbf{coMod}(C)$ of C-comodules. This implies that the forgetful functor $\mathbf{coMod}(C) \to \mathbf{Mod}(k)$ creates colimits. □

Theorem 3.6.4. *Let C be a k-coalgebra such that C is a flat k-module. Then the forgetful functor* $\mathbf{coMod}(C) \to \mathbf{Mod}(k)$ *creates finite limits.*

Proof. The proof is similar to the proof of Theorem 3.6.3. □

Corollary 3.6.5. *Let C be a coalgebra over k and assume that C is flat as a k-module. Then* $\mathbf{coMod}(C)$ *is an abelian category with small colimits.*

Proof. This follows from Theorems 3.6.3 and 3.6.4. □

Fact 3.6.6. *Let (C, Δ, ξ) be a k-coalgebra and let V be a C-comodule with respect to the coaction $d : V \to C \otimes_k V$. Suppose that C is flat over k. Then d is a monomorphism of C-comodules $(V, d) \hookrightarrow (C, \Delta)$.*

Proof. Since the diagram

$$\begin{tikzcd} V \arrow[r, "d"] \arrow[d, "d"'] & C \otimes_k V \arrow[d, "1_C \otimes_k d"] \\ C \otimes_k V \arrow[r, "\Delta \otimes_k 1_V"'] & C \otimes_k C \otimes_k V \end{tikzcd}$$

is commutative, it follows that $d : (V, d) \to (C, \Delta)$ is a morphism of C-comodules. Moreover, $(\xi \otimes_k 1_V) \cdot d$ is canonically isomorphic with 1_V. Thus d is a split monomorphism in the category of modules over k. By Theorem 3.6.4 we derive that d is a monomorphism of C-comodules. □

The next result is of fundamental importance.

Theorem 3.6.7. *Let C be a k-coalgebra that is free as a k-module. Suppose that V is a C-comodule over C. Then for every finitely generated k-submodule $U \subseteq V$ there exists a C-subcomodule W of V such that $U \subseteq W$ and W is a finitely generated k-module.*

The theorem follows from the following simple lemma.

Lemma 3.6.7.1. *Let C be a k-coalgebra over k that is free as a k-module. Suppose that V is a C-comodule over C and fix an element $v \in V$. Then there exists a C-subcomodule W of V such that $v \in W$ and W is a finitely generated k-module.*

Proof of the lemma. Let $\{e_j\}_{j \in J}$ be a free basis of C over k and let $d : V \to C \otimes_k V$ be a left coaction of C on V. Denote by $\Delta : C \to C \otimes_k C$ the comultiplication of C. Then we have

$$d(v) = \sum_{j \in J} e_j \otimes v_j$$

where $v_j \in V$ are zero for almost all $j \in J$. Next according to

$$(\Delta \otimes_k 1_V) \cdot d = (1_C \otimes_k d) \cdot d$$

we derive that equality

$$\sum_{j\in J} e_j \otimes d(v_j) = (1_C \otimes_k d)\big(d(v)\big) = (\Delta \otimes_k 1_V)\big(d(v)\big) = \sum_{j\in J} \Delta(e_j)\otimes v_j \subseteq \sum_{j\in J} C\otimes_k C \otimes_k k\cdot v_j$$

holds. This implies that $d(v_j) \subseteq C \otimes_k \left(\sum_{j\in J} k\cdot v_j\right)$. Hence the k-submodule W of V generated by v and $\{v_j\}_{j\in J}$ is a C-subcomodule of V. It is finitely generated as a k-module and $v\in W$. □

Proof of the theorem. Suppose that U is generated by $\{v_1,...,v_n\}$ as a k-module. For each i pick a C-subcomodule W_i of V such that W_i is finitely generated as a k-module and $v_i \in W_i$. This can be done by Lemma 3.6.7.1. Next

$$W = W_1 + ... + W_n$$

is a C-subcomodule of V that is finitely generated as a k-module and contains U. □

3.7 Linear representations and comodules

Let $\mathbf{M}$ be an affine monoid k-scheme and let $\rho : \mathbf{M} \to \mathcal{L}_V$ be a morphism of k-functors, where V is a k-module. Yoneda Lemma implies that ρ is determined by some element

$$d_\rho \in \mathrm{Hom}_k\left(V, k[\mathbf{M}]\otimes_k V\right)$$

Conversely, to a morphism $d : V \to k[\mathbf{M}]\otimes_k V$ there corresponds a morphism of k-functors $\rho_d : \mathbf{M} \to \mathcal{L}_V$ given by

$$\rho_d(f) = \big(A\otimes_k V \ni 1\otimes v \mapsto (f\otimes_k 1_V)(d(v)) \in A\otimes_k V\big)$$

for every k-algebra A and a morphism of k-algebras $f : k[\mathbf{M}] \to A$. Maps

$$\rho \mapsto d_\rho,\ d \mapsto \rho_d$$

are mutually inverse bijections. The following theorem is proved in [DG70, discussion 2.1 on page 173].

Theorem 3.7.1. *Let* $\mathbf{M}$ *be an affine monoid k-scheme. Then the correspondence*

$$\mathbf{Rep}(\mathbf{M}) \ni (V,\rho) \mapsto \left(V, d_\rho\right) \in \mathbf{coMod}\left(k[\mathbf{M}]\right)$$

is an isomorphism of categories over $\mathbf{Mod}(k)$.

We obtain an interesting consequence of Theorem 3.7.1.

Corollary 3.7.2. *Let k be a field. Let* (V,ρ) *be a linear representation of an affine monoid k-scheme* $\mathbf{M}$. *Then for every finitely generated k-subspace* $U\subseteq V$ *there exists a subrepresentation* W *of* (V,ρ) *such that* $U\subseteq W$ *and* W *is a finite dimensional k-space.*

Proof. This follows from Theorems 3.7.1 and 3.6.7. □

Proposition 3.7.3. *Let* $\mathbf{M}$ *be an affine monoid k-scheme and let* V *be a linear representation of* $\mathbf{M}$ *with coaction* $d : V \to k[\mathbf{M}]\otimes_k V$. *Define the morphism* $p : V \to k[\mathbf{M}]\otimes_k V$ *by formula* $p(v) = 1\otimes v$. *Then the following assertions hold.*

(1) *p is a coaction of $k[\mathbf{M}]$ on V and (V,p) is a trivial linear representation of $\mathbf{M}$.*

(2) *The sequence of k-modules*

$$0 \longrightarrow V^{\mathbf{M}} \longrightarrow V \xrightarrow{d-p} k[\mathbf{M}] \otimes_k V$$

is exact, where $V^{\mathbf{M}}$ is the k-module of invariants of a linear representation (V,d).

Proof. For the proof of **(1)** note that $\rho_p(f) = 1_{A\otimes_k V}$ for every $f : k[\mathbf{M}] \to A$. Thus $\rho_p(f)$ is the identity of $\mathcal{L}_V(A)$ and hence ρ_p is a trivial morphism of monoid k-functors. By Theorem 3.7.1 p is a coaction and (V,p) is the trivial representation of $\mathbf{M}$.
Next for the proof of **(2)** note that v is an invariant of (V,ρ_d) if and only if for every morphism $f : k[\mathbf{M}] \to A$ of k-algebras we have

$$(f \otimes_k 1_V)(d(v)) = 1 \otimes v$$

This holds if and only if

$$(f \otimes_k 1_V)(d(v)) = (f \otimes_k 1_V)(p(v))$$

for every f. Thus v is an invariant of (V,ρ_d) if and only if $d(v) = p(v)$. This proves **(2)**. $\square$

3.8 Functorial comparison between representations of a monoid and its group of units

In this section we assume that k is a field. We study the relation between the category $\mathbf{Rep}(\mathbf{M})$ of representations of an affine monoid k-scheme $\mathbf{M}$ and the category $\mathbf{Rep}(\mathbf{G})$ of representations of its group of units $\mathbf{G}$. Let $i : k[\mathbf{M}] \to k[\mathbf{G}]$ be the morphism of k-bialgebras induced by $\mathbf{G} \hookrightarrow \mathbf{M}$. Let us first note the following elementary result.

Fact 3.8.1. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Then i is an injective morphism of k-algebras.*

Proof. This follows from [GW10, Proposition 9.19]. $\square$

Fact 3.8.2. *The forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ creates colimits and finite limits.*

Proof. This follows from Theorems 3.6.3, 3.6.4 and the commutative triangle

$$\begin{tikzcd} \mathbf{Rep}(\mathbf{M}) \arrow[rr] \arrow[dr] & & \mathbf{Rep}(\mathbf{G}) \arrow[dl] \\ & \mathbf{Vect}_k & \end{tikzcd}$$

of functors. $\square$

The theorem below characterizes representations of $\mathbf{G}$ which are contained in the image of the forgetful functor $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$.

Theorem 3.8.3. *Assume that* $\mathbf{G}$ *is open and schematically dense in* $\mathbf{M}$*. Let* V *be a* $\mathbf{G}$*-representation. Then the following are equivalent.*

(i) *V is in the image of the forgetful functor* $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$*.*

(ii) *The coaction* $d : V \to k[\mathbf{G}] \otimes_k V$ *factors through* $i \otimes_k 1_V : k[\mathbf{M}] \otimes_k V \hookrightarrow k[\mathbf{G}] \otimes_k V$*.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 3.8.1 i is an injective morphism of k-algebras.
Clearly **(i)** $\Rightarrow$ **(ii)**. We prove the converse. Suppose that **(ii)** holds. Let $c : V \to k[\mathbf{M}] \otimes_k V$ be a unique morphism such that $d = (i \otimes_k 1_V) \cdot c$. It suffices to prove that c is the coaction of the bialgebra $k[\mathbf{M}]$ on V. Observe that

$$(i \otimes_k i \otimes_k 1_V) \cdot \left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (i \otimes_k d) \cdot c = \left(1_{k[\mathbf{G}]} \otimes_k d\right) \cdot d = (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot d =$$

$$= (\Delta_{\mathbf{G}} \otimes_k 1_V) \cdot \Big((i \otimes_k 1_V) \cdot c \Big) = ((\Delta_{\mathbf{G}} \cdot i) \otimes_k 1_V) \cdot c = (i \otimes_k i \otimes_k 1_V) \cdot (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

Since $i \otimes_k i \otimes_k 1_V$ is a monomorphism, we deduce that $\left(1_{k[\mathbf{M}]} \otimes_k c\right) \cdot c = (\Delta_{\mathbf{M}} \otimes_k 1_V) \cdot c$. Moreover, we have

$$(\xi_{\mathbf{G}} \otimes_k 1_V) \cdot d = (\xi_{\mathbf{G}} \otimes_k 1_V) \cdot (i \otimes_k 1_V) \cdot c = (\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$$

and hence $(\xi_{\mathbf{M}} \otimes_k 1_V) \cdot c$ is the canonical isomorphism $V \simeq k \otimes_k V$. Thus c is the coaction of $k[\mathbf{M}]$ and $d = (i \otimes_k 1_V) \cdot c$. Therefore, linear representation (V, d) of $\mathbf{G}$ is in the image of $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$. $\square$

Theorem 3.8.4. *Assume that* $\mathbf{G}$ *is open and schematically dense in* $\mathbf{M}$*. Then* $\mathbf{Rep}(\mathbf{M})$ *is a full subcategory of* $\mathbf{Rep}(\mathbf{G})$ *closed under subobjects and quotients.*

Proof. In the proof we denote by $\Delta_{\mathbf{M}}$ and $\Delta_{\mathbf{G}}$ comultiplications of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. We also denote by $\xi_{\mathbf{M}}$ and $\xi_{\mathbf{G}}$ counits of $k[\mathbf{M}]$ and $k[\mathbf{G}]$, respectively. According to Fact 3.8.1 i is an injective morphism of k-algebras.
We first prove that $\mathbf{Rep}(\mathbf{M})$ is a full subcategory of $\mathbf{Rep}(\mathbf{G})$. For this consider representations V, W of $\mathbf{M}$ and their morphism $f : V \to W$ as $\mathbf{G}$-representations. Let c_V and c_W be coactions of $k[\mathbf{M}]$ on V and W, respectively. Our goal is to prove that f is a morphism of $\mathbf{M}$-representations. Consider the diagram

$$\begin{tikzcd}
k[\mathbf{G}] \otimes_k V \arrow[r, "1_{k[\mathbf{G}]} \otimes_k f"] & k[\mathbf{G}] \otimes_k W \\
k[\mathbf{M}] \otimes_k V \arrow[u, "i \otimes_k 1_V"] \arrow[r, dotted, "1_{k[\mathbf{M}]} \otimes_k f"] & k[\mathbf{M}] \otimes_k W \arrow[u, "i \otimes_k 1_W"'] \\
V \arrow[u, "c_V"] \arrow[r, "f"'] & W \arrow[u, "c_W"']
\end{tikzcd}$$

in which the outer square and the top square are commutative. Our goal is to prove that the bottom square is commutative. We have

$$(i \otimes_k 1_W) \cdot c_W \cdot f = \left(1_{k[\mathbf{G}]} \otimes_k f\right) \cdot (i \otimes_k 1_V) \cdot c_V = (i \otimes_k 1_W) \cdot \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$$

Since $i \otimes_k 1_W$ is a monomorphism, we deduce that $c_W \cdot f = \left(1_{k[\mathbf{M}]} \otimes_k f\right) \cdot c_V$. Hence f is a morphism of $\mathbf{M}$-representations.
Next we prove that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ that is closed under subquotients. Consider an $\mathbf{M}$-representation V and its quotient $\mathbf{G}$-representation $q : V \twoheadrightarrow W$. We show that W is a quotient $\mathbf{M}$-representation of V. Let c_V be the coaction of $\mathbf{M}$ on V and let d_W be the coaction of $\mathbf{G}$ on W. We have a commutative diagram

$$\begin{tikzcd}
k[\mathbf{G}] \otimes_k V \arrow[r, two heads, "1_{k[\mathbf{G}]} \otimes_k q"] & k[\mathbf{G}] \otimes_k W \\
k[\mathbf{M}] \otimes_k V \arrow[u, "i \otimes_k 1_V"] & \\
V \arrow[u, "c_V"] \arrow[r, two heads, "q"'] & W \arrow[uu, "d_W"']
\end{tikzcd}$$

and hence $d_W(W)$ is in the image of $i \otimes_k 1_W$. Thus Theorem 3.8.3 implies that W is a representation of $\mathbf{M}$ and q is a morphism of $\mathbf{M}$-representations. This shows that $\mathbf{Rep}(\mathbf{M})$ is a subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under quotients. Next let $j : U \hookrightarrow V$ be a $\mathbf{G}$-subrepresentation of an $\mathbf{M}$-representation V. By what we proved above the cokernel $q : V \twoheadrightarrow W$ of j in $\mathbf{Rep}(\mathbf{G})$ is contained in $\mathbf{Rep}(\mathbf{M})$. Since both $\mathbf{Rep}(\mathbf{M})$ and $\mathbf{Rep}(\mathbf{G})$ are abelian (Corollary 3.6.5) and the forgetful functor $\mathbf{Rep}(\mathbf{M}) \rightarrow \mathbf{Rep}(\mathbf{G})$ is exact by Fact 3.8.2, we derive that the kernel of q in $\mathbf{Rep}(\mathbf{M})$ coincides with its kernel in $\mathbf{Rep}(\mathbf{G})$. Thus U is an $\mathbf{M}$-representation and $j : U \hookrightarrow V$ is a morphism of $\mathbf{M}$-representations. Hence $\mathbf{Rep}(\mathbf{M})$ is the subcategory of $\mathbf{Rep}(\mathbf{G})$ closed under subobjects. □

We give an example of a monoid k-scheme with non dense group of units and show that Theorem 3.8.4 does not hold for such monoids.

Example 3.8.5. Consider the subscheme $\mathbf{N}$ of $\mathbb{A}^2_k = \mathbb{A}^1_k \times_k \mathbb{A}^1_k$ defined by the equation

$$(x - y) \cdot x = 0$$

Note that $\mathbf{N}$ is a submonoid scheme of the product $\mathbb{A}^1_k \times_k \mathbb{A}^1_k$ with respect to coordinatewise multiplication. The origin of the affine plane $\mathbb{A}^2_k$ is the zero of $\mathbf{N}$. The unit group $\mathbf{N}^*$ of $\mathbf{N}$ is isomorphic with $\mathbb{G}_m$. Its closure is a submonoid scheme $\mathbf{L}$ of $\mathbf{N}$ isomorphic with the affine line $\mathbb{A}^1_k$ as a monoid k-scheme. Moreover, $\mathbf{L}$ is the irreducible component of $\mathbf{N}$. There is also another irreducible component, which we denote by L. Note that both $\mathbf{L}$ and L are isomorphic as k-schemes equipped with an action of $\mathbf{N}^*$.
Swapping $\mathbf{L}$ and L gives rise to a $\mathbf{N}^*$-equivariant automorphism of $\mathbf{N}$ which is not $\mathbf{N}$-equivariant. This implies that there exists an automorphism of $k[\mathbf{N}]$ as a linear representation of $\mathbf{N}^*$ which is not in the image of the functor $\mathbf{Rep}(\mathbf{N}) \rightarrow \mathbf{Rep}(\mathbf{N}^*)$. In particular, the

functor $\mathbf{Rep}(\mathbf{N}) \to \mathbf{Rep}(\mathbf{N}^*)$ is not full.
Note that $\mathbf{L} \hookrightarrow \mathbf{N}$ is a closed $\mathbf{N}^*$-stable subscheme, which is not $\mathbf{N}$-stable. Hence $k[\mathbf{N}] \twoheadrightarrow k[\mathbf{L}]$ is a quotient representation of $k[\mathbf{N}]$ as the representation of $\mathbf{N}^*$, but it is not its quotient as a representation of $\mathbf{N}$.

It turns out that under the assumption that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$ the inclusion of representations of $\mathbf{M}$ into representations of $\mathbf{G}$ admits left adjoint. This result plays the fundamental role in the proof of representability of Białynicki-Birula functor in the affine case.

Theorem 3.8.6. *Assume that $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$. Let V be a linear representation of $\mathbf{G}$. There exists an $\mathbf{M}$-representation W and a surjective morphism $q : V \twoheadrightarrow W$ of $\mathbf{G}$-representations such that for every $\mathbf{M}$-representation U and a morphism $f : V \to U$ of $\mathbf{G}$-representations there exists a unique morphism $\tilde{f} : W \to U$ of $\mathbf{M}$-representations making the triangle*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, two heads, "q"'] & U \\ W \arrow[ur, "\tilde{f}"'] & \end{tikzcd}$$

commutative.

Proof. Assume first that V is finite dimensional as a vector space over k. Let $\mathcal{K}$ be a set of $\mathbf{G}$-subrepresentations of V that consists of all $K \subseteq V$ such that linear representation V/K of $\mathbf{G}$ carries the structure of $\mathbf{M}$-representation. Clearly $\mathcal{K} \neq \emptyset$ because $V \in \mathcal{K}$. Since V is finite dimensional, there exists a finite subset $\{K_1, ..., K_n\} \subseteq \mathcal{K}$ such that

$$\bigcap_{i=1}^{n} K_i = \bigcap_{K \in \mathcal{K}} K$$

Then a morphism

$$V/\left(\bigcap_{K \in \mathcal{K}} K\right) \ni v \mapsto \left(v \operatorname{mod} K_i\right)_{1 \leq i \leq n} \in \bigoplus_{i=1}^{n} V/K_i$$

is a monomorphism and hence by Theorem 3.8.4 the quotient $W = V/\left(\bigcap_{K \in \mathcal{K}} K\right)$ is an $\mathbf{M}$-representation. Let $q : V \twoheadrightarrow W$ be the canonical epimorphism. Consider now a morphism $f : V \to U$ of $\mathbf{G}$-representations, where U is an $\mathbf{M}$-representation. Then $\operatorname{im}(f)$ is a $\mathbf{G}$-subrepresentation of U and by Theorem 3.8.4 we derive that $\operatorname{im}(f)$ is an $\mathbf{M}$-representation. This implies that $\ker(f)$ is in $\mathcal{K}$. Hence f factors through q. Thus there exists a unique morphism $\tilde{f} : W \to U$ of $\mathbf{G}$-representations such that $\tilde{f} \cdot q = f$. This completes the proof in the case when V is finite dimensional over k.
Now consider the general V. Let $\mathcal{F}$ be the set (partially ordered by inclusion) of all finite dimensional $\mathbf{G}$-representations of V. According to Corollary 3.7.2 we deduce that $V = \operatorname{colim}_{F \in \mathcal{F}} F$. By the case considered above we deduce that for every F in $\mathcal{F}$ there exists a universal morphism $q_F : F \to W_F$ of $\mathbf{G}$-representations into an $\mathbf{M}$-representation W_F. Note that if $F_1 \subseteq F_2$ are two elements of $\mathcal{F}$, then

$$\begin{tikzcd}
F_1 \arrow[r, two heads, "q_{F_1}"] \arrow[d, hook] & W_{F_1} \arrow[d] \\
F_2 \arrow[r, two heads, "q_{F_2}"'] & W_{F_2}
\end{tikzcd}$$

Thus $\{W_F\}_{F\in\mathcal{F}}$ together with morphisms $W_{F_1} \to W_{F_2}$ for $F_1 \subseteq F_2$ in $\mathcal{F}$ form a diagram indexed by the poset $\mathcal{F}$. The category $\mathbf{Rep}(\mathbf{M})$ has small colimits by Corollary 3.6.5 and we define $W = \mathrm{colim}_{F\in\mathcal{F}} W_F$. This is also a colimit of this diagram in the category $\mathbf{Rep}(\mathbf{G})$ by Fact 3.8.2. We also define $q = \mathrm{colim}_{F\in\mathcal{F}} q_F : V = \mathrm{colim}_{F\in\mathcal{F}} F \to W$. Since a colimit of a family of epimorphisms is an epimorphism, we derive that q is an epimorphism of $\mathbf{G}$-representations. Suppose now that $f : V \to U$ is a morphism of $\mathbf{G}$-representations and U is an $\mathbf{M}$-representation. Then $f_{|F}$ uniquely factors through q_F for every F in $\mathcal{F}$. Hence by universal property of colimits we derive that f factors through q in a unique way. This completes the proof. □

3.9 Irreducible representations

In this section we assume that k is a field.

Definition 3.9.1. Let V be a linear representation of an affine monoid k-scheme $\mathbf{M}$. Then V is *irreducible* if it has no proper nonzero subrepresentations.

Corollary 3.9.2. *Let V be an irreducible representation of an affine monoid k-scheme $\mathbf{M}$. Then V is of finite dimension over k.*

Proof. This is a consequence of Corollary 3.7.2. □

Fact 3.9.3. *Let V, W be irreducible representations of an affine monoid k-scheme $\mathbf{M}$. If $f : V \to W$ is a morphism linear representations, then either $f = 0$ or f is an isomorphism.*

Proof. Note that $\ker(f)$ is a subrepresentation of an irreducible representation V. Thus either $\ker(f) = 0$ or $\ker(f) = V$. If $\ker(f) = 0$, then f is injective and $\mathrm{im}(f) \subseteq W$ is nonzero. Since $\mathrm{im}(f)$ is a subrepresentation of an irreducible representation W, we derive that $\mathrm{im}(f) = W$. Hence f is an isomorphism. If $\ker(f) = V$, then f is equal to zero. □

Definition 3.9.4. Let V be a linear representation of an affine monoid k-scheme $\mathbf{M}$. Then V is *completely reducible* if it is the sum of its irreducible subrepresentations.

Theorem 3.9.5. *Let V be a representation of an affine monoid k-scheme $\mathbf{M}$. Then the following are equivalent.*

(i) *V is completely reducible.*

(ii) *Every monomorphism of linear representations with V as the codomain splits.*

Moreover, the class of completely reducible representations of $\mathbf{M}$ is closed under subrepresentations and quotients.

For a better logical organization of the proof we extracted from it the following result.

Lemma 3.9.5.1. *Let V be a linear representation of an affine monoid k-scheme* $\mathbf{M}$. *Assume that every monomorphism of linear representations with V as the codomain splits. Let W be a subrepresentation of V. Then every monomorphism of linear representations with W as the codomain splits.*

Proof of the lemma. Let $j : W \hookrightarrow V$ be the inclusion. Suppose also that $i : U \hookrightarrow W$ is a monomorphism of representations. Then the assumption implies that $j \cdot i$ splits. This means that there exists a morphism $r : V \twoheadrightarrow U$ of representations such that $r \cdot (j \cdot i) = 1_U$. Then $r \cdot j : W \twoheadrightarrow U$ is a left inverse of i. Thus i splits. Since i is an arbitrary monomorphism with W as the codomain, we infer that the assertion holds. $\square$

Proof of the theorem. Assume that **(i)** holds and let W be a subrepresentation of V. Consider the family

$$\mathcal{U} = \{U \subseteq V \,|\, U \text{ is a subrepresentation of } V \text{ and } U \cap W = \{0\}\}$$

By Zorn's lemma there exists a maximal element of $\mathcal{U}$ with respect to inclusion. Suppose that U is maximal in $\mathcal{U}$. Pick an irreducible subrepresentation K of V. If $K \cap (U + W) = \{0\}$, then $(U + K) \cap W = \{0\}$. Since $U \subseteq U + K$ and $U + K$ is a subrepresentation of V, it follows from the fact that U is maximal in $\mathcal{U}$ that $K \subseteq U$, but this is contradiction with the fact that $K \cap (U + W) = \{0\}$. Therefore, $K \cap (U + W) \neq \{0\}$. Since $K \cap (U + W)$ is a nonzero subrepresentation of K, we deduce that $K = K \cap (U + W)$. Hence K is a subrepresentation of $U + W$. Since V is the sum of its irreducible subrepresentations and K was chosen arbitrarily, we derive that $V \subseteq U + W$. Hence V is a direct sum of W and U. Thus the inclusion $W \hookrightarrow V$ splits. This completes the proof of **(i)** $\Rightarrow$ **(ii)**.
Next we prove that **(ii)** $\Rightarrow$ **(i)**. By Corollary 3.7.2 it suffices to show that every finite dimensional subrepresentation of V is the sum of its irreducible subrepresentations. By Lemma 3.9.5.1 it suffices to assume that V is finite dimensional. Let W be the sum of all irreducible subrepresentations of V. Then V is a direct sum of W and some subrepresentation U. Clearly U is finite dimensional. If U is nontrivial, then it has a minimal nonzero subrepresentation. This subrepresentation is an irreducible subrepresentation of V not contained in W. This is a contradiction. Hence $U = \{0\}$ and thus V is completely reducible.
Note that by Fact 3.9.3 the quotient of a representation that is a sum of its irreducible subrepresentation also admits this property. By Lemma 3.9.5.1 the property **(ii)** is inherited by subrepresentations. This completes the proof. $\square$

Definition 3.9.6. Let $\mathbf{M}$ be an affine monoid k-scheme. We denote by $\mathbf{Irr}(\mathbf{M})$ the collection of all isomorphism classes of irreducible representations of $\mathbf{M}$.

Definition 3.9.7. Let V be a representation of an affine monoid k-scheme $\mathbf{M}$. Consider the sum $\mathrm{soc}(V)$ of all irreducible subrepresentations of V. Then $\mathrm{soc}(V)$ is the largest completely reducible subrepresentation of V. We call it *the socle of V*.

Theorem 3.9.8. *Let V be a representation of an affine monoid k-scheme. For each $\lambda \in \mathbf{Irr}(\mathbf{M})$ let $V[\lambda]$ be a sum of all irreducible subrepresentation of V which are contained in λ. Then*

$$\mathrm{soc}(V) = \bigoplus_{\lambda \in \mathbf{Irr}(\mathbf{M})} V[\lambda]$$

The theorem is a consequence of the following result.

Lemma 3.9.8.1. *Let V be a representation of $\mathbf{M}$ such that*

$$V = \sum_{i=1}^{n} K_i$$

where K_i are irreducible for each $1 \leq i \leq n$. If K is an irreducible subrepresentation of V, then K is isomorphic with one of $\{K_1, ..., K_n\}$.

Proof of the lemma. Since K is a subrepresentation of V, we derive by Theorem 3.9.5 that there exists an epimorphism $q : V \twoheadrightarrow K$ such that $q_{|K} = 1_K$. Then for some i the morphism $q_{|K_i}$ is nontrivial. By Fact 3.9.3 this implies that K and K_i are isomorphic. □

Proof of the theorem. Note that

$$\mathrm{soc}(V) = \sum_{\lambda \in \mathbf{Irr}(\mathbf{M})} V[\lambda]$$

by definition of $\mathrm{soc}(V)$. Moreover, if $\lambda \cap \big(\lambda_1 \cup ... \cup \lambda_n\big) = \varnothing$, then subrepresentation

$$V[\lambda] \cap \big(V[\lambda_1] + ... + V[\lambda_n]\big)$$

must be zero. Indeed, if it is nonzero, then according to Theorem 3.9.5 it would be nonzero and completely reducible. Hence it would contain some irreducible subrepresentation K and then by Lemma 3.9.8.1 we would have $K \in \lambda \cap \big(\lambda_1 \cup ... \cup \lambda_n\big)$. This is a contradiction. □

Definition 3.9.9. Let V be a representation of an affine monoid k-scheme $\mathbf{M}$. Then the decomposition of Theorem 3.9.8 is called *the isotypic decomposition of* $\mathrm{soc}(V)$ and for every $\lambda \in \mathbf{Irr}(\lambda)$ subrepresentations $V[\lambda]$ is called *the isotypic component of V of type λ.*

Fact 3.9.10. *Let $\mathbf{M}$ be an affine monoid k-scheme. Let $f : V \to W$ be a morphism of representations of $\mathbf{M}$. Then*

$$f\big(V[\lambda]\big) \subseteq W[\lambda]$$

for every λ in $\mathbf{Irr}(\mathbf{M})$.

Proof. Fix $\lambda \in \mathbf{Irr}(\mathbf{M})$. By Fact 3.9.3 $f\big(V[\lambda]\big)$ is the sum of irreducible subrepresentations contained in λ. Thus $f\big(V[\lambda]\big) \subseteq W[\lambda]$. □

Proposition 3.9.11. *Let $\mathbf{M}, \mathbf{N}$ be affine monoid k-schemes. Suppose that V is a representation of both $\mathbf{M}$ and $\mathbf{N}$ and assume that their actions on V commute. Assume that V is completely reducible as a representation of $\mathbf{N}$ and consider the isotypic decomposition*

$$V = \bigoplus_{\lambda \in \mathbf{Irr}(\mathbf{N})} V[\lambda]$$

Then for every λ in $\mathbf{Irr}(\mathbf{N})$ the subspace $V[\lambda]$ is an $\mathbf{M}$-subrepresentation of V.

Proof. Consider morphisms $\rho : \mathbf{M} \to \mathcal{L}_V$ and $\delta : \mathbf{N} \to \mathcal{L}_V$ determining the structure of V as representation of $\mathbf{M}$ and $\mathbf{N}$, respectively. Fix k-algebra A and $m \in \mathbf{M}(A)$. Consider $A \otimes_k V$ as a tensor product (Example 3.5.6) of $\mathbf{N}$-representation V with A as the trivial $\mathbf{N}$-representation. We claim that $\rho(m) : A \otimes_k V \to A \otimes_k V$ is an endomorphism of this $\mathbf{N}$-representation. For this consider k-algebra B and $n \in \mathbf{N}(B)$. Since actions of $\mathbf{M}$ and $\mathbf{N}$ on V commute, we derive that

$$\big(1_B \otimes_k \rho(m)\big) \cdot \big(1_A \otimes_k \delta(n)\big) = \big(1_A \otimes_k \delta(n)\big) \cdot \big(1_B \otimes_k \rho(m)\big)$$

Since this holds for every k-algebra B and every $n \in \mathbf{N}(B)$, we deduce that indeed $\rho(m)$ is an endomorphism of $A \otimes_k V$ as a representation of $\mathbf{N}$. Next we have

$$(A \otimes_k V)[\lambda] = A \otimes_k V[\lambda]$$

for every $\lambda \in \Lambda$. Thus by Fact 3.9.10 we have

$$\rho(m)\,(A \otimes_k V[\lambda]) \subseteq A \otimes_k V[\lambda]$$

for every λ in $\mathbf{Irr}(\mathbf{N})$. This holds for every k-algebra A and $m \in \mathbf{M}(A)$. Hence $V[\lambda]$ is an $\mathbf{M}$-subrepresentation of V. □

Corollary 3.9.12. *Let $\mathbf{M}$ be an affine monoid k-scheme and let $\mathbf{G}$ be its group of units. If $\mathbf{G} \hookrightarrow \mathbf{M}$ is open and schematically dense, then $\mathbf{Rep}(\mathbf{M}) \to \mathbf{Rep}(\mathbf{G})$ sends irreducible representations to irreducible representations and the induced map $\mathbf{Irr}(\mathbf{M}) \to \mathbf{Irr}(\mathbf{G})$ of classes is an injection.*

Proof. If V is irreducible as a representation of $\mathbf{M}$, then by Theorem 3.8.4 it is also irreducible as a representation of $\mathbf{G}$. Now if V, W are two irreducible representations of $\mathbf{M}$ which are isomorphic as representations of $\mathbf{G}$, then again by Theorem 3.8.4 they are isomorphic as representations of $\mathbf{M}$. Thus the induced map $\mathbf{Irr}(\mathbf{M}) \to \mathbf{Irr}(\mathbf{G})$ is injective. □

Definition 3.9.13. Let $\mathbf{M}$ be an affine monoid k-scheme. We say that category $\mathbf{Rep}(\mathbf{M})$ is *semisimple* if every representation of $\mathbf{M}$ is completely reducible.

Corollary 3.9.14. *Let $\mathbf{M}$ be an affine monoid k-scheme and let $\mathbf{G}$ be its group of units. If $\mathbf{G} \hookrightarrow \mathbf{M}$ is open and schematically dense and $\mathbf{Rep}(\mathbf{G})$ is semisimple, then $\mathbf{Rep}(\mathbf{M})$ is semisimple.*

Proof. Let V be a representation of $\mathbf{M}$. Then V is a sum of its irreducible $\mathbf{G}$-subrepresentations. By Theorem 3.8.4 we infer that all these irreducible subrepresentations of V with respect to $\mathbf{G}$ are $\mathbf{M}$-representations of V. According to Corollary 3.9.12 we derive that V is a sum of its irreducible $\mathbf{M}$-subrepresentations. Hence V is completely reducible representation of $\mathbf{M}$. □

3.10 Actions of monoid k-functors and fixed points

This section is devoted to introducing actions of monoid k-functors and related topics.

Definition 3.10.1. Let $\mathfrak{M}$ be a monoid k-functor and let $\mathfrak{X}$ be a k-functor. Suppose that $\alpha : \mathfrak{M} \times X \to X$ is a morphism of k-functors such that there are commutative diagrams

$$\begin{tikzcd} \mathfrak{M}\times\mathfrak{M}\times\mathfrak{X} \arrow[r, "\mu\times 1_{\mathfrak{X}}"] \arrow[d, "1_{\mathfrak{M}}\times\alpha"'] & \mathfrak{M}\times\mathfrak{X} \arrow[d, "\alpha"] \\ \mathfrak{M}\times\mathfrak{X} \arrow[r, "\alpha"'] & \mathfrak{X} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1}\times\mathfrak{X} \arrow[r, "e\times 1_{\mathfrak{X}}"] \arrow[dr, "1_{\mathfrak{X}}"'] & \mathfrak{M}\times\mathfrak{X} \arrow[d, "\alpha"] \\ & \mathfrak{X} \end{tikzcd}$$

where $\mu : \mathfrak{M}\times\mathfrak{M} \to \mathfrak{M}$ is the multiplication and $e : \mathbf{1} \to \mathfrak{X}$ is the identity of $\mathfrak{M}$. Then α is called *an action of* $\mathfrak{M}$ *on* $\mathfrak{X}$.

Definition 3.10.2. Let $\mathfrak{M}$ be a monoid k-functor and let $\mathfrak{X}_1, \mathfrak{X}_2$ be k-functors equipped with actions α_1, α_2 of $\mathfrak{M}$, respectively. Consider a morphism $\sigma : \mathfrak{X}_1 \to \mathfrak{X}_2$ of k-functors such that the diagram

$$\begin{tikzcd} \mathfrak{M}\times\mathfrak{X}_1 \arrow[r, "1_{\mathfrak{M}}\times\sigma"] \arrow[d, "\alpha_1"'] & \mathfrak{M}\times\mathfrak{X}_2 \arrow[d, "\alpha_2"] \\ \mathfrak{X}_1 \arrow[r, "\sigma"'] & \mathfrak{X}_2 \end{tikzcd}$$

is commutative. Then σ is called *an* $\mathfrak{M}$*-equivariant morphism of k-functors*.

Definition 3.10.3. Let $\mathfrak{X}$ be a k-functor equipped with an action $\alpha : \mathfrak{M}\times\mathfrak{X} \to \mathfrak{X}$ of a monoid k-functor $\mathfrak{M}$. Consider a k-subfunctor $\mathfrak{Z}$ of $\mathfrak{X}$. Then $\mathfrak{Z}$ is called $\mathfrak{M}$*-stable* if the restriction of α to $\mathfrak{M}\times\mathfrak{Z}$ factors through $\mathfrak{Z}$.

Using our convention (Remark 2.3.8) all these notions apply to monoid k-schemes. In particular, we have the following notion.

Definition 3.10.4. Let X be a k-scheme equipped with an action $a : \mathbf{M}\times_k X \to X$ of a monoid k-scheme $\mathbf{M}$. Consider a locally closed subscheme Z of X. Then Z is called $\mathbf{M}$*-stable* if the restriction of a to $\mathbf{M}\times_k Z$ factors scheme-theoreticaly through Z.

Definition 3.10.5. Let $\mathbf{M}$ be a monoid k-scheme and let $a : \mathbf{M}\times_k X \to X$ be an action of $\mathbf{M}$ on a k-scheme X. Then we define a subpresheaf $X^{\mathbf{M}}$ of X by

$$X^{\mathbf{M}}(Y) = \left\{ f \in \mathrm{Mor}_k(Y, X) \,\middle|\, f \text{ is } \mathbf{M}\text{-equivariant for } Y \text{ with the trivial } \mathbf{M}\text{-action} \right\}$$

for every k-scheme Y. Then $X^{\mathbf{M}}$ is called *the fixed point presheaf of* $\mathbf{M}$*-scheme* X.

Fact 3.10.6. *Let* $\mathbf{M}$ *be a monoid k-scheme and let X be a k-scheme with an action of* $\mathbf{M}$*. Then* $X^{\mathbf{M}}$ *is a Zariski sheaf.*

Proof. Let a be an action on X and $\mathrm{pr}_Z : \mathbf{M}\times_k Z \to Z$ denote the projection on Z for every k-scheme Z.
Consider a k-scheme Y and let $Y = \bigcup_{i\in I} Y_i$ be an open cover of Y. Suppose that $\{f_i : Y_i \to X\}_{i\in I}$ is a family of morphisms such that for each i morphism f_i is $\mathbf{M}$-equivariant when Y_i is considered with the trivial action of $\mathbf{M}$. Moreover, assume that $f_{i|Y_i\cap Y_j} = f_{j|Y_i\cap Y_j}$ for any pair $i, j \in I$. Since X is a Zariski sheaf (Remark 2.3.8), we derive that there exists a

unique morphism $f : Y \to X$ of k-schemes such that $f_{|Y_i} = f_i$ for $i \in I$. It suffices to verify that f is $\mathbf{M}$-equivariant as a morphism defined on Y with the trivial action of $\mathbf{M}$. Since $\mathbf{M} \times_k Y = \bigcup_{i \in I} \mathbf{M} \times_k Y_i$ is an open cover and

$$\left(f \cdot \mathrm{pr}_Y\right)_{|\mathbf{M} \times_k Y_i} = f_i \cdot \mathrm{pr}_{Y_i} = a \cdot (1_{\mathbf{M}} \times_k f_i) = \left(a \cdot (1_{\mathbf{M}} \times_k f)\right)_{|\mathbf{M} \times_k Y_i}$$

we derive that $f \cdot \mathrm{pr}_Y = a \cdot (1_{\mathbf{M}} \times_k f)$. This completes the proof. □

3.11 Locally linear M-schemes

The class of locally linear schemes generalizes affine k-schemes equipped with actions of monoid k-schemes. Similarly to affine k-schemes with actions of monoid k-schemes locally linear schemes can be studied efficiently by means of representations theory.

Definition 3.11.1. Let $\mathbf{M}$ be a monoid k-scheme and let X be a k-scheme with an action of $\mathbf{M}$. Suppose that each point of X admits an open affine $\mathbf{M}$-stable neighborhood. Then we say that X is *a locally linear* $\mathbf{M}$*-scheme*.

Proposition 3.11.2. *Assume that k is a field. Let $\mathbf{M}$ be a monoid k-scheme and let X be a k-scheme with an action of $\mathbf{M}$. Suppose that Z is a closed $\mathbf{M}$-stable subscheme of X defined by the ideal with nilpotent sections. Consider an open subset U of X. Then the following are equivalent.*

- **(i)** *U is $\mathbf{M}$-stable.*
- **(ii)** *The intersection $U \cap Z$ is $\mathbf{M}$-stable.*

Proof. Let $a : \mathbf{M} \times_k X \to X$ be the action of $\mathbf{M}$ on X. Fix an open subset U of X. If U is $\mathbf{M}$-stable, then $U \cap Z$ is $\mathbf{M}$-stable and this proves that **(i)** ⇒ **(ii)**. So suppose that $U \cap Z$ is $\mathbf{M}$-stable. Since ideal of Z has nilpotent sections and k is a field, we derive that closed immersions $U \cap Z \hookrightarrow U$ and $\mathbf{M} \times_k (U \cap Z) \hookrightarrow \mathbf{M} \times_k U$ induce homeomorphisms on topological spaces. Consider the commutative diagram

$$\begin{tikzcd} \mathbf{M} \times_k U \arrow[r, "a_{|\mathbf{M} \times_k U}"] & X \\ \mathbf{M} \times_k (U \cap Z) \arrow[u, hook] \arrow[r] & U \cap Z \arrow[u, hook] \end{tikzcd}$$

where the bottom horizontal arrow is the induced action on $U \cap Z$ and vertical morphisms are homeomorphisms. The commutativity of the diagram implies that $a(\mathbf{M} \times_k U)$ is contained set-theoretically in U. Since U is open in X, we derive that morphism of schemes $a_{|\mathbf{M} \times_k U}$ factors through U. Hence U is $\mathbf{M}$-stable. This completes the proof of **(ii)** ⇒ **(i)**. □

Corollary 3.11.3. *Assume that k is a field. Let $\mathbf{M}$ be a monoid k-scheme and let X be a k-scheme with an action of $\mathbf{M}$. Suppose that Z is a closed $\mathbf{M}$-stable subscheme of X defined by the nilpotent ideal. Consider an open subset U of X. Then the following are equivalent.*

- **(i)** *U is $\mathbf{M}$-stable and affine.*

(ii) *$U \cap Z$ is* $\mathbf{M}$*-stable and affine.*

Proof. Since ideal of Z is nilpotent, we derive that U is affine if and only if $U \cap Z$ is affine. This combined with Proposition 3.11.2 yields the result. □

Corollary 3.11.4. *Assume that k is a field. Let* $\mathbf{M}$ *be a monoid k-scheme and let X be a k-scheme with an action of* $\mathbf{M}$*. Suppose that Z is a closed* $\mathbf{M}$*-stable subscheme of X defined by the nilpotent ideal. Then X is locally linear* $\mathbf{M}$*-scheme if and only if Z is locally linear* $\mathbf{M}$*-scheme.*

Proof. This is a consequence of Corollary 3.11.3. □

3.12 Quasi-coherent G-sheaves on locally linear G-schemes

Line bundles with $\mathbf{G}$-equivariant structure were studied extensively by Mumford in his geometric invariant theory [MFK94, Definition 1.6 on page 30]. Here we study $\mathbf{G}$-equivariant sheaves on locally linear $\mathbf{G}$-schemes for an affine group k-scheme $\mathbf{G}$.

Definition 3.12.1. Let $\mathbf{G}$ be a group k-scheme and let X be a k-scheme with an action $a : \mathbf{G} \times_k X \to X$ of $\mathbf{G}$. We denote by $\pi : \mathbf{G} \times_k X \to X$ the projection. Consider a pair $(\mathcal{F}, \tau)$ consisting of a quasi-coherent sheaf $\mathcal{F}$ on X and an isomorphism $\tau : a^*\mathcal{F} \to \pi^*\mathcal{F}$. We call it *a quasi-coherent* $\mathbf{G}$*-sheaf on X* if the following equality

$$(\mu \times 1_X)^* \tau = \pi_{23}^* \tau \cdot (1_{\mathbf{G}} \times_k a)^* \tau$$

holds, where $\mu : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ is the multiplication on $\mathbf{G}$ and $\pi_{2,3} : \mathbf{G} \times_k \mathbf{G} \times_k X \to \mathbf{G} \times_k X$ is the projection on the last two factors.

Definition 3.12.2. Let $\mathbf{G}$ be a group k-scheme and let X be a k-scheme with an action a of $\mathbf{G}$. We denote by $\pi : \mathbf{G} \times_k X \to X$ the projection. Let $(\mathcal{F}_1, \tau_1)$ and $(\mathcal{F}_2, \tau_2)$ be quasi-coherent $\mathbf{G}$-sheaves on X. Suppose that $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism of quasi-coherent sheaves on X such that the square

$$\begin{tikzcd} a^*\mathcal{F}_1 \arrow[r, "\tau_1"] \arrow[d, "a^*\phi"'] & \pi^*\mathcal{F}_1 \arrow[d, "\pi^*\phi"] \\ a^*\mathcal{F}_2 \arrow[r, "\tau_2"'] & \pi^*\mathcal{F}_2 \end{tikzcd}$$

is commutative. Then ϕ is *a morphism of quasi-coherent* $\mathbf{G}$*-sheaves on X*. We denote by $\mathfrak{Qcoh}_{\mathbf{G}}(X)$ the category of quasi-coherent $\mathbf{G}$-sheaves and call it *the category of quasi-coherent* $\mathbf{G}$*-sheaves on X*.

Definition 3.12.3. Let $\mathbf{G}$ be a monoid k-scheme and let X be a locally noetherian k-scheme with an action of $\mathbf{G}$. Then a quasi-coherent $\mathbf{G}$-sheaf $(\mathcal{F}, \tau)$ is *coherent* $\mathbf{G}$*-sheaf* if $\mathcal{F}$ is coherent. Coherent $\mathbf{G}$-sheaves form a full subcategory of $\mathfrak{Qcoh}_{\mathbf{G}}(X)$. We denote it by $\mathfrak{Coh}_{\mathbf{G}}(X)$ and call it *the category of coherent* $\mathbf{G}$*-sheaves on X*.

Remark 3.12.4. The notion of quasi-coherent $\mathbf{G}$-sheaf is often considered nonintuitive. Some discussion that is useful in understanding the concept is contained in [MFK94, page 30 immediately after Definition 1.6]. There is also a highly abstract level of considerations which sheds light on it. We discuss it now without going into technical details. Let us start with a formally less complex structures than schemes. Suppose that $\mathbf{G}$ is a topological group and X is a topological space on which $\mathbf{G}$ acts. Consider also a sheaf $\mathcal{F}$ of sets on X. The construction ([MM94, Chapter II, Section 6]) of *the space étale of* $\mathcal{F}$ shows that one can consider $\mathcal{F}$ (via appropriate equivalence of categories) as a topological space $|\mathcal{F}| \to X$ over X. Next if $a : \mathbf{G} \times X \to X$ is an action, then the pullback $a^{-1}(\mathcal{F})$ of $\mathcal{F}$ corresponds to a cartesian square

$$\begin{tikzcd} |a^{-1}(\mathcal{F})| \arrow[r, "\tilde{a}"] \arrow[d] & |\mathcal{F}| \arrow[d] \\ \mathbf{G} \times X \arrow[r, "a"'] & X \end{tikzcd}$$

Moreover, if $\pi : \mathbf{G} \times X \to X$ is the projection, then $|\pi^{-1}(\mathcal{F})| = \mathbf{G} \times |F|$ and an invertible morphism $\tau : a^{-1}\mathcal{F} \to \pi^{-1}\mathcal{F}$ gives rise to a homeomorphism $|\tau| : |a^{-1}\mathcal{F}| \to |\pi^{-1}\mathcal{F}| = \mathbf{G} \times |F|$ over $\mathbf{G} \times X$. Now it turns out that τ satisfies the equality

$$(\mu \times 1_X)^{-1}\tau = \pi_{23}^{-1}\tau \cdot (1_{\mathbf{G}} \times_k a)^{-1}\,\tau$$

(where μ and π_{23} have their usual denotations) if and only if in the commutative diagram

$$\begin{tikzcd} \mathbf{G} \times |\mathcal{F}| = |\pi^{-1}(\mathcal{F})| \arrow[r, "|\tau^{-1}|"] \arrow[dr] & |a^{-1}(\mathcal{F})| \arrow[r, "\tilde{a}"] \arrow[d] & |\mathcal{F}| \arrow[d] \\ & \mathbf{G} \times X \arrow[r, "a"'] & X \end{tikzcd}$$

the composition of two top horizontal maps is a continuous map that defines an action of $\mathbf{G}$ on $|\mathcal{F}|$. Thus morphisms τ satisfying equality above correspond to these actions of $\mathbf{G}$ on $|\mathcal{F}|$ which are defined over a.
Now in case of quasi-coherent sheaves there is no geometric construction of *étale space*, that will make every quasi-coherent sheaf on X it into a k-scheme over X. Despite of this there is an abstract machinery of *fibered categories* (see also Section 7.2) that enables with respect to some abstracted properties to conceive a quasi-coherent sheaf $\mathcal{F}$ on a k-scheme X as some analogon of a space over X. In his excellent notes [FGI05, Part 1, Subsection 3.2.1] on fibered categories Vistoli constructs *a fibered category* $\mathfrak{Qcoh} \to \mathbf{Sch}_k$ *of quasi-coherent sheaves over k-schemes*. Then in [FGI05, Part 1, Subsection 3.8] he explains how; given a category, an object B in its base equipped with an action of a group object G and an object E over B; to define an action of G on E compatible with the action of G on B. Application of this notion to $\mathfrak{Qcoh} \to \mathbf{Sch}_k$, k-scheme X with a scheme group action $\mathbf{G}$ and a quasi-coherent sheaf $\mathcal{F}$ on X yields precisely the structure of quasi-coherent $\mathbf{G}$-sheaf with $\mathcal{F}$ as the underlying sheaf.

Remark 3.12.5. Let X be a k-scheme equipped with an action of a group k-scheme $\mathbf{G}$. Then there exists a structure of monoidal category on $\mathfrak{Qcoh}_{\mathbf{G}}(X)$ such that the forgetful functor $\mathfrak{Qcoh}_{\mathbf{G}}(X) \to \mathfrak{Qcoh}(X)$ is a strict monoidal functor. Moreover, if $f : X \to Y$ is a $\mathbf{G}$-equivariant

morphism of k-schemes equipped with $\mathbf{G}$-actions, then $f^* : \mathfrak{Qcoh}(Y) \to \mathfrak{Qcoh}(X)$ induces a cocontinuous, monoidal functor $f^* : \mathfrak{Qcoh}_{\mathbf{G}}(Y) \to \mathfrak{Qcoh}_{\mathbf{G}}(X)$. Now if X is locally noetherian, then $\mathfrak{Coh}_{\mathbf{G}}(X)$ is a monoidal subcategory of $\mathfrak{Qcoh}_{\mathbf{G}}(X)$ and if $f : X \to Y$ is a $\mathbf{G}$-equivariant morphism of locally noetherian k-schemes with $\mathbf{G}$-actions, then $f^* : \mathfrak{Qcoh}_{\mathbf{G}}(Y) \to \mathfrak{Qcoh}_{\mathbf{G}}(X)$ restricts to a functor $f^* : \mathfrak{Coh}_{\mathbf{G}}(Y) \to \mathfrak{Coh}_{\mathbf{G}}(X)$.
The properties above are elementary and their proofs are straightforward. One can also employ the machinery of fibered categories (see Remark 3.12.4) to prove them. This essentially boils down to the statement that the category $\mathfrak{Qcoh} \to \mathbf{Sch}_k$ is fibered in monoidal categories.

Proposition 3.12.6. *Let $\mathbf{G}$ be a group k-scheme and let X be a k-scheme equipped with an action a of $\mathbf{G}$. Suppose that $\tau : a^*\mathcal{F} \to \pi^*\mathcal{F}$ is a morphism of quasi-coherent sheaves on $\mathbf{G} \times_k X$. Then the following are equivalent.*

(i) *The equalities*

$$(\mu \times 1_X)^*\tau = \pi_{23}^*\tau \cdot (1_{\mathbf{G}} \times_k a)^*\,\tau,\ \langle e \cdot q, 1_X\rangle^*\tau = 1_{\mathcal{F}}$$

hold, where $q : X \to \operatorname{Spec} k$ is the unique morphism and $e : \operatorname{Spec} k \to \mathbf{G}$ is the identity.

(ii) *$(\mathcal{F}, \tau)$ is a quasi-coherent $\mathbf{G}$-sheaf.*

Proof. Assume that **(i)** holds. Let $p : \mathbf{G} \to \operatorname{Spec} k$ be the unique morphism. Since $\mathbf{G}$ is a group k-scheme, there exists a morphism $i : \mathbf{G} \to \mathbf{G}$ of k-schemes such that

$$\mu \cdot \langle 1_{\mathbf{G}}, i\rangle = e \cdot p = \mu \cdot \langle i, 1_{\mathbf{G}}\rangle$$

and $i \cdot i = 1_{\mathbf{G}}$. Then

$$1_{\pi^*\mathcal{F}} = \pi^*\langle e \cdot q, 1_X\rangle^*\tau = ((e \cdot p) \times_k 1_X)^*\,\tau = \left(\langle i, 1_{\mathbf{G}}\rangle \times_k 1_X\right)^*(\mu \times_k 1_X)^*\tau =$$

$$= \left(\langle i, 1_{\mathbf{G}}\rangle \times_k 1_X\right)^*\left(\pi_{23}^*\tau \cdot (1_{\mathbf{G}} \times_k a)^*\,\tau\right) = \left(\langle i, 1_{\mathbf{G}}\rangle \times_k 1_X\right)^*\pi_{23}^*\tau \cdot \left(\langle i, 1_{\mathbf{G}}\rangle \times_k 1_X\right)^*\,(1_{\mathbf{G}} \times_k a)^*\,\tau =$$

$$= \tau \cdot \left(\langle i, 1_{\mathbf{G}}\rangle \times_k 1_X\right)^*\,(1_{\mathbf{G}} \times_k a)^*\,\tau$$

Therefore, τ is a split epimorphism. Similarly we have

$$1_{a^*\mathcal{F}} = a^*\langle e \cdot q, 1_X\rangle^*\tau = \langle 1_{\mathbf{G}}, a\rangle^*\,((e \cdot p) \times_k 1_X)^*\,\tau =$$

$$= \langle 1_{\mathbf{G}}, a\rangle^*\left(\langle 1_{\mathbf{G}}, i\rangle \times_k 1_X\right)^*(\mu \times_k 1_X)^*\tau = \langle 1_{\mathbf{G}}, a\rangle^*\left(\langle 1_{\mathbf{G}}, i\rangle \times_k 1_X\right)^*\left(\pi_{23}^*\tau \cdot (1_{\mathbf{G}} \times_k a)^*\,\tau\right) =$$

$$= \langle 1_{\mathbf{G}}, a\rangle^*\left(\langle 1_{\mathbf{G}}, i\rangle \times_k 1_X\right)^*\pi_{23}^*\tau \cdot \langle 1_{\mathbf{G}}, a\rangle^*\left(\langle 1_{\mathbf{G}}, i\rangle \times_k 1_X\right)^*\,(1_{\mathbf{G}} \times_k a)^*\,\tau =$$

$$= \langle 1_{\mathbf{G}}, a\rangle^*\left(\langle 1_{\mathbf{G}}, i\rangle \times_k 1_X\right)^*\pi_{23}^*\tau \cdot \tau$$

Thus τ is a split monomorphism. Therefore, if **(i)** holds, we deduce that τ is an isomorphism and hence $(\mathcal{F}, \tau)$ is a quasi-coherent $\mathbf{G}$-sheaf.
Assume now that **(ii)** holds. Then $(\mu \times 1_X)^*\tau = \pi_{23}^*\tau \cdot (1_{\mathbf{G}} \times_k a)^*\,\tau$. Thus

$$\langle e \cdot q, e \cdot q, 1_X\rangle^*(\mu \times 1_X)^*\tau = \langle e \cdot q, 1_X\rangle^*\tau$$

and

$$\langle e \cdot q, e \cdot q, 1_X\rangle^*\left(\pi_{23}^*\tau \cdot (1_{\mathbf{G}} \times_k a)^*\,\tau\right) = \langle e \cdot q, 1_X\rangle^*\tau \cdot \langle e \cdot q, 1_X\rangle^*\tau$$

Hence $\langle e \cdot q, 1_X\rangle^*\tau = \langle e \cdot q, 1_X\rangle^*\tau \cdot \langle e \cdot q, 1_X\rangle^*\tau$. Since τ is an isomorphism, we derive that $\langle e \cdot q, 1_X\rangle^*\tau = 1_{\mathcal{F}}$. □

Let $\mathbf{G}$ be an affine group k-scheme. We describe quasi-coherent $\mathbf{G}$-sheaves on locally linear $\mathbf{G}$-schemes.

Theorem 3.12.7. *Let $\mathbf{G}$ be an affine group k-scheme and let X be a k-scheme equipped with an action a of $\mathbf{G}$ that makes X a locally linear $\mathbf{G}$-scheme. Let $\pi : \mathbf{G} \times_k X \to X$ be the projection. Suppose that $\mathcal{F}$ is a quasi-coherent sheaf on X. Assume that $\gamma : \mathcal{F} \to a_*\pi^*\mathcal{F}$ is a morphism of quasi-coherent sheaves on X. Then the following are equivalent.*

(i) *For every $\mathbf{G}$-stable open affine subscheme U of X consider the morphism*

$$\mathcal{F}(U) \to k[\mathbf{G}] \otimes_k \mathcal{F}(U)$$

determined as the composition of $\Gamma(U, \gamma)$ with the identification

$$\Gamma(\mathbf{G} \times_k U, \pi^*\mathcal{F}) = k[\mathbf{G}] \otimes_k \mathcal{F}(U)$$

Then this morphism is a coaction of $k[\mathbf{G}]$ on $\mathcal{F}(U)$.

(ii) *Let τ be the image of γ under the isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, a_*\pi^*\mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbf{G}\times_k X}}(a^*\mathcal{F}, \pi^*\mathcal{F})$$

for $a^* \dashv a_*$. Then $(\mathcal{F}, \tau)$ is a quasi-coherent $\mathbf{G}$-sheaf on X.

Proof. Let $\mu : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ be the multiplication and $e : \operatorname{Spec} k \to \mathbf{G}$ be the unit of the group k-scheme structure on $\mathbf{G}$. Moreover, we denote by $\pi_{23} : \mathbf{G} \times_k \mathbf{G} \times_k X \to \mathbf{G} \times_k X$ the projection on the last two factors and by $q : X \to \operatorname{Spec} k$ the unique morphism.
Let τ be the image of γ under the bijection

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, a_*\pi^*\mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbf{G}\times_k X}}(a^*\mathcal{F}, \pi^*\mathcal{F})$$

for $a^* \dashv a_*$. Fix an open $\mathbf{G}$-stable affine subscheme U of X. Let c be the morphism

$$\mathcal{F}(U) \to k[\mathbf{G}] \otimes_k \mathcal{F}(U)$$

determined as the composition of $\Gamma(U, \gamma)$ with the identification

$$\Gamma(\mathbf{G} \times_k U, \pi^*\mathcal{F}) = k[\mathbf{G}] \otimes_k \mathcal{F}(U)$$

Next observe that $\gamma = a_*\tau \cdot \eta_{\mathcal{F}}$, where $\eta_{\mathcal{F}} : \mathcal{F} \to a_*a^*\mathcal{F}$ is the unit of $a^* \dashv a_*$. Thus c is the composition of

$$\Gamma(\mathbf{G} \times_k U, \tau) \cdot \Gamma(U, \eta_{\mathcal{F}})$$

with the identification $\Gamma(\mathbf{G} \times_k U, \pi^*\mathcal{F}) = k[\mathbf{G}] \otimes_k \mathcal{F}(U)$. Note that $\Gamma(U, \eta_{\mathcal{F}})(s) = a^*s$ for every s in $\mathcal{F}(U)$. Fix now s in $\mathcal{F}(U)$. Suppose that

$$c(s) = \sum_{i=1}^{n} a_i \otimes s_i$$

where $a_i \in k[\mathbf{G}]$ and $s_i \in \mathcal{F}(U)$ for all i. Then

$$\big(1_{k[\mathbf{G}]} \otimes_k c\big)\big(c(s)\big) = \sum_{i=1}^{n} a_i \otimes c(s_i) = \sum_{i=1}^{n}\Bigg(\Gamma\left(\mathbf{G}\times_k \mathbf{G}\times_k U, \pi_{23}^{*}\tau\right)\left(a_i \otimes a^{*}s_i\right)\Bigg) =$$

$$= \Gamma\left(\mathbf{G}\times_k \mathbf{G}\times_k U, \pi_{23}^{*}\tau\right)\big(\left(1_{\mathbf{G}}\times_k a\right)^{*} c(s)\big) =$$

$$= \Bigg(\Gamma\left(\mathbf{G}\times_k \mathbf{G}\times_k U, \pi_{23}^{*}\tau\right)\cdot \Gamma\big(\mathbf{G}\times_k \mathbf{G}\times_k U, \left(1_{\mathbf{G}}\times_k a\right)^{*}\tau\big)\Bigg)\big(\left(1_{\mathbf{G}}\times_k a\right)^{*} a^{*}s)\big) =$$

$$= \Gamma\big(\mathbf{G}\times_k \mathbf{G}\times_k U, \pi_{23}^{*}\tau\cdot\left(1_{\mathbf{G}}\times_k a\right)^{*}\tau\big)\big(\left(1_{\mathbf{G}}\times_k a\right)^{*} a^{*}s)\big)$$

and

$$\big(\Delta_{\mathbf{G}} \otimes_k 1_{\mathcal{F}(U)}\big)\big(c(s)\big) = \left(\mu\times_k 1_X\right)^{*} c(s) = \Gamma\left(\mathbf{G}\times_k \mathbf{G}\times_k U, \left(\mu\times_k 1_X\right)^{*}\tau\right)\big(\left(\mu\times_k 1_X\right)^{*} a^{*}s\big)$$

where $\Delta_{\mathbf{G}}$ is the comultiplication of $k[\mathbf{G}]$. Since s is an arbitrary section of $\mathcal{F}$ over U, we derive that

$$\big(1_{k[\mathbf{G}]} \otimes_k c\big)\cdot c = \big(\Delta_{\mathbf{G}} \otimes_k 1_{\mathcal{F}(U)}\big)\cdot c$$

if and only if

$$\Gamma\big(\mathbf{G}\times_k \mathbf{G}\times_k U, \pi_{23}^{*}\tau\cdot\left(1_{\mathbf{G}}\times_k a\right)^{*}\tau\big) = \Gamma\left(\mathbf{G}\times_k \mathbf{G}\times_k U, \left(\mu\times_k 1_X\right)^{*}\tau\right)$$

Next suppose that $\xi_{\mathbf{G}} : k \to k[\mathbf{G}]$ is the counit of $k[\mathbf{G}]$. Then

$$\sum_{i=1}^{n}\xi_{\mathbf{G}}(a_i)\cdot s_i = \langle e\cdot q, 1_X\rangle^{*}c(s) = \Gamma\big(U, \langle e\cdot q, 1_X\rangle^{*}\tau\big)\big(\langle e\cdot q, 1_X\rangle^{*}a^{*}s\big) = \Gamma\big(U, \langle e\cdot q, 1_X\rangle^{*}\tau\big)(s)$$

Since s is arbitrary section of $\mathcal{F}$ over U, we derive that $\big(\xi_{\mathbf{G}} \otimes_k 1_{\mathcal{F}(U)}\big)\cdot c$ is isomorphic with $1_{\mathcal{F}(U)}$ if and only if

$$\Gamma\big(U, \langle e\cdot q, 1_X\rangle^{*}\tau\big) = 1_{\mathcal{F}(U)}$$

Thus c is a coaction of $k[\mathbf{G}]$ if and only if

$$\Gamma\big(\mathbf{G}\times_k \mathbf{G}\times_k U, \pi_{23}^{*}\tau\cdot\left(1_{\mathbf{G}}\times_k a\right)^{*}\tau\big) = \Gamma\left(\mathbf{G}\times_k \mathbf{G}\times_k U, \left(\mu\times_k 1_X\right)^{*}\tau\right)$$

and

$$\Gamma\big(U, \langle e, 1_X\rangle^{*}\tau\big) = 1_{\mathcal{F}(U)}$$

Now X is a locally linear $\mathbf{G}$-scheme. Hence X has an open cover by $\mathbf{G}$-stable open affine subsets like U. From this assumption we deduce that **(i)** is equivalent with the fact that formulas

$$\pi_{23}^{*}\tau\cdot\left(1_{\mathbf{G}}\times_k a\right)^{*}\tau = \left(\mu\times_k 1_X\right)^{*}\tau,\ \langle e, 1_X\rangle^{*}\tau = 1_{\mathcal{F}}$$

hold. By Proposition 3.12.6 it follows that these these formulas hold if and only if **(ii)** holds. Thus assertions **(i)** and **(ii)** are equivalent. □

Remark 3.12.8. Theorem 3.12.7 gives rise to the description of the category $\mathfrak{Qcoh}_{\mathbf{G}}(X)$, where X is a k-scheme equipped with an action $a : \mathbf{G}\times_k X \to X$ of affine group k-scheme $\mathbf{G}$ that makes it into a $\mathbf{G}$-linear scheme. We give now details of this description. Denote by π :

$\mathbf{G} \times_k X \to X$ the projection. Objects of $\mathfrak{Qcoh}_{\mathbf{G}}(X)$ are pairs $(\mathcal{F}, \gamma)$ consisting of a quasi-coherent sheaf $\mathcal{F}$ on X and a morphism $\gamma : \mathcal{F} \to a_*\pi^*\mathcal{F}$ of quasi-coherent sheaves on X such that for every open $\mathbf{G}$-stable affine subscheme U of X morphism

$$\Gamma(U, \gamma) : \mathcal{F}(U) \to k[\mathbf{G}] \otimes_k \mathcal{F}(U)$$

is a coaction of the bialgebra $k[\mathbf{G}]$. Now if $(\mathcal{F}_1, \gamma_1)$ and $(\mathcal{F}_2, \gamma_2)$ are two objects of $\mathfrak{Qcoh}_{\mathbf{G}}(X)$, then a morphism $\phi : (\mathcal{F}_1, \gamma_1) \to (\mathcal{F}_2, \gamma_2)$ is a morphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ of quasi-coherent sheaves on X such that the square

$$\begin{tikzcd} \mathcal{F}_1 \arrow[r, "\gamma_1"] \arrow[d, "\phi"'] & a_*\pi^*\mathcal{F}_1 \arrow[d, "a_*\pi^*\phi"] \\ \mathcal{F}_2 \arrow[r, "\gamma_2"'] & a_*\pi^*\mathcal{F}_2 \end{tikzcd}$$

is commutative. Moreover, if X is locally noetherian, then analogical description is valid for $\mathfrak{Coh}_{\mathbf{G}}(X)$.

The next two examples are consequences of Remark 3.12.8.

Example 3.12.9. Consider $\operatorname{Spec} k$ as a k-scheme with the trivial action of an affine group k-scheme $\mathbf{G}$. Then $\mathfrak{Qcoh}_{\mathbf{G}}(\operatorname{Spec} k)$ is isomorphic with $\mathbf{Rep}(\mathbf{G})$. If k is a field, then $\mathfrak{Coh}_{\mathbf{G}}(\operatorname{Spec} k)$ is isomorphic with the category $\mathbf{Repf}(\mathbf{G})$ of finite dimensional representations of $\mathbf{G}$.

The example above can be generalized.

Example 3.12.10. Let $\mathbf{G}$ be an affine group k-scheme and let X be a k-scheme equipped with the action of $\mathbf{G}$ given by the projection $\pi : \mathbf{G} \times_k X \to X$. Suppose that $\mathcal{F}$ is a quasi-coherent sheaf on X. Then to give a structure of $\mathbf{G}$-sheaf on $\mathcal{F}$ is the same as to give a morphism $\mathcal{F} \to \pi^*\mathcal{F}$ of quasi-coherent sheaves on X such that for every open affine subscheme U the induced morphism

$$\mathcal{F}(U) \to k[\mathbf{G}] \otimes_k \mathcal{F}(U)$$

is the coaction of $k[\mathbf{G}]$. In other words to give a structure of $\mathbf{G}$-sheaf on $\mathcal{F}$ is the same as to give a structure of $\mathbf{G}$-representation on $\mathcal{F}(U)$ for every open affine subscheme U of X in such a way that the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ becomes a morphism of $\mathbf{G}$-representations for every pair $U \subseteq V$ of open affine subschemes of X. In this way we obtain a description of $\mathfrak{Qcoh}_{\mathbf{G}}(X)$ and $\mathfrak{Coh}_{\mathbf{G}}(X)$ if X is locally noetherian.

Chapter 4

Algebraic Groups

4.1 Introduction

We introduce elements of the theory of algebraic groups. There are two main results of this chapter. The first is a classical theorem which states that the image of an algebraic group under a morphism of algebraic groups is a closed algebraic subgroup. The other concerns the representability of $X^{\mathbf{G}}$ for group schemes over k. Note that classically [DG70, Thèoréme 3.6 on page 165] this result was known for separated schemes. Result that we present here (Theorem 4.5.1) is slightly more general and its proof is based on some mixture of ideas of Drinfeld ([Dri13, Proposition 1.2.2]) and the original approach of Gabriel and Demazure. In the last two sections of this chapter we study linearly reductive groups and tori.
Throughout this chapter we assume that k is a field.

4.2 Functions on products of quasi-compact and semi-separated schemes

Definition 4.2.1. Let Y be a scheme and let X be a Y-scheme. If the diagonal $X \to X \times_Y X$ is affine, then we say that X is *semi-separated* over Y.

Remark 4.2.2. Let Y be a scheme. Every separated Y-scheme is semi-separated.

Example 4.2.3 (Semi-separated scheme that is not separated)**.** Let o be the origin of the affine line $\mathbb{A}^1_k$. Consider the following pushout diagram in the category of k-schemes.

$$\begin{tikzcd} \mathbb{A}^1_k \setminus \{o\} \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^1_k \arrow[d] \\ \mathbb{A}^1_k \arrow[r] & X \end{tikzcd}$$

Then X is an affine line with double origin. The diagonal $X \to X \times_k X$ is affine but not a closed immersion. Hence X is semi-separated but not separated.

Theorem 4.2.4. *Let X, Y be quasi-compact and semi-separated k-schemes. Denote by π_X and π_Y*

projections from $X \times_k Y$ to X and Y, respectively. Let $\mathcal{F}$ and $\mathcal{G}$ be quasi-coherent sheaves on X and Y, respectively. Then the canonical morphism

$$\Gamma(X, \mathcal{F}) \otimes_k \Gamma(Y, \mathcal{G}) \ni s \otimes t \mapsto \pi_X^* s \otimes \pi_Y^* t \in \Gamma\left(X \times_k Y, \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right)$$

is an isomorphism.

The theorem follows from the following result.

Lemma 4.2.4.1. *Let X, Y be k-schemes and let $\{V_i\}_{i=1}^n$ be a finite open cover of Y. Suppose that the canonical morphism*

$$\Gamma(X, \mathcal{F}) \otimes_k \Gamma\left(V_i \cap V_j, \mathcal{G}\right) \to \Gamma\left(X \times_k \left(V_i \cap V_j\right), \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right)$$

is an isomorphism for any two (not necessarily distinct) $i, j \in \{1, ..., n\}$. Then the canonical morphism

$$\Gamma(X, \mathcal{F}) \otimes_k \Gamma(Y, \mathcal{G}) \to \Gamma\left(X \times_k Y, \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right)$$

is an isomorphism.

Proof of the lemma. For each $i \in \{1, ..., n\}$ we have the restriction

$$r_i : \Gamma\left(X \times_k Y, \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right) \to \Gamma\left(X \times_k V_i, \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right)$$

and we denote by p_i the restriction $\Gamma(Y, \mathcal{G}) \to \Gamma(V_i, \mathcal{G})$ tensored with $\Gamma(X, \mathcal{O}_X)$ over k. For $i, j \in \{1, ..., n\}$ we have the restriction

$$r_{i,j} : \Gamma\left(X \times_k V_i, \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right) \to \Gamma\left(X \times_k \left(V_i \cap V_j\right), \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right)$$

and we denote by $p_{i,j}$ the restriction $\Gamma(V_i, \mathcal{G}) \to \Gamma\left(V_i \cap V_j, \mathcal{G}\right)$ tensored with $\Gamma(X, \mathcal{F})$ over k. Consider the commutative diagram

$$\begin{tikzcd}
\Gamma\left(X \times_k Y, \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right) \arrow[r, "\langle r_i \rangle_{i=1}^n"] & \bigoplus_{i=1}^n \Gamma\left(X \times_k V_i, \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right) \arrow[r, shift left, "\langle r_{i,j} \cdot \mathrm{pr}_i \rangle_{1 \le i < j \le n}"] \arrow[r, shift right, "\langle r_{j,i} \cdot \mathrm{pr}_j \rangle_{1 \le i < j \le n}"'] & \bigoplus_{1 \le i < j \le n} \Gamma\left(X \times_k \left(V_i \cap V_j\right), \pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}\right) \\
\Gamma(X, \mathcal{F}) \otimes_k \Gamma(Y, \mathcal{G}) \arrow[u] \arrow[r, "\langle p_i \rangle_{i=1}^n"] & \bigoplus_{i=1}^n \Gamma(X, \mathcal{F}) \otimes_k \Gamma(V_i, \mathcal{G}) \arrow[u, "\simeq"] \arrow[r, shift left, "\langle p_{i,j} \cdot \mathrm{pr}_i \rangle_{1 \le i < j \le n}"] \arrow[r, shift right, "\langle p_{j,i} \cdot \mathrm{pr}_j \rangle_{1 \le i < j \le n}"'] & \bigoplus_{1 \le i < j \le n} \Gamma(X, \mathcal{F}) \otimes_k \Gamma\left(V_i \cap V_j, \mathcal{G}\right) \arrow[u, "\simeq"']
\end{tikzcd}$$

in which vertical arrows are canonically defined. Moreover, by assumptions right and middle vertical arrows are isomorphisms. Note also that both rows are kernel diagrams. Indeed, for the top row this follows from the sheaf property of $\pi_X^* \mathcal{F} \otimes_{\mathcal{O}_{X \times_k Y}} \pi_Y^* \mathcal{G}$ and for the bottom row this follows from the fact that $\Gamma(X, \mathcal{F})$ is flat over k (k is a field) together with the sheaf property of $\mathcal{G}$. These imply that the left vertical arrow is an isomorphism and this completes the proof. $\square$

Proof of the theorem. The statement holds, if X, Y are affine. Note that semi-separatedness of a scheme over a field (commutative ring) is equivalent to the fact that intersection of every pair of its open affine subschemes is affine. Now Lemma 4.2.4.1 implies that the result holds if X is affine and Y is quasi-compact and semi-separated over k. Next by symmetry in Lemma 4.2.4.1, we derive that the result holds if X, Y are quasi-compact and semi-separated over k. $\square$

Corollary 4.2.5. *Let X, Y be quasi-compact and semi-separated k-schemes. Denote by π_X and π_Y projections from $X \times_k Y$ to X and Y, respectively. Then the canonical morphism*

$$\Gamma(X, \mathcal{O}_X) \otimes_k \Gamma(Y, \mathcal{O}_Y) \ni f \otimes_k g \mapsto \pi_X^{\#}(f) \cdot \pi_Y^{\#}(g) \in \Gamma\left(X \times_k Y, \mathcal{O}_{X \times_k Y}\right)$$

is an isomorphism.

Corollary 4.2.6. *Let $\mathbf{Sch}_k^{\mathrm{qc,ss}}$ be the category of quasi-compact and semi-separated schemes over k. Then the contravariant functor*

$$\mathbf{Sch}_k^{\mathrm{qc,ss}} \ni X \mapsto \Gamma(X, \mathcal{O}_X) \in \left(\mathbf{Alg}_k\right)^{\mathrm{op}}$$

preserves products.

Proof. This is a reformulation of Corollary 4.2.5. □

4.3 General properties of groups schemes over a field

In this section we prove some elementary properties of group schemes over a field.

Proposition 4.3.1. *Let $\mathbf{G}$ be a group scheme over k. Then $\mathbf{G}$ is a separated k-scheme.*

Proof. Consider a morphism $f : \mathbf{G} \times_k \mathbf{G} \to \mathbf{G}$ given on A-points g_1, g_2 of $\mathbf{G}$ by formula

$$(g_1, g_2) \mapsto g_1 \cdot g_2^{-1}$$

where A is a k-algebra. Note that we have a cartesian square

$$\begin{tikzcd} \mathbf{G} \arrow[r] \arrow[d, "\delta_{\mathbf{G}}"'] & \operatorname{Spec} k \arrow[d, "e"] \\ \mathbf{G} \times_k \mathbf{G} \arrow[r, "f"'] & \mathbf{G} \end{tikzcd}$$

where $\delta_{\mathbf{G}}$ is the diagonal of $\mathbf{G}$, the top horizontal arrow $\mathbf{G} \to \operatorname{Spec} k$ is the structure morphism and $e : \operatorname{Spec} k \to \mathbf{G}$ is the identity of $\mathbf{G}$. Since base change of a closed immersion is a closed immersion, we derive that $\delta_{\mathbf{G}}$ is a closed immersion if e is a closed immersion. Since $\mathbf{G}$ is a k-scheme and k is a field, it follows that every morphism $\operatorname{Spec} k \to \mathbf{G}$ of k-schemes is a closed immersion (every k-point in a scheme over k is closed). In particular, e is a closed immersion and hence $\mathbf{G}$ is separated. □

Remark 4.3.2. Let $\mathfrak{G}$ be a group k-functor and let $\alpha : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{X}$ be an action of $\mathfrak{G}$ on $\mathfrak{X}$. Consider an isomorphism $\phi : \mathfrak{G} \times \mathfrak{X} \to \mathfrak{G} \times \mathfrak{X}$ of k-functors given by

$$\mathfrak{G}(A) \times \mathfrak{X}(A) \ni (g, x) \mapsto (g, g^{-1}x) \in \mathfrak{G}(A) \times \mathfrak{X}(A)$$

for every k-algebra A. Then the triangle

$$\begin{tikzcd} \mathfrak{G} \times \mathfrak{X} \arrow[rr, "\phi"] \arrow[dr, "\pi_{\mathfrak{X}}"'] & & \mathfrak{G} \times \mathfrak{X} \arrow[dl, "a"] \\ & \mathfrak{X} & \end{tikzcd}$$

is commutative.

Corollary 4.3.3. *Let* $\mathbf{G}$ *be a group scheme over* k *and let* $a : \mathbf{G} \times_k X \to X$ *be an action of* $\mathbf{G}$ *on a* k*-scheme* X*. Then* a *is isomorphic with the projection* $\pi_X : \mathbf{G} \times_k X \to X$*.*

Proof. This is a consequence of Remark 4.3.2. □

Corollary 4.3.4. *Let* $\mathbf{G}$ *be a group scheme over* k *and let* $a : \mathbf{G} \times_k X \to X$ *be an action of* $\mathbf{G}$ *on* k*-scheme* X*. Then* a *is faithfully flat.*

Proof. This is a direct consequence of Corollary 4.3.3 and the fact that each group scheme $\mathbf{G}$ over a field k is faithfully flat. □

4.4 Algebraic groups and their actions

There is some ambiguity in literature concerning the notion of algebraic group. Some discussion related to this topic can be found in [Mil17, Notes on page 12]. In this work we decided to define this notion as in [Mil17] and [DG70].

Definition 4.4.1. Let $\mathbf{G}$ be a group scheme over k. If $\mathbf{G}$ is locally of finite type over k, then we say that $\mathbf{G}$ is *a locally algebraic group over* k. We say that $\mathbf{G}$ is *an algebraic group over* k if it is of finite type over k.

Corollary 4.4.2. *Let* $\mathbf{G}$ *be a locally algebraic group over* k *and let* $a : \mathbf{G} \times_k X \to X$ *be an action of* $\mathbf{G}$ *on* k*-scheme* X*. Then* a *is universally open.*

Proof. By Corollary 4.3.3 the action a is isomorphic with the projection $\pi_X : \mathbf{G} \times_k X \to X$. Since $\mathbf{G}$ is locally algebraic group over k the projection π_X is locally of finite type and flat. Thus by [GW10, Theorem 14.33] π_X is universally open. Hence also a is universally open. □

Remark 4.4.3. According to [GW10, Corollary 5.45] the projection $\pi_X : Y \times_k X \to X$ is universally open regardless of finiteness assumptions on Y.

Corollary 4.4.4. *Let* $\mathbf{G}$ *be a locally algebraic group over* k *and let* $a : \mathbf{G} \times_k X \to X$ *be an action of* $\mathbf{G}$ *on* k*-scheme* X*. If* U *is an open subscheme of* X*, then* $a\left(\mathbf{G} \times_k U\right)$ *is the smallest open* $\mathbf{G}$*-stable subscheme of* X *containing* U*.*

Proof. First note that by Corollary 4.4.2 $a(\mathbf{G} \times_k U)$ is indeed an open subscheme of X. Denote it by $\mathbf{G} \cdot U$. Observe that

$$a\left(\mathbf{G} \times_k \mathbf{G} \cdot U\right) = a\Big(\left(1_{\mathbf{G}} \times_k a\right)\left(\mathbf{G} \times_k \mathbf{G} \times_k U\right)\Big) =$$

$$= a\Big(\big(\mu \times_k 1_X\big)\big(\mathbf{G} \times_k \mathbf{G} \times_k U\big)\Big) = a\big(\mathbf{G} \times_k U\big) = \mathbf{G} \cdot U$$

This implies that $a_{|\mathbf{G}\times_k \mathbf{G}\cdot U}$ factors through $\mathbf{G} \cdot U$ and hence this open subscheme is $\mathbf{G}$-stable. It remains to prove that $\mathbf{G} \cdot U$ is contained in every open $\mathbf{G}$-stable subscheme W of X which contains U. Note that for such W we have

$$\mathbf{G} \cdot U = a(\mathbf{G} \times_k U) \subseteq a(\mathbf{G} \times_k W) \subseteq W$$

Thus the proof is complete. □

Definition 4.4.5. Let $\mathbf{G}$ be a locally algebraic group over k and let $a : \mathbf{G} \times_k X \to X$ be an action of $\mathbf{G}$ on k-scheme X. Fix an open subset U of X. Then we denote by $\mathbf{G} \cdot U$ the smallest open $\mathbf{G}$-stable subscheme of X. We call it *the* $\mathbf{G}$*-stable hull of* U.

Definition 4.4.6. A morphism of schemes $f : X \to Y$ is *an fpqc-morphism* if f is faithfully flat and for every open affine subscheme V of Y there exists a quasi-compact open subscheme U of X such that $f(U) = V$.

The notion of fpqc-morphism is introduced in [FGI05, Definition 2.34]. Its importance lies in the fact that most interesting classes of morphisms of schemes descend along fpqc-morphism ([FGI05, Proposition 2.36]).

Corollary 4.4.7. *Let* $\mathbf{G}$ *be a locally algebraic group over* k *and let* $a : \mathbf{G} \times_k X \to X$ *be an action of* $\mathbf{G}$ *on* k*-scheme* X*. If* U *is an open subscheme of* X*, then the restriction* $\mathbf{G} \times_k U \to \mathbf{G} \cdot U$ *of* a *is an fpqc-morphism.*

Proof. Corollaries 4.3.4 and 4.4.3 show that $\mathbf{G} \times_k U \to \mathbf{G} \cdot U$ is faithfully flat and open. Hence according to [FGI05, (iii) of Proposition 2.35] the morphism in question is fpqc. □

Theorem 4.4.8. *Let* $\mathbf{G}$ *be a locally algebraic group over* k*, let* X, Y *be* k*-schemes with* $\mathbf{G}$*-actions and let* $f : X \to Y$ *be a* $\mathbf{G}$*-equivariant morphism. Suppose that* $\mathbf{P}$ *is a property of morphisms of* k*-schemes such that the following conditions hold.*

- **(1)** $\mathbf{P}$ *is local on the base.*
- **(2)** $\mathbf{P}$ *is closed under base change.*
- **(3)** $\mathbf{P}$ *descends along fpqc base change.*

Then there exists the largest open subset of Y *such that the restriction* $f^{-1}(V) \to V$ *of* f *is in* $\mathbf{P}$ *and it is* $\mathbf{G}$*-stable.*

Proof. Note that the existence of V follows from **(1)**. We denote by $\tilde{f}$ the restriction of f to $f^{-1}(V) \to V$. We also denote by $\hat{f} : \mathbf{G} \cdot f^{-1}(V) \to \mathbf{G} \cdot V$ the restriction of f. Since the square

$$\begin{tikzcd}
\mathbf{G} \times_k f^{-1}(V) \arrow[r, "\text{action of } \mathbf{G}"] \arrow[d, "1_{\mathbf{G}} \times_k \tilde{f}"'] & X \arrow[d, "f"] \\
\mathbf{G} \times_k V \arrow[r, "\text{action of } \mathbf{G}"'] & Y
\end{tikzcd}$$

is cartesian (this can be checked on k-functors of points), we derive by Corollary 4.4.4 that the square

$$\begin{tikzcd} \mathbf{G}\times_k f^{-1}(V) \arrow[r, "\text{action of } \mathbf{G}"] \arrow[d, "1_{\mathbf{G}}\times_k \tilde{f}"'] & \mathbf{G}\cdot f^{-1}(V) \arrow[d, "\hat{f}"] \\ \mathbf{G}\times_k V \arrow[r, "\text{action of } \mathbf{G}"'] & \mathbf{G}\cdot V \end{tikzcd}$$

is cartesian. The assumption **(2)** implies that $1_{\mathbf{G}}\times_k \tilde{f}$ is in $\mathbf{P}$. Since the bottom horizontal morphism is fpqc by Corollary 4.4.7, we deduce by **(3)** that $\hat{f}$ is in $\mathbf{P}$. Since V is the largest open subset of Y such that the restriction $f^{-1}(V)\to V$ of f is in $\mathbf{P}$ and

$$f^{-1}\left(\mathbf{G}\cdot V\right) = \mathbf{G}\cdot f^{-1}(V)$$

we derive that $\mathbf{G}\cdot V\subseteq V$. Hence $V=\mathbf{G}\cdot V$, which means that V is $\mathbf{G}$-stable. □

Now we prove the fundamental result.

Theorem 4.4.9. *Let $f:\mathbf{H}\to\mathbf{G}$ be a morphism of locally algebraic groups over k. Suppose that f is of finite type. Let $i:\mathbf{K}\to\mathbf{G}$ be the scheme-theoretic image of f and let $g:\mathbf{H}\to\mathbf{K}$ be the unique morphism of schemes such that $f=i\cdot g$. Then the following assertions hold.*

- **(1)** $\mathbf{K}$ *is a closed subgroup k-scheme of* $\mathbf{G}$.
- **(2)** *g is a surjective morphism of group schemes over k.*

Proof. Since f is quasi-compact, we deduce that $i:\mathbf{K}\hookrightarrow\mathbf{G}$ is a closed immersion determined by the kernel of $f^{\#}:\mathcal{O}_{\mathbf{G}}\to f_*\mathcal{O}_{\mathbf{H}}$ ([GW10, Proposition 10.30]) and $g^{\#}:\mathcal{O}_{\mathbf{K}}\to g_*\mathcal{O}_{\mathbf{H}}$ is an injective morphism of sheaves. Moreover, the fact that f is quasi-compact implies that g is quasi-compact. Fix two affine open subschemes V,W of $\mathbf{K}$. We derive by Corollary 4.2.5 and Proposition 4.3.1 that the square

$$\begin{tikzcd} \mathcal{O}_{\mathbf{K}\times_k\mathbf{K}}(V\times_k W) \arrow[r, "(g\times_k g)^{\#}_{V\times_k W}"] & \mathcal{O}_{\mathbf{H}\times_k\mathbf{H}}\left(g^{-1}(V)\times_k g^{-1}(W)\right) \\ \mathcal{O}_{\mathbf{K}}(V)\otimes_k\mathcal{O}_{\mathbf{K}}(W) \arrow[u, "\simeq"] \arrow[r, "g^{\#}_V\otimes_k g^{\#}_W"'] & \mathcal{O}_{\mathbf{H}}\left(g^{-1}(V)\right)\otimes_k\mathcal{O}_{\mathbf{H}}\left(g^{-1}(W)\right) \arrow[u, "\simeq"'] \end{tikzcd}$$

is commutative, where vertical arrows are canonical isomorphisms. This implies that the morphism $(g\times_k g)^{\#}$ of sheaves is injective. Consider the commutative diagram

$$
\begin{tikzcd}
\mathbf{H}\times_k\mathbf{H} \arrow[r, "g\times_k g"] \arrow[d, "\nu_{\mathbf{H}}"'] & \mathbf{K}\times_k\mathbf{K} \arrow[r, hook, "i\times_k i"] & \mathbf{G}\times_k\mathbf{G} \arrow[d, "\nu_{\mathbf{G}}"] \\
\mathbf{H} \arrow[r, "g"'] & \mathbf{K} \arrow[r, hook, "i"'] & \mathbf{G}
\end{tikzcd}
$$

where $\nu_{\mathbf{G}}$ and $\nu_{\mathbf{H}}$ are morphisms determined by formula $(x_1, x_2) \mapsto x_1^{-1} \cdot x_2$ on k-functors of points. Commutativity of the diagram implies that we have equality

$$\Big((\nu_{\mathbf{G}})_* (i \times_k i)_* (g \times_k g)^{\#} \Big) \cdot \Big((\nu_{\mathbf{G}})_* (i \times_k i)^{\#} \Big) \cdot \nu_{\mathbf{G}}^{\#} = \big(i_* g_* (\nu_{\mathbf{H}})^{\#} \big) \cdot \big(i_* g^{\#} \big) \cdot i^{\#}$$

of morphism of sheaves. This equality together with injectivity of $(g \times_k g)^{\#}$ implies that the kernel of

$$\big(\nu_{\mathbf{G}} \cdot (i \times_k i) \big)^{\#} = \Big((\nu_{\mathbf{G}})_* (i \times_k i)^{\#} \Big) \cdot \nu_{\mathbf{G}}^{\#}$$

contains $\ker(i^{\#})$. Thus $\nu_{\mathbf{G}} \cdot (i \times_k i)$ factors through i. Hence there exists a unique morphism ν such that the square

$$
\begin{tikzcd}
\mathbf{K}\times_k\mathbf{K} \arrow[r, hook, "i\times_k i"] \arrow[d, "\nu"'] & \mathbf{G}\times_k\mathbf{G} \arrow[d, "\nu_{\mathbf{G}}"] \\
\mathbf{K} \arrow[r, hook, "i"'] & \mathbf{G}
\end{tikzcd}
$$

is commutative. This implies that $i : \mathbf{K} \hookrightarrow \mathbf{G}$ is a closed subgroup k-scheme of $\mathbf{G}$. Indeed, if k_1, k_2 are A-points of $\mathbf{K}$ for some A-algebra, then the commutativity of the square above implies that the A-point $k_1^{-1} \cdot k_2$ of $\mathbf{G}$ is the A-point of $\mathbf{K}$ and this is well known criterion for subgroup. Since i is a monomorphism and

$$i \cdot \nu \cdot (g \times_k g) = \nu_{\mathbf{G}} \cdot (i \times_k i) \cdot (g \times_k g) = i \cdot g \cdot \nu_{\mathbf{H}}$$

we derive that $\nu \cdot (g \times_k g) = g \cdot \nu_{\mathbf{H}}$. Hence g is a morphism of group schemes over k. It remains to prove that $g : \mathbf{H} \to \mathbf{K}$ is surjective. Recall that g is of finite type and $g^{\#}$ is injective. Note that these properties are preserved under base change to an algebraic closure of k. Moreover, the surjectivity of morphism descends along faithfully flat base change. Thus we may assume that k is algebraically closed. By [GW10, Theorem 10.20] and the fact that g is of finite type, we deduce that $g(\mathbf{H})$ is a constructible subset locally on $\mathbf{K}$. Since $g^{\#}$ is injective, we derive that set-theoretic image $g(\mathbf{H}) \subseteq \mathbf{K}$ is dense. Thus $g(\mathbf{H})$ is dense and locally constructible. Hence there exists an open and dense subset V of $\mathbf{K}$ such that $V \subseteq g(\mathbf{H})$. Since k is algebraically closed and $\mathbf{K}$ is locally algebraic, we may pick a k-point v in V. Since $V \subseteq g(\mathbf{H})$, we deduce that $v \in g(\mathbf{H})$ and thus $v^{-1} \in g(\mathbf{H})$. Hence

$$W = v^{-1} \cdot V \subseteq g(\mathbf{H}) \cdot g(\mathbf{H}) \subseteq g(\mathbf{H})$$

Thus W is an open neighborhood of the identity in $\mathbf{K}$, dense in $\mathbf{K}$ and contained in $g(\mathbf{H})$. Next

$$g(\mathbf{H}) \subseteq g(\mathbf{H}) \cdot W \subseteq g(\mathbf{H}) \cdot g(\mathbf{H}) \subseteq g(\mathbf{H})$$

Thus $g(\mathbf{H})$ is open in $\mathbf{K}$. Now if $u \in \mathbf{K} \setminus g(\mathbf{H})$ is a k-point, then

$$u \cdot g(\mathbf{H}) \cap g(\mathbf{H}) = \emptyset$$

as two distinct left cosets of an open subgroup $g(\mathbf{H})$ in $\mathbf{K}$ are disjoint. This is contradiction, because $u \cdot g(\mathbf{H})$ is an open neighborhood of u and $g(\mathbf{H})$ is dense in $\mathbf{K}$. Therefore, $g(\mathbf{H})$ is an open subset of $\mathbf{K}$ that contains all its k-points. Since k is algebraically closed and $\mathbf{K}$ is locally algebraic, this implies that the closed subset $\mathbf{K} \setminus g(\mathbf{H})$ is empty. Thus g is surjective. □

4.5 Representability of fixed points for group schemes over a field

Theorem 4.5.1. *Let $\mathbf{G}$ be a group scheme over k and let $a : \mathbf{G} \times_k X \to X$ be an action of $\mathbf{G}$ on a k-scheme X. Suppose that one of the following assertions hold.*

- **(i)** *X is separated.*
- **(ii)** *$\mathbf{G}$ is a geometrically connected, locally algebraic group.*

Then the fixed point functor $X^{\mathbf{G}}$ is a closed subscheme of X.

The following result is based on Theorem 2.5.2 and plays the fundamental role in the proof.

Lemma 4.5.1.1. *Let X, Y be k-schemes and let $a : Y \times_k X \to X$ be a morphism of k-schemes. Suppose that one of the following assertions hold.*

- **(1)** *X is separated.*
- **(2)** *For every open subscheme U of X we have $a(Y \times_k U) \subseteq U$*

Consider k-subfunctor X^a of X given by formula

$$A \mapsto \left\{ f : \operatorname{Spec} A \to X \,\middle|\, a \cdot (1_Y \times_k f) = \operatorname{pr}_X \cdot (1_Y \times_k f) \right\}$$

where A is a k-algebra and $\operatorname{pr}_X : Y \times_k X \to X$ is the projection. Then X^a is representable by a closed subscheme of X.

Proof of the lemma. In the proof we identify k-schemes with their k-functors of points (Remark 2.3.8). We use internal homs for k-functors representable by k-schemes. Note that by Fact 2.3.10 they exists.
Assume first that X is separated. Consider a morphism

$$\langle a, \operatorname{pr}_X \rangle : Y \times_k X \to X \times_k X$$

By Theorem 2.2.6 we deduce that $\langle a, \operatorname{pr}_X \rangle$ corresponds to a morphism $\sigma : X \to \mathcal{M}\mathrm{or}_k(Y, X \times X)$ of k-functors. Since X is separated, the diagonal $\delta_X : X \to X \times_k X$ is a closed immersion. This implies that δ_X is a closed immersion of k-functors. The fact that Y is locally free over k (Remark 2.5.3) and Theorem 2.5.2 imply that

$$\mathcal{M}\mathrm{or}_k(1_Y, \delta_X) : \mathcal{M}\mathrm{or}_k(Y, X) \hookrightarrow \mathcal{M}\mathrm{or}_k(Y, X \times X)$$

is a closed immersion of k-functors. Consider now a cartesian square

$$\begin{tikzcd} X^a \arrow[r] \arrow[d, "j"'] & \mathcal{M}\mathrm{or}_k(Y, X) \arrow[d, "\mathcal{M}\mathrm{or}_k(1_Y, \delta_X)"] \\ X \arrow[r, "\sigma"'] & \mathcal{M}\mathrm{or}_k(Y, X \times X) \end{tikzcd}$$

of k-functors. Fact 2.4.5 implies that $j : X^a \to X$ is a closed immersion of k-functors. Observe that $j : X^a \hookrightarrow X$ is precisely the inclusion of the k-subfunctor described in the statement. Corollary 2.4.11 completes the proof of **(1)**.
Now suppose that $a(Y \times_k U) \subseteq U$ for every open subscheme U of X. For every open subscheme denote by $a_U : Y \times_k U \to U$ the restriction of a. Let $\mathcal{U}$ be an open affine cover of X. Let $j : X^a \hookrightarrow X$ be a monomorphism of k-functors in the statement. For each U in $\mathcal{U}$ we have a cartesian square

$$\begin{tikzcd} U^{a_U} \arrow[r, hook] \arrow[d, hook, "j_U"'] & X^a \arrow[d, hook, "j"] \\ U \arrow[r, hook] & X \end{tikzcd}$$

where $U \hookrightarrow X$ is the inclusion and $U^{a_U} \hookrightarrow X^a$ interprets U^{a_U} as an open k-subfunctor of X^a. Since each U is separated, by virtue of **(1)** each $j_U : U^{a_U} \hookrightarrow U$ is a closed immersion of k-schemes. Since $\mathcal{U}$ is an open cover of X, it follows by simple argument that $j : X^a \hookrightarrow X$ is a closed immersion of k-functors. By virtue of Corollary 2.4.11 this proves **(2)**. □

Lemma 4.5.1.2. *Let $f : \mathbf{H} \to \mathbf{G}$ be a morphism of locally algebraic groups over k. Suppose that the following assertions hold.*

(1) *The morphism*

$$\widehat{\mathcal{O}_{\mathbf{G},e_{\mathbf{G}}}} \to \widehat{\mathcal{O}_{\mathbf{H},e_{\mathbf{H}}}}$$

induced by $f^\#$ is an isomorphism.

(2) *f is a monomorphism of k-schemes.*

Then f is an open immersion.

Proof of the lemma. Note that f is locally of finite type. The assertion **(1)** implies that f is étale in $e_{\mathbf{H}}$. Let K be an algebraic closure of k and let us use the following notation:

$$\mathbf{G}_K = \operatorname{Spec} K \times_k \mathbf{G},\ \mathbf{H}_K = \operatorname{Spec} K \times_k \mathbf{H},\ f_K = 1_{\operatorname{Spec} K} \times_k f$$

Consider the étale locus U of f_k. Then U is an open subscheme of $\mathbf{H}_K$ containing the identity of $\mathbf{G}_K$. Moreover, for every K-point h of $\mathbf{H}_K$ we have a commutative square

$$\begin{tikzcd} \mathbf{H}_K \arrow[r, "f_K"] \arrow[d, "h\cdot(-)"'] & \mathbf{G}_K \arrow[d, "f_K(h)\cdot(-)"] \\ \mathbf{H}_K \arrow[r, "f_K"'] & \mathbf{G}_K \end{tikzcd}$$

where $h \cdot (-)$ and $f_K(h) \cdot (-)$ are isomorphisms of K-schemes. This proves that $h \cdot U \subseteq U$. Hence U contains all K-rational points of $\mathbf{H}_K$. Therefore, the complement of U in $\mathbf{H}_K$ is a closed subset and does not contain K-points. Since $\mathbf{H}_K$ is a scheme locally of finite type over an algebraically closed field K, this proves that the complement of U is empty. Hence $U = \mathbf{H}_K$. This shows that f_K is étale and by faithfully flat descent also f is étale. Since étale monomorphisms are open immersions, we derive that f is an open immersion. $\square$

Proof of the theorem. If **(1)** holds, then the statement follows directly from Lemma 4.5.1.1 if $Y = \mathbf{G}$ and a is the action of $\mathbf{G}$ on X.
Suppose now that **(2)** holds. That is $\mathbf{G}$ is a geometrically connected, locally algebraic group over k. In the proof we use Theorem 2.3.6 and view $X^{\mathbf{G}}$ as a Zariski local k-functor. For each $n \in \mathbb{N}$ we define

$$\mathbf{G}_n = \operatorname{Spec} \mathcal{O}_{\mathbf{G},e_{\mathbf{G}}}/\mathfrak{m}_{e_{\mathbf{G}}}^{n+1}$$

where e is the unit of $\mathbf{G}$. Then $\mathbf{G}_n$ is the n-th infinitesimal neighborhood of e in $\mathbf{G}$. Denote by $p_n : \mathbf{G}_n \times_k X \to X$ the projection on the second factor. Let $a_n : \mathbf{G}_n \times_k X \to X$ be the morphism induced by a. Note that for every open subscheme U of X we have $a_n(\mathbf{G}_n \times_k U) \subseteq U$. By Lemma 4.5.1.1 it follows that the k-functor given by

$$\mathbf{Alg}_k \ni A \mapsto \left\{ f : \operatorname{Spec} A \to X \,\middle|\, a_n \cdot (1_{\mathbf{G}_n} \times_k f) = \mathrm{pr}_n \cdot (1_{\mathbf{G}_n} \times_k f) \right\} \in \mathbf{Set}$$

is representable by a closed subscheme Z_n of X. Consider now the quasi-coherent ideal $\mathcal{I}_n$ of Z_n inside X. Define

$$\mathcal{I} = \sum_{n \in \mathbb{N}} \mathcal{I}_n$$

Let $i : Z \hookrightarrow X$ be a closed subscheme of X determined by $\mathcal{I}$. This means that Z is the scheme-theoretic intersection inside X of closed subschemes Z_n for $n \in \mathbb{N}$. We show that Z represents the fixed point functor. For this assume that A is a k-algebra and $f : \operatorname{Spec} A \to X$ is a morphism of k-schemes such that f is an A-point of the fixed point functor. This is equivalent with

$$a \cdot (1_{\mathbf{G}} \times_k f) = \mathrm{pr}_X \cdot (1_{\mathbf{G}} \times_k f)$$

From this equality we deduce that

$$a_n \cdot (1_{\mathbf{G}_n} \times_k f) = \mathrm{pr}_n \cdot (1_{\mathbf{G}_n} \times_k f)$$

for every $n \in \mathbb{N}$ and hence f factors through Z_n for every $n \in \mathbb{N}$. Hence $f^{-1}(\mathcal{I}_n)A = 0$ for every $n \in \mathbb{N}$. Thus $f^{-1}(\mathcal{I})A = 0$ and we derive that f factors through Z. This proves that the fixed point functor is a k-subfunctor of the functor of points of Z. It suffices to prove that Z is $\mathbf{G}$-fixed. For this consider the morphism $a_{|\mathbf{G}\times_k Z} : \mathbf{G} \times_k Z \to X$. By Theorem 2.2.6 it corresponds to a morphism $\sigma : \mathbf{G} \to \mathcal{M}\mathrm{or}_k(Z, X)$ of k-functors. The fact that Z is locally free over k (Remark 2.5.3) and Theorem 2.5.2 imply that $\mathcal{M}\mathrm{or}_k(1_Z, i)$ is a closed immersion of

k-functors. Therefore, the composition of a closed immersion $\mathbf{1} \hookrightarrow \mathcal{M}\mathrm{or}_k(Z, Z)$ determined by 1_Z ($\mathbf{1}$ is the terminal k-functor) with $\mathcal{M}\mathrm{or}_k(1_Z, i)$ is a closed immersion of k-functors (by Fact 2.4.5). Consider a cartesian square

$$\begin{tikzcd}
\mathfrak{H} \arrow[r] \arrow[dd, "j"'] & \mathbf{1} \arrow[d, "\text{determined by } 1_Z"] \\
 & \mathcal{M}\mathrm{or}_k(Z, Z) \arrow[d, "\mathcal{M}\mathrm{or}_k(1_Z, i)"] \\
\mathbf{G} \arrow[r, "\sigma"'] & \mathcal{M}\mathrm{or}_k(Z, X)
\end{tikzcd}$$

We derive that $j : \mathfrak{H} \hookrightarrow \mathbf{G}$ is a closed immersion of k-functors. Note that an A-point g of $\mathbf{G}$ is contained in $\mathfrak{H}(A)$ if and only if the action of g on X_A restricts to identity on its A-subfunctor Z_A. From this description it follows that $\mathfrak{H}$ is a subgroup k-functor of $\mathbf{G}$, which fixes Z inside X. We denote by $\mathbf{H}$ locally algebraic group over k representing $\mathfrak{H}$. We deduce that $j : \mathbf{H} \hookrightarrow \mathbf{G}$ is a closed immersion of locally algebraic groups. By definition of $i : Z \hookrightarrow X$, we derive that the morphism of local k-algebras

$$\widehat{\mathcal{O}_{\mathbf{G}, e_{\mathbf{G}}}} \to \widehat{\mathcal{O}_{\mathbf{H}, e_{\mathbf{H}}}}$$

induced by $j^{\#}$ is an isomorphism. Hence by Lemma 4.5.1.2 j is an open immersion of locally algebraic groups. Thus j is both open and closed immersion. Since $\mathbf{G}$ is geometrically connected, we deduce that j is an isomorphism. Thus j is an isomorphism and this means (by virtue of the description of A-points of $\mathfrak{H}$ above) that Z is fixed by $\mathbf{G}$. □

4.6 Linearly reductive and reductive algebraic groups

In this section we recall an important class of affine group k-schemes defined by the semisimplicity of their categories of linear representations. Let $\mathbf{G}$ be an affine group k-scheme and let $\Delta_{\mathbf{G}} : k[\mathbf{G}] \to k[\mathbf{G}] \otimes_k k[\mathbf{G}]$ be the corresponding comultiplication. If K is a field over k and $\mathbf{G}$ is a group scheme over k, then we denote $\operatorname{Spec} K \times_k \mathbf{G}$ by $\mathbf{G}_K$. Recall also that for any two linear representations of $\mathbf{G}$ there exists a tensor product (Example 3.5.6).

Proposition 4.6.1 ([Jan03, 3.4 and 3.7]). *Let $\mathbf{G}$ be an affine group k-scheme. Then a functor*

$$\mathbf{Vect}_k \ni V \mapsto (k[\mathbf{G}] \otimes_k V, \Delta_{\mathbf{G}} \otimes_k 1_V) \in \mathbf{Rep}(\mathbf{G})$$

is right adjoint to the forgetful functor $\mathbf{Rep}(\mathbf{G}) \to \mathbf{Vect}_k$.

Corollary 4.6.2. *Let $\mathbf{G}$ be an affine group k-scheme. For every vector k-space V the $\mathbf{G}$-representation*

$$(k[\mathbf{G}] \otimes_k V, \Delta_{\mathbf{G}} \otimes_k 1_V)$$

is an injective object of $\mathbf{Rep}(\mathbf{G})$.

Proof. The functor

$$\mathbf{Vect}_k \ni V \mapsto (k[\mathbf{G}] \otimes_k V, \Delta_{\mathbf{G}} \otimes_k V) \in \mathbf{Rep}(\mathbf{G})$$

is right adjoint to the forgetful functor $\mathbf{Rep}(\mathbf{G}) \to \mathbf{Vect}_k$ by Proposition 4.6.1. Moreover, the forgetful functor $\mathbf{Rep}(\mathbf{G}) \to \mathbf{Vect}_k$ is exact. Right adjoint to an exact functor between abelian categories sends injective objects to injective objects. Since every vector space over k is injective, we derive that $(k[\mathbf{G}] \otimes_k V, \Delta_{\mathbf{G}} \otimes_k 1_V)$ is an injective object in $\mathbf{Rep}(\mathbf{G})$ for every vector space V over k. □

Proposition 4.6.3 ([Jan03, 3.7])**.** *Let V be a representation of an affine group k-scheme $\mathbf{G}$ with respect to coaction $c : V \to k[\mathbf{G}] \otimes_k V$. Denote by V_{tr} the trivial representation of $\mathbf{G}$ with V as the underlying vector k-space. Then*

$$k[\mathbf{G}] \otimes_k V \ni f \otimes v \mapsto f \cdot c(v) \in k[\mathbf{G}] \otimes_k V_{tr}$$

is an isomorphism of tensor products of $\mathbf{G}$-representations.

Corollary 4.6.4. *Let $\mathbf{G}$ be an affine group k-scheme and let V be a $\mathbf{G}$-representation. Then the tensor product $k[\mathbf{G}] \otimes_k V$ of $\mathbf{G}$-representations is an injective object in $\mathbf{Rep}(\mathbf{G})$.*

Proof. This is a consequence of Corollary 4.6.2 and Proposition 4.6.3. □

Theorem 4.6.5. *Let $\mathbf{G}$ be an affine group k-scheme. Then the following are equvalent.*

- **(i)** $\mathbf{Rep}(\mathbf{G})$ *is semisimple.*
- **(ii)** *Let k be a trivial $\mathbf{G}$-representation and let $i : k \hookrightarrow k[\mathbf{G}]$ be the inclusion of constant functions. Then i is a split monomorphism of $\mathbf{G}$-representations.*

Proof. In order to prove **(i)** ⇒ **(ii)** it suffices to note that in semisimple abelian categories all monomorphisms split.
Suppose that **(ii)** holds. Pick a $\mathbf{G}$-representation V. Since $i : k \hookrightarrow k[\mathbf{G}]$ is a split monomorphism, we derive that the tensor product

$$i \otimes_k V : k \otimes_k V \hookrightarrow k[\mathbf{G}] \otimes_k V$$

is a split monomorphism. By Corollary 4.6.4 we know that $k[\mathbf{G}] \otimes_k V$ is an injective object of $\mathbf{Rep}(\mathbf{G})$. Hence $V \simeq k \otimes_k V$ is an injective object of $\mathbf{Rep}(\mathbf{G})$, because it is a direct summand of $k[\mathbf{G}] \otimes_k V$. Thus every object of $\mathbf{Rep}(\mathbf{G})$ is injective and hence every object of $\mathbf{Rep}(\mathbf{G})$ is completely reducible by Theorem 3.9.5. This completes the proof of the implication **(ii)** ⇒ **(i)**. □

Proposition 4.6.6. *Let $\mathbf{G}$ be an affine group k-scheme and let V be a representation of $\mathbf{G}$. Then for every field K over k the natural map of vector spaces over K*

$$K \otimes_k V^{\mathbf{G}} \to (K \otimes_k V)^{\mathbf{G}_K}$$

is an isomorphism.

Proof. By Proposition 3.7.3 we have a left exact sequence of k-vector spaces defining invariants

$$0 \longrightarrow V^{\mathbf{G}} \longrightarrow V \xrightarrow{c-p} k[\mathbf{G}] \otimes_k V$$

where $c : V \to k[\mathbf{G}] \otimes_k V$ is the coaction and $p : V \to k[\mathbf{G}] \otimes_k V$ is the trivial coaction defined by formula $p(v) = 1 \otimes v$ for every $v \in V$. Now tensoring the sequence with K over k yields a left exact sequence

$$0 \longrightarrow K \otimes_k V^{\mathbf{G}} \longrightarrow K \otimes_k V \xrightarrow{c_K - p_K} (K \otimes_k k[\mathbf{G}]) \otimes_K (K \otimes_k V)$$

where c_K is the coaction on $K \otimes_k V$ induced by c and p_K is the trivial coaction on $K \otimes_k V$. This shows that $K \otimes_k V^{\mathbf{G}} \to (K \otimes_k V)^{\mathbf{G}_K}$ is an isomorphism. □

Theorem 4.6.7. *Let* $\mathbf{G}$ *be an affine group k-scheme. Then the following are equvalent.*

(i) $\mathbf{Rep}(\mathbf{G})$ *is semisimple.*

(ii) *The category* $\mathbf{Rep}(\mathbf{G}_K)$ *is semisimple for every field extension K of k.*

(iii) *The category* $\mathbf{Rep}(\mathbf{G}_K)$ *is semisimple for some field extension K of k.*

(iv) *The category* $\mathbf{Rep}(\mathbf{G}_K)$ *is semisimple for some algebraically closed field extension K of k.*

Proof. Implications **(i)** $\Rightarrow$ **(ii)**, **(ii)** $\Rightarrow$ **(iii)** and **(iii)** $\Rightarrow$ **(iv)** follow from Theorem 4.6.5, since split monomorphisms are preserved by flat base changes.
Suppose now that **(iv)** holds. Fix a finite dimensional representation V of $\mathbf{G}$. Let W be an arbitrary subrepresentation of V and denote by $j : W \hookrightarrow V$ the inclusion. Let $\mathrm{Hom}_k(V, W)$ and $\mathrm{Hom}_k(W, W)$ be equipped with the usual structures (Example 3.5.10) of representations of $\mathbf{G}$, then $\mathrm{Hom}_k(j, 1_W)$ is a surjective morphism of representations. This means that

$$\mathrm{Hom}_K(1_K \otimes_k j, 1_K \otimes_k 1_W) : \mathrm{Hom}_K(K \otimes_k V, K \otimes_K W) \twoheadrightarrow \mathrm{Hom}_K(K \otimes_k W, K \otimes_k W)$$

is surjective morphism of representations of $\mathbf{G}_K$. Since $\mathbf{Rep}(\mathbf{G}_K)$ is semisimple, we derive that the functor of invariants $(-)^{\mathbf{G}_K}$ is exact. Hence the induced morphism

$$\mathrm{Hom}_K(1_K \otimes_k j, 1_K \otimes_k 1_W)^{\mathbf{G}_K} : \mathrm{Hom}_K(K \otimes_k V, K \otimes_K W)^{\mathbf{G}_K} \twoheadrightarrow \mathrm{Hom}_K(K \otimes_k W, K \otimes_k W)^{\mathbf{G}_K}$$

of K-vector spaces is surjective. According to Proposition 4.6.6, we deduce that the morphism

$$1_K \otimes_k \mathrm{Hom}(j, 1_W) : K \otimes_k \mathrm{Hom}_k(V, W)^{\mathbf{G}} \twoheadrightarrow K \otimes_k \mathrm{Hom}_k(W, W)^{\mathbf{G}}$$

is surjective. By faithfully flat descent and Proposition 3.5.11, we derive that

$$\mathrm{Hom}_{\mathbf{G}}(j, 1_W) : \mathrm{Hom}_{\mathbf{G}}(V, W) = \mathrm{Hom}_k(V, W)^{\mathbf{G}} \twoheadrightarrow \mathrm{Hom}_k(W, W)^{\mathbf{G}} = \mathrm{Hom}_{\mathbf{G}}(W, W)$$

is surjective. This means that there exists a morphism of representations $r : V \to W$ such that $r \cdot j = 1_W$. Hence j splits. Therefore, every representation of $\mathbf{G}$ of finite dimension over k is completely reducible. Since every representation of $\mathbf{G}$ is a sum of its finitely dimensional subrepresentations (Corollary 3.7.2), we derive that every representation of $\mathbf{G}$ is a sum of its completely reducible subrepresentations. Hence every representation of $\mathbf{G}$ is completely reducible and this means that $\mathbf{Rep}(\mathbf{G})$ is semisimple abelian category. □

Definition 4.6.8. Let $\mathbf{G}$ be an affine group k-scheme and suppose that equivalent conditions of Theorem 4.6.7 are satisfied for $\mathbf{G}$. Then we say that $\mathbf{G}$ is *a fully reducible group over k*.

Definition 4.6.9. Let $\mathbf{G}$ be a fully reducible group k-scheme. Suppose that $\mathbf{G}$ is smooth and algebraic over k. Then we say that $\mathbf{G}$ is *a linearly reductive group over k*.

Recall [Mil17, 6.46] that there is an important class of *reductive algebraic groups* extending classical *semisimple algebraic groups*.

Theorem 4.6.10 ([Mil17, Corollary 22.43, 22.46])**.** *Let* $\mathbf{G}$ *be a geometrically integral algebraic group over k. If* $\mathbf{G}$ *is linearly reductive, then it is reductive.*

4.7 Tori

Definition 4.7.1. Let T be an affine algebraic group over k. Suppose that there exists $n \in \mathbb{N}$ such that for every algebraically closed extension K of k there exists an isomorphism

$$T_K \simeq \operatorname{Spec} K \times_k \underbrace{\mathbb{G}_m \times_k \mathbb{G}_m \times_k \ldots \times_k \mathbb{G}_m}_{n \text{ times}}$$

of group schemes over K. Then T is called *a torus over k*.

Example 4.7.2. If $T \simeq \underbrace{\mathbb{G}_m \times_k \mathbb{G}_m \times_k \ldots \times_k \mathbb{G}_m}_{n \text{ times}}$, then T is a torus. We call tori T of this form *split tori*. Clearly

$$k[T] = k[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$$

as k-algebra and for every $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$ we define $\chi^m = x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}$. Then

$$k[T] = \bigoplus_{m \in \mathbb{Z}^m} k \cdot \chi^m$$

and χ^m can be identified with a character $T \to \mathbb{G}_m$ given by

$$k[\mathbb{G}_m] = k[x, x^{-1}] \ni x \mapsto \chi^m \in k[T]$$

This gives rise to a decomposition of $k[T]$ as the representation of T into one-dimensional representations of T. Since every one-dimensional representation is irreducible, we derive that $k[T]$ is completely reducible representation of T. Hence by Fact 3.6.6 and Theorem 3.9.5 we deduce that every representation of T is completely reducible and thus T is linearly reductive algebraic group. Moreover, this also implies that every irreducible representation of T corresponds to a character and hence $\mathbf{Irr}(T) = \mathbb{Z}^n$. Note that by Remark 3.5.7 there is a canonical structure of an abstract monoid on a set of characters of of T. Thus $\mathbf{Irr}(T)$ has a structure of a monoid. It is easy to verify that $\mathbf{Irr}(T) \simeq \mathbb{Z}^n$ as an abstract monoids, where $\mathbb{Z}^n$ as the free abelian group of rank n is an abstract monoid.

Example 4.7.3. Let

$$\mathbf{S}^1 = \operatorname{Spec} k[x, y]/(x^2 + y^2 - 1)$$

be a scheme over k. Then for every k-algebra A we have

$$\mathbf{S}^1(A) = \left\{(u, v) \in A \times_k A \,\middle|\, u^2 + v^2 = 1\right\}$$

There is also a morphism $\mathbf{S}^1 \times \mathbf{S}^1 \to \mathbf{S}^1$ of k-functors given by

$$\mathbf{S}^1(A) \times \mathbf{S}^1(A) \ni ((u_1, v_1), (u_2, v_2)) \mapsto (u_1 u_2 - v_1 v_2, u_1 v_2 + u_2 v_1) \in \mathbf{S}^1(A)$$

for every k-algebra A. This makes $\mathbf{S}^1$ into a group k-functor. Thus $\mathbf{S}^1$ with the group structure described above is an affine algebraic group over k. We call it *the circle group over k*.
Now suppose that $\mathrm{char}(k) \neq 2$ and K is an algebraically closed extension of k. Consider an element $i \in K$ such that $i^2 = -1$. For every K-algebra A we have a map

$$\mathbf{S}^1(A) \ni (u, v) \mapsto u + iv \in A^*$$

First note that this map is bijective. Indeed, its inverse is given by

$$A^* \ni a \mapsto \left(\frac{1}{2}(a + a^{-1}), \frac{1}{2i}(a - a^{-1})\right) \in \mathbf{S}^1(A)$$

Moreover, the map $\mathbf{S}^1(A) \to A^*$ is a homomorphism of abstract groups. Thus $\mathbf{S}^1$ resricted to the category $\mathbf{Alg}_K$ of K-algebras is isomorphic with $\mathrm{Spec}\, K \times_k \mathbb{G}_m$ as a group k-functor. Hence

$$\mathrm{Spec}\, K \times_k \mathbf{S}^1 \simeq \mathrm{Spec}\, K \times_k \mathbb{G}_m$$

as algebraic group schemes over K. Hence $\mathbf{S}^1$ is a torus over k.
Now assume that $k = \mathbb{R}$. Then abstract groups

$$\mathbf{S}^1(\mathbb{R}) = \left\{z \in \mathbb{C} \,\middle|\, |z| = 1\right\} \subseteq \mathbb{C}^*, \, \mathbb{R}^*$$

are not isomorphic. Indeed, the left hand side group has infinite torsion subgroup and the right hand side group has torsion subgroup equal to $\{-1, 1\}$. This implies that over $\mathbb{R}$ algebraic groups $\mathbf{S}^1$ and $\mathbb{G}_m$ are not isomorphic. Hence $\mathbf{S}^1$ is not a split torus over $\mathbb{R}$.

Corollary 4.7.4. *Let T be a torus over k. Then T is a linearly reductive algebraic group.*

Proof. Pick an algebraically closed extension K of k. Then T_K is a split torus and hence by Example 4.7.2 it is a linearly reductive group. Theorem 4.6.7 implies that T is fully reducible. It is also smooth over k as T_K is smooth. Hence T is a linearly reductive group over k. □

Chapter 5

Algebraic Monoids

5.1 Introduction

Thanks to our previous discussion (Section 3.8) it is useful to know when the group of units is open and schematically dense in a monoid k-scheme. This is addressed in Theorem 5.2.4 and Corollary 5.2.5. The other results of this chapter concern the representability of $X^{\mathbf{M}}$ for a geometrically integral, algebraic monoid over k. We include here a short discussion of toric monoids (affine toric varieties). The most important part of this chapter is devoted to the wide class of *Kempf monoids*. This class contains properly reductive monoids with zero (Corollary 5.5.4 and Example 5.5.5) and is precisely the class of monoids for which we generalize Białynicki-Birula decomposition.
In this chapter k is a field.

5.2 The unit group of an algebraic monoid

Definition 5.2.1. Let $\mathbf{M}$ be a monoid k-scheme. If $\mathbf{M}$ is of finite type over k, then we say that $\mathbf{M}$ is *an algebraic monoid over k*.

Fact 5.2.2. *Let $\mathbf{M}$ be an algebraic monoid over k. Then its group of units $\mathbf{G}$ is an algebraic group over k.*

Proof. Recall from the proof of Proposition 3.2.6 that $\mathbf{G}$ fits into a cartesian square

$$\begin{tikzcd}
\mathbf{G} \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "e"] \\
\mathbf{M} \times_k \mathbf{M} \arrow[r, "\mu"'] & \mathbf{M}
\end{tikzcd}$$

where μ is the multiplication and $e : \operatorname{Spec} k \to \mathbf{M}$ is the unit of $\mathbf{M}$. Thus $\mathbf{G}$ is a closed subscheme of a scheme $\mathbf{M} \times_k \mathbf{M}$ of finite type over k. Thus $\mathbf{G}$ is of finite type over k. $\square$

Out first goal is to prove that under quite general assumptions the group of units of an algebraic monoid is open. For this we prove a basic result on generic finiteness.

Theorem 5.2.3. *Let $f : X \to Y$ be a dominant morphism of finite type between irreducible noetherian schemes. Suppose that η is a generic point and assume that the generic fiber $f^{-1}(\eta)$ is finite. Then there exists an open and nonempty subset V of Y such that the restriction $f^{-1}(V) \to V$ of V is finite.*

For the proof we need the following local version of the theorem.

Lemma 5.2.3.1. *Let A be a noetherian ring such that* Spec *A is irreducible and let B be an A-algebra of finite type. Suppose that $\mathfrak{p}$ is the unique minimal prime ideal of A and assume that $k(\mathfrak{p}) \otimes_A B$ is finite over $k(\mathfrak{p})$, where $k(\mathfrak{p})$ denotes the residue field of $\mathfrak{p}$ in A. Then there exists $s \in A \smallsetminus \mathfrak{p}$ such that B_s is a finite A_s-module.*

Proof of the lemma. Let $b_1, ..., b_n$ be generators of B as an A-algebra. Then

$$\overline{b_i} = b_i \bmod \mathfrak{p}B$$

for $1 \leq i \leq n$ are generators of $B/\mathfrak{p}B$ as an $A/\mathfrak{p}$ algebra. Since $k(\mathfrak{p}) \otimes_A B$ is finite over $k(\mathfrak{p})$ for each i there exists a positive integer m_i and a polynomial

$$f_i(x) = s_{i,m_i}x^{m_i} + s_{i,m_i-1}x^{m_i-1} + ... + s_{i,0} \in (A/\mathfrak{p})[x]$$

such that $s_{i,m_i} \neq 0$ and $f_i(\overline{b_i}) = 0$. Let $s \in A$ be an element such that

$$s \bmod \mathfrak{p} = s_{1,m_1} \cdot s_{2,m_2} \cdot ... \cdot s_{n,m_n}$$

Clearly $s \in A \smallsetminus \mathfrak{p}$ and $B_s/(\mathfrak{p}B)_s = (B/\mathfrak{p}B)_s$ is a finite A_s-algebra. Hence there exists a finite A_s-submodule M of B_s such that

$$B_s = M + (\mathfrak{p}B)_s = M + \mathfrak{p}B_s$$

Since A is noetherian and $\mathfrak{p}$ is the unique minimal ideal, we deduce that $\mathfrak{p}$ is nilpotent. Hence there exists $N \in \mathbb{N}$ such that $\mathfrak{p}^N = 0$. Thus

$$B_s = M + \mathfrak{p}B_s = M + \mathfrak{p}M + \mathfrak{p}^2B_s = ... = M + \mathfrak{p}M + ... + \mathfrak{p}^{N-1}M + \mathfrak{p}^N B_s = M + \mathfrak{p}M + ... + \mathfrak{p}^{N-1}M$$

is a finite A_s-module. □

Proof of the theorem. Pick an open, nonempty, affine neighborhood W of η. Since f is of finite type, we derive that

$$f^{-1}(W) = \bigcup_{i=1}^{n} U_i$$

where each U_i is nonempty open affine subscheme of X and moreover, the morphism $U_i \to W$ induced by f is of finite type. According to Lemma 5.2.3.1 for each i there exists an open, affine and nonempty subscheme $W_i \subseteq W$ such that the morphism $f^{-1}(W_i) \cap U_i \to W_i$ induced by f is finite. Thus replacing W by the intersection of $W_1, ..., W_n$ we may assume that each $U_i \to W$ is finite. Consider

$$F = f^{-1}(W) \smallsetminus U_1$$

Then F is a closed subset of $f^{-1}(W)$ and it does not contain the generic point ξ of X. Since each restriction $U_i \to W$ of f is finite, we derive that $f\big(U_i \cap F\big)$ is closed in W for every $1 \leq i \leq n$ and does not contain $\eta = f(\xi)$. This last assertion follows from the fact that f is dominant. Thus $f(F)$ is a closed subset of W and $\eta \notin f(F)$. Hence $V = W \smallsetminus f(F)$ is an open neighborhood of η and $f^{-1}(V) \subseteq U_1$. Thus the restriction $f^{-1}(V) \to V$ of f is finite. □

Theorem 5.2.4. *Let* $\mathbf{M}$ *be a geometrically integral algebraic monoid* k*-scheme. Suppose that* $\mathbf{G}$ *is the group of units of* $\mathbf{M}$ *and* $i : \mathbf{G} \hookrightarrow \mathbf{M}$ *is the canonical monomorphism. Then* i *is an open immersion.*

Proof. Assume that k is algebraically closed. Denote by $\mu : \mathbf{M} \times_k \mathbf{M} \to \mathbf{M}$ and $e : \operatorname{Spec} k \to \mathbf{M}$ the multiplication and the unit, respectively. By Fact 5.2.2 $\mathbf{G}$ is an algebraic group over k. Since $\mathbf{M}$ is integral and of finite type over k, we derive that $\mathbf{M} \times_k \mathbf{M}$ is integral and

$$\dim\left(\mathbf{M} \times_k \mathbf{M}\right) = 2 \cdot \dim\left(\mathbf{M}\right)$$

Moreover, μ is surjective (which can be checked on k-functors of points). Pick any irreducible component Z of $\mu^{-1}(e)$. By [GW10, Lemma 14.109] we deduce

$$\dim\left(Z\right) \geq \dim\left(\mu^{-1}(\eta)\right)$$

where η is the generic point of $\mathbf{M}$. Since

$$\dim\left(\mu^{-1}(\eta)\right) = \dim\left(\mathbf{M} \times_k \mathbf{M}\right) - \dim\left(\mathbf{M}\right) = 2 \cdot \dim\left(\mathbf{M}\right) - \dim\left(\mathbf{M}\right) = \dim\left(\mathbf{M}\right)$$

we deduce that $\dim(Z) \geq \dim(\mathbf{M})$. Moreover, we have $\mathbf{G} \simeq \mu^{-1}(e)$ as k-schemes and this isomorphism is given by the restriction $\pi : \mu^{-1}(e) \to \mathbf{G}$ to $\mu^{-1}(e)$ of the projection $\mathrm{pr} : \mathbf{M} \times_k \mathbf{M} \to \mathbf{M}$ on the first factor (this can be checked on k-functors of points, see also Proposition 3.2.6). Thus each irreducible component Z of $\mathbf{G}$ is of dimension at least $\dim(\mathbf{M})$. Now we fix an irreducible component Z of $\mathbf{G}$ and consider it as a closed subscheme of $\mathbf{G}$ with reduced structure. Then the morphism $i_{|Z} : Z \hookrightarrow \mathbf{M}$ is a monomorphism of finite type and $\dim(Z) \geq \dim(\mathbf{M})$. Hence $i_{|Z}$ is dominant. Since i is a monomorphism, this implies that $\mathbf{G}$ has only one irreducible component and $i : \mathbf{G} \hookrightarrow \mathbf{M}$ is dominant. By Theorem 5.2.3 there exists an open and nonempty subset V of $\mathbf{M}$ such that the morphism $i^{-1}(V) \hookrightarrow V$ induced by i is finite. Finite monomorphisms are closed immersions. Dominant, closed immersions with codomain an integral scheme are isomorphisms. Thus $i^{-1}(V) \to V$ is an isomorphism. Now observe that there exists a canonical action of $\mathbf{G}$ on $\mathbf{M}$ and $i : \mathbf{G} \hookrightarrow \mathbf{M}$ is $\mathbf{G}$-equivariant with respect to this action. Thus Theorem 4.4.8 implies that there exists the largest open subscheme W of $\mathbf{M}$ such that $i^{-1}(W) \to W$ is an isomorphism of k-schemes and moreover, W is $\mathbf{G}$-stable. Since $V \subseteq W$, we derive that W is nonempty. Hence $i^{-1}(W)$ is a nonempty $\mathbf{G}$-stable open subscheme of $\mathbf{G}$. Therefore, $i^{-1}(W) = \mathbf{G}$ and $i : \mathbf{G} \hookrightarrow \mathbf{M}$ is an open immersion. If k is not algebraically closed, then we pick an algebraically closed extension K of k and consider $1_{\operatorname{Spec} K} \times_k i$. This is an open immersion according to the case considered above. By faithfully flat descent i is an open immersion. □

The more general result for algebraically closed fields is [Bri14, Theorem 1]. It seems that Theorem 5.2.4 is a consequence of this more general theorem. Since for our purposes the case of geometrically integral monoids suffices, we decide for self-containment to give its proof.

Corollary 5.2.5. *Let* $\mathbf{M}$ *be a geometrically integral, algebraic monoid over k. Then the inclusion* $\mathbf{G} \hookrightarrow \mathbf{M}$ *of the group of units is schematically dense open immersion.*

Let us also note the following theorem, which proof is exactly the analogue of the proof of the fact that every affine algebraic group is linear.

Theorem 5.2.6 ([DG70, Corollaire 3.5 on page 183])**.** *Let* $\mathbf{M}$ *be an affine, algebraic monoid k-scheme. There exists a finite dimensional vector space V over k and a closed immersion*

$$\mathbf{M} \hookrightarrow \mathbf{L}(V)$$

of algebraic monoids.

We state the following result, which is both useful and interesting.

Theorem 5.2.7 ([Bri14, Theorem 2])**.** *Let* $\mathbf{M}$ *be a geometrically integral algebraic monoid over a field k and let* $\mathbf{G}$ *be an group of units of* $\mathbf{M}$*. If* $\mathbf{G}$ *is affine, then* $\mathbf{M}$ *is affine.*

Definition 5.2.8. Let $\mathbf{M}$ be a geometrically integral algebraic monoid over k and let $\mathbf{G}$ be its group of units. If $\mathbf{G}$ is (linearly) reductive, then $\mathbf{M}$ is called *a (linearly) reductive monoid over k*.

Corollary 5.2.9. *Let* $\mathbf{M}$ *be a linearly reductive monoid over k. Then* $\mathbf{M}$ *is reductive.*

Proof. This follows from definition and Theorem 4.6.10. □

By definition every reductive group is affine. Hence using Theorem 5.2.7 we deduce the following result.

Corollary 5.2.10. *Let* $\mathbf{M}$ *be a reductive monoid over k. Then* $\mathbf{M}$ *is affine.*

5.3 Representability of fixed points for algebraic monoids

Proposition 5.3.1. *Let* $\mathbf{M}$ *be a monoid k-scheme and let* $\mathbf{G}$ *be its group of units. Suppose that X is a k-scheme with an action of* $\mathbf{M}$*. If* $\mathbf{G}$ *is open and schematically dense in* $\mathbf{M}$*, then subpresheaves* $X^{\mathbf{M}}$ *and* $X^{\mathbf{G}}$ *of X are equal.*

Proof. Consider a morphism $f : Y \to X$ of k-schemes which is $\mathbf{G}$-equivariant when Y is considered with the trivial action of $\mathbf{G}$. It suffices to show that f is $\mathbf{M}$-equivariant when Y is considered with the trivial action of $\mathbf{M}$. Since both $X^{\mathbf{G}}$ and $X^{\mathbf{M}}$ are Zariski sheaves by Fact 3.10.6, we deduce that it suffices to assume that $f(Y)$ is contained in some affine open subscheme U of X. Consider now the kernel (the equalizer) $Z \hookrightarrow \mathbf{M} \times_k X$ of a pair

$$\mathbf{M} \times_k Y \underset{f \cdot \mathrm{pr}_Y}{\overset{a \cdot (1_{\mathbf{M}} \times_k f)}{\rightrightarrows}} X$$

In general this kernel is a locally closed subscheme of X, but since $f(Y) \subseteq U$ and U is separated, we deduce that Z is actually a closed subscheme of $\mathbf{M} \times_k X$. Moreover, by assumption f is $\mathbf{G}$-equivariant. Hence $\mathbf{G} \times_k X$ is contained in Z. According to the fact that k is a field, we infer that $\mathbf{G} \times_k X$ is open and schematically dense in $\mathbf{M} \times_k X$. Thus Z is equal to $\mathbf{M} \times_k X$. This means that f is $\mathbf{M}$-equivariant. □

Corollary 5.3.2. *Let $\mathbf{M}$ be a geometrically integral, algebraic monoid k-scheme and let $\mathbf{G}$ be its group of units. Then $X^{\mathbf{M}} = X^{\mathbf{G}}$ and the canonical inclusion $X^{\mathbf{M}} \hookrightarrow X$ is a closed immersion.*

Proof. By Fact 5.2.2 $\mathbf{G}$ is an algebraic group over k. According to Corollary 5.2.5 we derive that $\mathbf{G} \hookrightarrow \mathbf{M}$ is schematically dense, open immersion. Thus Proposition 5.3.1 imply that subpresheaves $X^{\mathbf{G}}$ and $X^{\mathbf{M}}$ of X are equal. Corollary 5.2.5 and the fact that $\mathbf{M}$ is geometrically integral imply that $\mathbf{G}$ is geometrically integral and hence it is geometrically connected. Thus by Theorem 4.5.1 we deduce that $X^{\mathbf{G}} \hookrightarrow X$ is a closed immersion of k-schemes. □

5.4 Toric monoids

Definition 5.4.1. Let T be a torus over k and let $\overline{T}$ be a geometrically integral, algebraic monoid having T as the group of units. Then $\overline{T}$ is *a toric monoid over k.*

Corollary 5.4.2. *Let $\overline{T}$ be a toric monoid over k. Then $\overline{T}$ is a linearly reductive monoid over k.*

Proof. This follows from Corollary 4.7.4. □

Corollary 5.4.3. *Let $\overline{T}$ be a toric monoid over k. Then $\overline{T}$ is an affine algebraic monoid over k.*

Proof. This follows from Corollaries 5.4.2 and 5.2.9. □

Theorem 5.4.4. *Let $\overline{T}$ be a toric monoid over k with group of units T and let K be an algebraically closed extension of k. Suppose that N is a dimension of T.*

(1) *The group of characters of T_K is isomorphic to $\mathbb{Z}^N$ and there exists an abstract submonoid S of $\mathbb{Z}^N$ such that the open immersion*

$$T_K = \mathrm{Spec}\left(\bigoplus_{m \in \mathbb{Z}^N} K \cdot \chi^m\right) \hookrightarrow \mathrm{Spec}\left(\bigoplus_{m \in S} K \cdot \chi^m\right) = \overline{T}_K$$

is induced by the inclusion $S \hookrightarrow \mathbb{Z}^N$.

(2) *Let $\{V_\lambda\}_{\lambda \in \mathbf{Irr}(T)}$ be a set of irreducible representation of T such that V_λ is in isomorphism class λ. For every λ there exists a finite subset A_λ of $\mathbb{Z}^N$ such that*

$$K \otimes_k V_\lambda = \bigoplus_{m \in A_\lambda} K \cdot \chi^m$$

If λ is in $\mathbf{Irr}(\overline{T})$, then A_λ is a subset of S. Moreover, we have

$$\mathbb{Z}^N = \coprod_{\lambda \in \mathbf{Irr}(T)} A_\lambda$$

and $A_{\lambda_0} = \{0\}$, where λ_0 is the class of the trivial representation of T.

(3) *If $\overline{T}$ has a zero, then there exists a homomorphism $f : \mathbb{Z}^N \to \mathbb{Z}$ of abelian groups such that $f_{|S \smallsetminus \{0\}} > 0$. In particular, f induces a closed immersion*

$$\operatorname{Spec} K \times_k \mathbb{G}_m = \operatorname{Spec} K[\mathbb{Z}] \hookrightarrow \operatorname{Spec}\left(\bigoplus_{m \in \mathbb{Z}^N} K \cdot \chi^m\right) = T_K$$

of group K-schemes that extends to a zero preserving closed immersion $\mathbb{A}^1_K \hookrightarrow \overline{T}_K$ of monoid K-schemes.

Proof. Since T is a torus, we know that

$$T_K = \operatorname{Spec} K \times_k \underbrace{\mathbb{G}_m \times_k \mathbb{G}_m \times_k \ldots \times_k \mathbb{G}_m}_{N \text{ times}} = \operatorname{Spec}\left(\bigoplus_{m \in \mathbb{Z}^N} K \cdot \chi^m\right)$$

by Example 4.7.2 and hence

$$\overline{T}_K = \operatorname{Spec}\left(\bigoplus_{s \in S} K \cdot \chi^s\right)$$

for some abstract submonoid S of $\mathbb{Z}^N$. Moreover, the open immersion $T_K \hookrightarrow \overline{T}_K$ is induced by the inclusion $S \hookrightarrow \mathbb{Z}^N$. This proves **(1)**.
We have an identification

$$k[T] = \bigoplus_{\lambda \in \mathbf{Irr}(T)} V_\lambda^{n_\lambda}$$

of T-representations, where $n_\lambda \in \mathbb{N} \smallsetminus \{0\}$ for each λ. Thus

$$\bigoplus_{m \in \mathbb{Z}^N} K \cdot \chi^m = K \otimes_k k[T] = \bigoplus_{\lambda \in \mathbf{Irr}(T)} (K \otimes_k V_\lambda)^{n_\lambda}$$

This implies that $n_\lambda = 1$ for every λ and moreover, we derive that

$$K \otimes_k V_\lambda = \bigoplus_{m \in A_\lambda} K \cdot \chi^m$$

for some finite set $A_\lambda \subseteq \mathbb{Z}^N$. We also have $A_{\lambda_0} = \{0\}$ and $A_\lambda \subseteq S \smallsetminus \{0\}$ for $\lambda \in \mathbf{Irr}(\overline{T})$. This proves **(2)**.
Since $\overline{T}$ admits a zero, we derive that

$$\mathfrak{m} = \bigoplus_{m \in S \smallsetminus \{0\}} K \cdot \chi^s \subseteq \bigoplus_{m \in \mathbb{Z}^N} K \cdot \chi^m$$

is an ideal. This implies that $S \smallsetminus \{0\}$ is closed under addition. In particular, there exists a homomorphism of abelian groups $f : \mathbb{Z}^N \to \mathbb{Z}$ such that $f_{|S \smallsetminus \{0\}} > 0$. This implies **(3)**. □

5.5 The class of Kempf monoids

In this section we introduce important class of monoid k-schemes, which contains all reductive monoids over k. We recall first the classical result concerning good quotients with respect to actions of linearly reductive groups on affine algebraic schemes over k.

Theorem 5.5.1. *[[BBCM13, Theorem 5.4 and discussion below its statement]] Let X be an affine k-scheme of finite type equipped with an action of a linearly reductive algebraic group* $\mathbf{G}$*. Consider the morphism $\pi : X \to Y$ of affine k-schemes induced by the inclusion $\Gamma(X, \mathcal{O}_X)^{\mathbf{G}} \hookrightarrow \Gamma(X, \mathcal{O}_X)$. Then the following assertions hold.*

(1) *Y is of finite type over k.*

(2) *If Z_1 and Z_2 are disjoint,* $\mathbf{G}$*-stable and closed subschemes of X, then $\pi(Z_1)$ and $\pi(Z_2)$ are disjoint.*

(3) *π is surjective.*

(4) *If we consider Y as a k-scheme with the trivial* $\mathbf{G}$*-action, then π is* $\mathbf{G}$*-equivariant morphism.*

(5) *If $p : X \to W$ is a* $\mathbf{G}$*-equivariant morphism and W is a k-scheme with the trivial* $\mathbf{G}$*-action, then p uniquely factors through π.*

Now we are ready to prove the following result.

Theorem 5.5.2. *Let* $\mathbf{M}$ *be a reductive algebraic monoid over k and let* $\mathbf{G}$ *be a group of units of* $\mathbf{M}$*. Assume that* $\mathbf{M}$ *admits a zero* $\mathbf{o}$*. Then there exists a central torus T in* $\mathbf{G}$ *such that $\mathbf{o} \in \mathbf{cl}(T)$.*

Proof. By assumption $\mathbf{G}$ is a reductive group. According to [Mil17, Corollary 17.62 and Notation 12.29] its centre $Z(\mathbf{G})$ is an algebraic group of multiplicative type and the largest subtorus T of $Z(\mathbf{G})$ is the solvable radical $R(\mathbf{G})$ of $\mathbf{G}$. In particular, the quotient group $\mathbf{G}/T$ has trivial solvable radical and hence it is a semisimple algebraic group. Now T is linearly reductive (Corollary 4.7.4). Thus by Theorem 5.5.1 we obtain a quotient $\pi : \mathbf{M} \twoheadrightarrow \mathbf{Q}$ of $\mathbf{M}$ by the action of T. Note also that T is central in $\mathbf{M}$ as it is central in $\mathbf{G}$. Next the fact that T is central in $\mathbf{M}$, the fact that $\mathbf{M}$ is geometrically integral and Theorem 5.5.1 imply that $\mathbf{Q}$ is a geometrically integral, affine and algebraic monoid k-scheme with zero. Moreover, π is a surjective morphism of algebraic monoids over k. According to Theorem 5.2.4 we derive that the group of units $\mathbf{Q}^*$ is an open subscheme of $\mathbf{Q}$. From the fact that $\mathbf{G} \hookrightarrow \mathbf{M}$ is dominant we derive that the restriction $\pi_{|\mathbf{G}} : \mathbf{G} \to \mathbf{Q}$ is dominant. Thus π induces a dominant morphism of geometrically integral algebraic groups $\mathbf{G} \to \mathbf{Q}^*$. Next Theorem 4.4.9 implies that $\pi(\mathbf{G}) = \mathbf{Q}^*$. Theorem 5.2.6 implies that there exists a closed immersion of monoids $i : \mathbf{Q} \hookrightarrow \mathbf{L}(V)$ for some finite dimensional vector k-space V. Thus $i \cdot \pi_{|\mathbf{G}}$ composed with the determinant $\det : \mathbf{L}(V) \to \mathbb{A}^1_k$ (Example 3.4.10) induces a character of $\mathbf{G}$ (that is a morphism of algebraic groups $\mathbf{G} \to \mathbb{G}_m$) that factors through the quotient morphism $\mathbf{G} \twoheadrightarrow \mathbf{G}/T$, but $\mathbf{G}/T$ is a semisimple algebraic group and hence it has only trivial characters. Therefore, the character of $\mathbf{G}$ constructed above is trivial. Hence $i(\mathbf{Q}^*) = i \cdot \pi(\mathbf{G})$ is contained in the algebraic subgroup $\mathbf{SL}(V)$ of $\mathbf{L}(V)$ (Example 3.4.12). Next i induces a morphism of algebraic groups $\mathbf{Q}^* \hookrightarrow \mathbf{SL}(V)$ and by Theorem 4.4.9 we infer that $i(\mathbf{Q}^*)$ is closed in $\mathbf{SL}(V)$. Since $\mathbf{SL}(V)$ is closed in $\mathbf{L}(V)$, we derive that $i(\mathbf{Q}^*)$ is closed in $\mathbf{L}(V)$ and hence it is also closed in $\mathbf{Q}$. On the other hand we proved that is open in $\mathbf{Q}$. Monoid $\mathbf{Q}$ is integral and hence it is connected. Thus $\mathbf{Q}^* = \mathbf{Q}$ which means that $\mathbf{Q}$ is a group k-scheme. Moreover, $\mathbf{Q}$ is a monoid k-scheme with zero. This is only possible if $\mathbf{Q}$ is $\operatorname{Spec} k$. Therefore, the categorical quotient $\pi : \mathbf{M} \to \mathbf{Q}$ consists of a single k-rational point. Thus by **(2)** in Theorem 5.5.1 the closure of every orbit of T in $\mathbf{M}$ contains the zero $\mathbf{o}$. In particular, $\mathbf{o} \in \mathbf{cl}(T)$. □

This theorem motivates the following definition.

Definition 5.5.3. Let $\mathbf{M}$ be a geometrically integral, affine algebraic monoid over k. Assume that $\mathbf{M}$ admits a zero $\mathbf{o}$ and let $\mathbf{G}$ be the group of units of $\mathbf{M}$. Suppose that there exists a central subtorus T of $\mathbf{G}$ such that its closure contains $\mathbf{o}$. Then we say that $\mathbf{M}$ is *a Kempf monoid over k.*

Let us note for the future reference the following reformulation of Theorem 5.5.2.

Corollary 5.5.4. *Let $\mathbf{M}$ be a reductive monoid over k. Then $\mathbf{M}$ is a Kempf monoid.*

Now we give an example of a Kempf monoid which is not reductive.

Example 5.5.5 (Kempf monoid with nonreductive group of units)**.** Let n be a positive integer. Consider the algebraic group $\mathbf{B}_n$ of invertible upper triangular $n \times n$ matrices. Its k-functor is given as follows

$$\mathbf{Alg}_k \ni A \mapsto \{M \in \mathrm{M}_{n\times n}(A)\,|\,M \text{ is upper triangular and invertible}\} \in \mathbf{Grp}$$

Let $\overline{\mathbf{B}}_n$ be the closure of $\mathbf{B}_n$ in the algebraic monoid of all $n \times n$ matrices $\mathbf{M}_n$. Then $\overline{\mathbf{B}}_n$ is an affine, geometrically integral algebraic monoid over k with zero (it contains zero matrix). Actually $\overline{\mathbf{B}}_n$ (or better to say its k-functor of points) consists of all upper triangular $n \times n$ matrices. The group of units of $\overline{\mathbf{B}}_n$ is $\mathbf{B}_n$ and hence it is solvable. Moreover, the center of $\mathbf{B}_n$ contains the one-dimensional split torus $\mathbb{G}_m$ consisting of scalar matrices in $\mathbf{M}_n$. The closure of this torus in $\overline{\mathbf{B}}_n$ contains zero matrix and hence $\overline{\mathbf{B}}_n$ is the Kempf monoid.

Let us discuss some properties of Kempf monoids. We first note the following.

Proposition 5.5.6. *Let $\mathbf{M}$ be a Kempf monoid over k and let T be a central torus of $\mathbf{M}$ such that $\mathbf{cl}(T)$ contains $\mathbf{o}$. Then the closure $\overline{T}$ of T in $\mathbf{M}$ with reduced subscheme structure is a closed toric submonoid k-scheme of $\mathbf{M}$ containing zero.*

Proof. The multiplication μ on $\mathbf{M}$ induces a morphism $\mu_{|\overline{T}\times_k\overline{T}} : \overline{T} \times_k \overline{T} \to \mathbf{M}$. Since scheme-theoretic image of $\mu(T \times_k T)$ is contained in $\overline{T}$ and $T \times_k T$ is open and schematically dense in $\overline{T} \times_k \overline{T}$, we deduce that $\mu_{|\overline{T}\times_k\overline{T}}$ factors through the closed subscheme $\overline{T}$. Thus μ restricts to a multiplication $\nu : \overline{T} \times_k \overline{T} \to \overline{T}$ and hence $\overline{T} \hookrightarrow \mathbf{M}$ is closed immersion of monoid k-schemes. Clearly $\overline{T}$ is geometrically integral as a scheme-theoretic closure of a geometrically integral scheme T. The fact that the zero $\mathbf{o}$ of $\mathbf{M}$ is contained in $\overline{T}$ follows by definition. □

Corollary 5.5.7. *Let $\mathbf{M}$ be a Kempf monoid over k. Fix an algebraically closed field K over k. Then there exists a closed immersion*

$$i : \mathbb{A}^1_K \hookrightarrow \operatorname{Spec} K \times_k \mathbf{M}$$

of monoid K-schemes sending the zero of $\mathbb{A}^1_K$ to the zero of $\mathbf{M}_K = \operatorname{Spec} K \times_k \mathbf{M}$.

Proof. This follows from Proposition 5.5.6 and **(3)** in Theorem 5.4.4. □

Theorem 5.5.8. *Let $\mathbf{M}$ be a Kempf monoid over k with group $\mathbf{G}$ of units and let $j : Z \hookrightarrow \mathbf{M}$ be a locally closed $\mathbf{G}$-stable subscheme of $\mathbf{M}$. Then the following are equivalent.*

- **(i)** *For every $n \in \mathbb{N}$ the n-th infinitesimal neighborhood $\mathbf{M}_n$ of the zero $\mathbf{o}$ in $\mathbf{M}$ is contained in Z.*

(ii) *j is an isomorphism.*

We first consider the following special case.

Lemma 5.5.8.1. *Let U be an open $\mathbf{G}$-stable subscheme of $\mathbf{M}$. If $\mathbf{o}$ is a point of U, then $U = \mathbf{M}$.*

Proof of the lemma. Fix $i : \mathbb{A}^1_K \hookrightarrow \operatorname{Spec} K \times_k \mathbf{M}$ as in Corollary 5.5.7. Denote

$$\operatorname{Spec} K \times_k \mathbf{M}, \operatorname{Spec} K \times_k \mathbf{G}, \operatorname{Spec} K \times_k U$$

by $\mathbf{M}_K$, $\mathbf{G}_K$, U_K, respectively. We also denote by $\mathbf{o}_K$ the zero of $\mathbf{M}_K$ (it is a K-point lying over $\mathbf{o}$). Note that $i(\mathbb{G}_{m,K}) \subseteq \mathbf{G}_K$. Fix a field L over K and a morphism $j : \operatorname{Spec} L \hookrightarrow \mathbf{M}_K$. Next consider the composition

$$\begin{tikzcd} \mathbb{A}^1_L = \mathbb{A}^1_K \times_K \operatorname{Spec} L \arrow[r, hook, "i \times_K j"] \arrow[rr, bend left, "f"] & \mathbf{M}_K \times_k \mathbf{M}_K \arrow[r, hook, "\mu_K"'] & \mathbf{M}_K \end{tikzcd}$$

where the second morphism $\mu_K : \mathbf{M}_K \times_k \mathbf{M}_K \to \mathbf{M}_K$ is the multiplication. Clearly f is $\mathbb{G}_{m,L}$-equivariant. Hence $f^{-1}(U_K)$ is an open $\mathbb{G}_{m,L}$-stable subscheme of $\mathbb{A}^1_L$. It contains the zero of $\mathbb{A}^1_L$ because $\mathbf{o}_K \in U_K$ by assumption. Since the only open $\mathbb{G}_{m,L}$-stable subscheme of $\mathbb{A}^1_L$ containing the zero is $\mathbb{A}^1_L$, we derive that $f^{-1}(U_K) = \mathbb{A}^1_L$. Thus the image of j is in U_K. Hence $U_K = \mathbf{M}_K$ because $j : \operatorname{Spec} L \to \mathbf{M}_K$ and L are arbitrary. By faithfully flat descent, we derive that $U = \mathbf{M}$. □

Proof of the theorem. Assume that **(i)** holds. Since $\mathbf{o}$ is a point in Z, we have a surjective morphism $j^{\#} : \mathcal{O}_{\mathbf{M},\mathbf{o}} \twoheadrightarrow \mathcal{O}_{Z,\mathbf{o}}$ of local rings. Both schemes $Z, \mathbf{M}$ are noetherian and hence we have a commutative square

$$\begin{tikzcd} \widehat{\mathcal{O}_{\mathbf{M},\mathbf{o}}} \arrow[r, "\widehat{j^{\#}}"] & \widehat{\mathcal{O}_{Z,\mathbf{o}}} \\ \mathcal{O}_{\mathbf{M},\mathbf{o}} \arrow[r, two heads, "j^{\#}"'] \arrow[u, hook] & \mathcal{O}_{Z,\mathbf{o}} \arrow[u, hook] \end{tikzcd}$$

where vertical morphisms are injective. Since $\mathbf{M}_n \subseteq Z$ for every $n \in \mathbb{N}$, we derive that $\widehat{j^{\#}}$ is an isomorphism. Hence $j^{\#}$ is injective and thus it is an isomorphism. This implies that there exists an open neighborhood V of $\mathbf{o}$ in $\mathbf{M}$ such that $V \subseteq Z$. Let $\mathbf{G} \cdot V$ be the $\mathbf{G}$-stable hull of V in $\mathbf{M}$. From the fact that j is $\mathbf{G}$-equivariant, we deduce that $\mathbf{G} \cdot V \subseteq Z$. By Lemma 5.5.8.1 we infer that $\mathbf{G} \cdot V = \mathbf{M}$ because $\mathbf{o} \in V \subseteq \mathbf{G} \cdot V$. This shows that $Z = \mathbf{M}$. Thus we have **(i)** $\Rightarrow$ **(ii)**. The implication **(ii)** $\Rightarrow$ **(i)** is obvious. □

We use the proposition below frequently in the following chapters.

Proposition 5.5.9. *Let $\mathbf{M}$ be a monoid k-scheme with zero $\mathbf{o}$ and let X be an $\mathbf{M}$-scheme. Then the following assertions hold.*

(1) *Assume that $X^{\mathbf{M}}$ is a closed subscheme of X. The multiplication by zero $\mathbf{o}\cdot(-) : X \to X$ factors through $X^{\mathbf{M}}$ inducing an $\mathbf{M}$-equivariant retraction $r_{\mathbf{M}} : X \twoheadrightarrow X^{\mathbf{M}}$.*

(2) *Assume that $X^{\mathbf{M}}$ is a closed subscheme of X. If $\mathbf{N}$ is a submonoid k-scheme of $\mathbf{M}$ and $\mathbf{o}$ is a k-point of $\mathbf{N}$, then $r_{\mathbf{M}} = r_{\mathbf{N}}$. In particular, $X^{\mathbf{M}}$ and $X^{\mathbf{N}}$ coincide.*

(3) *If $\mathbf{M}$ is a Kempf monoid and X is a locally linear $\mathbf{M}$-scheme, then $r_{\mathbf{M}}$ is an affine morphism.*

(4) *If $\mathbf{M}$ is a Kempf monoid, X is a locally noetherian, locally linear $\mathbf{M}$-scheme and ideal of $X^{\mathbf{M}}$ in X is nilpotent, then $r_{\mathbf{M}}$ is a finite morphism.*

Proof. The multiplication $\mathbf{o}\cdot(-) : X \to X$ factors as an $\mathbf{M}$-equivariant epimorphism $X \twoheadrightarrow X^{\mathbf{M}}$ composed with a closed immersion $X^{\mathbf{M}} \hookrightarrow X$ (this can be checked on k-functors of points). The $\mathbf{M}$-equivariant epimorphism $X \twoheadrightarrow X^{\mathbf{M}}$ corresponds to an $\mathbf{M}$-equivariant morphism $r_{\mathbf{M}} : X \twoheadrightarrow X^{\mathbf{M}}$ of k-schemes such that $r_{\mathbf{M}}$ restricted to $X^{\mathbf{M}}$ is the identity $1_{X^{\mathbf{M}}}$. This proves **(1)**.
For the proof of **(2)** note that $\mathbf{o}\cdot(-) : X \to X$ is defined in exactly the same manner for $\mathbf{M}$ and $\mathbf{N}$ (provided that $\mathbf{o}$ is a k-point of $\mathbf{N}$). Thus $r_{\mathbf{M}} = r_{\mathbf{N}}$.
Note that if $\mathbf{M}$ is a Kempf monoid, then by Corollary 5.3.2 $X^{\mathbf{M}}$ is a closed subscheme of X. In particular, $r_{\mathbf{M}}$ is well defined by **(1)**.
Now we prove **(3)**. Suppose that $\mathbf{M}$ is a Kempf monoid and X is a locally linear $\mathbf{M}$-scheme. We prove that $\mathbf{o}\cdot(-) : X \to X$ is an affine morphism. Since X is a locally linear $\mathbf{M}$-scheme, it suffices to prove that the preimage under $\mathbf{o}\cdot(-) : X \to X$ of an open affine $\mathbf{M}$-stable subscheme U of X is affine. We prove that this preimage is equal to U. For this pick a point x in X such that $\mathbf{o}\cdot x \in U$. Let $j : \operatorname{Spec} k(x) \hookrightarrow X$ be the inclusion of x into X. Consider the $\mathbf{M}$-equivariant morphism $f : \mathbf{M}_{k(x)} \to X$ given by the composition

$$\begin{tikzcd} \mathbf{M}_{k(x)} = \mathbf{M}\times_k \operatorname{Spec} k(x) \arrow[r, hook, "1_{\mathbf{M}}\times_k j"] \arrow[rr, bend left, "f"] & \mathbf{M}\times_k X \arrow[r, hook, "a"'] & X \end{tikzcd}$$

where $a : \mathbf{M}\times_k X \to X$ is the action. Since $\mathbf{o}\cdot x \in U$, we derive that $f^{-1}(U)$ contains the zero of the Kempf monoid $\mathbf{M}_{k(x)}$ over $k(x)$. Moreover, $f^{-1}(U)$ is open and $\mathbf{M}$-stable subscheme of $\mathbf{M}_{k(x)}$. Thus Theorem 5.5.8 implies that $f^{-1}(U) = \mathbf{M}_{k(x)}$ and hence $x \in U$. This proves that the preimage under $\mathbf{o}\cdot(-) : X \to X$ of U is a subset of U. On the other hand we have $\mathbf{o}\cdot U \subseteq U$ as U is $\mathbf{M}$-stable. Thus this preimage is equal to U. Therefore, $\mathbf{o}\cdot(-) : X \to X$ is affine. Since the composition of $r_{\mathbf{M}}$ with a closed immersion $X^{\mathbf{M}} \hookrightarrow X$ is $\mathbf{o}\cdot(-)$ and hence an affine morphism, we derive that $r_{\mathbf{M}}$ is affine. This completes the proof of **(3)**.
We prove **(4)**. From **(3)** we know that $r_{\mathbf{M}}$ is an affine morphism. Hence $r_{\mathbf{M}} : X \twoheadrightarrow X^{\mathbf{M}}$ corresponds to some quasi-coherent algebra $\mathcal{A}$ on $X^{\mathbf{M}}$. Moreover, the embedding $X^{\mathbf{M}} \hookrightarrow X$ corresponds to the surjection $\mathcal{A} \twoheadrightarrow \mathcal{O}_{X^{\mathbf{M}}}$ with nilpotent ideal $\mathcal{I} \subseteq \mathcal{A}$. Assume that $\mathcal{I}^n = 0$. Then we have a filtration

$$0 = \mathcal{I}^n \subseteq \mathcal{I}^{n-1} \subseteq \ldots \subseteq \mathcal{I} \subseteq \mathcal{A}$$

with factors $\mathcal{I}^k/\mathcal{I}^{k+1}$ for $k = 0, 1, \ldots, n-1$. Since X is locally noetherian, we derive that each $\mathcal{I}^k/\mathcal{I}^{k+1}$ is a finite type $\mathcal{A}$-module. Hence each factor is a finite type module over $\mathcal{A}/\mathcal{I} = \mathcal{O}_{X^{\mathbf{M}}}$.

Thus $\mathcal{A}$ has the finite filtration which factors are coherent sheaves on $X^{\mathbf{M}}$. Therefore, $\mathcal{A}$ is a coherent algebra on $X^{\mathbf{M}}$ and this shows that $r_{\mathbf{M}}$ is finite. $\square$

Chapter 6

Formal M-Schemes And Their Algebraizations

6.1 Introduction

This is the first of two chapters in this work, where essentially new results are presented. In this chapter we study formal schemes

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \ldots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \ldots$$

such that each Z_n is equipped with action of a monoid k-scheme $\mathbf{M}$, all closed immersions $Z_n \hookrightarrow Z_{n+1}$ are $\mathbf{M}$-equivariant and $Z_n^{\mathbf{M}} = Z_0$ for every $n \in \mathbb{N}$. We call them *formal* $\mathbf{M}$*-schemes*. The basic problem in formal geometry since Grothendieck's introduction of this subject concerns algebraization of formal objects (for the excellent exposition of this topic we refer to [FGI05, Part 4]). In this chapter we solve the problem of algebraization of formal $\mathbf{M}$-schemes in the case when $\mathbf{M}$ is a Kempf monoid over k.
In this chapter k is a field.

6.2 Some 2-categorical limits

In this technical section we discuss certain categorical 2-limits. Framework introduced here enables to define in a precise manner categories of equivariant coherent sheaves on formal schemes. Since we are using monoidal categories and functors, we refere the reader to [ML98, Chapter VII] for definitions of these notions.
Consider a category $\mathcal{C}$ and an endofunctor $T : \mathcal{C} \to \mathcal{C}$. Our goal is to construct certain 2-categorical limit associated with a pair $(\mathcal{C}, T)$. Consider pairs (X, u) consisting of an object X of $\mathcal{C}$ and an isomorphism $u : T(X) \to X$ in $\mathcal{C}$. If (X, u) and (Y, w) are two such pairs, then a morphism $f : (X, u) \to (Y, u)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that the following square

$$\begin{tikzcd}
T(X) \arrow[r, "u"] \arrow[d, "T(f)"'] & X \arrow[d, "f"] \\
T(Y) \arrow[r, "w"'] & Y
\end{tikzcd}$$

is commutative. This data give rise to a category $\mathcal{C}(T)$. There exists a forgetful functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ that sends a morphism $f : (X, u) \to (Y, w)$ to $f : X \to Y$. Moreover, there exists a natural isomorphism $\sigma : T \cdot \pi \Rightarrow \pi$ such that the component of σ on an object (X, u) of $\mathcal{C}(T)$ is u. The next result states that the data above form a certain 2-categorical limit.

Theorem 6.2.1. *Let $(\mathcal{C}, T)$ be a pair consisting of a category and an endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ is a functor and $\tau : T \cdot P \Rightarrow P$ is a natural isomorphisms. Then there exists a unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$.*

Proof. Suppose that $F : \mathcal{D} \to \mathcal{C}(T)$ is a functor such that $P = \pi \cdot F$ and $\sigma_F = \tau$. Pick an object X of $\mathcal{D}$. Then we have $\pi\left(F(X)\right) = P(X)$ and $\sigma_{F(X)} = \tau_X$. This implies that

$$F(X) = \left(P(X), \tau_X : T(P(X)) \to P(X)\right)$$

Next if $f : X \to Y$ is a morphism in $\mathcal{D}$, then we derive that $\pi(F(f)) = P(f)$. Hence $F(f) = P(f)$. This implies that there exists at most one functor F satisfying the properties above. Note also that formulas

$$F(X) = \left(P(X), \tau_X : T(P(X)) \to P(X)\right), \, F(f) = P(f)$$

for an object X in $\mathcal{D}$ and a morphism $f : X \to Y$ in $\mathcal{D}$, give rise to a functor that satisfy $P = \pi \cdot F$ and $\sigma_F = \tau$. This establishes existence and the uniqueness of F. $\square$

Assume now that the pair $(\mathcal{C}, T)$ consists of a monoidal category $\mathcal{C}$ and a monoidal endofunctor T. Then there exists a canonical monoidal structure on $\mathcal{C}(T)$. We define $(-) \otimes_{\mathcal{C}(T)} (-)$ by the formula

$$(X, u) \otimes_{\mathcal{C}(T)} (Y, w) = \left(X \otimes_{\mathcal{C}} Y, (u \otimes_{\mathcal{C}} w) \cdot m_{X,Y}\right)$$

where

$$m_{X,Y} : T\left(X \otimes_{\mathcal{C}} Y\right) \to T(X) \otimes_{\mathcal{C}} T(Y)$$

is the tensor preserving isomorphism of T. We also define the unit

$$I_{\mathcal{C}(T)} = \left(I, T(I) \simeq I\right)$$

where isomorphism $T(I) \simeq I$ is precisely the unit preserving isomorphism of the monoidal functor T. The associativity natural isomorphism for $(-) \otimes_{\mathcal{C}(T)} (-)$ and right, left units for $I_{\mathcal{C}(T)}$ in $\mathcal{C}(T)$ are associativity natural isomorphism and right, left units for $\mathcal{C}$, respectively. The structure makes a functor $\pi : \mathcal{C}(T) \to \mathcal{C}$ strict monoidal and σ a monoidal natural isomorphism. The next result states that the data with these extra monoidal structure form a 2-categorical limit in the 2-category of monoidal categories.

Theorem 6.2.2. *Let $(\mathcal{C}, T)$ be a pair consisting of a monoidal category and its monoidal endofunctor $T : \mathcal{C} \to \mathcal{C}$. Suppose that $\mathcal{D}$ is a monoidal category, $P : \mathcal{D} \to \mathcal{C}$ is a monoidal functor and $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphism. Then there exists a unique monoidal functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ as monoidal functors and monoidal transformations.*

Proof. As follows from (the proof of) Theorem 6.2.1

$$F(X) = (P(X), \tau_X : T(P(X)) \to P(X)), \, F(f) = P(f)$$

for an object X in $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$. Suppose now that F admits a structure of a monoidal functor such that $P = \pi \cdot F$ as monoidal functors. Let

$$\left\{m^F_{X,Y} : F(X \otimes_{\mathcal{D}} Y) \to F(X) \otimes_{\mathcal{C}(T)} F(Y)\right\}_{X,Y \in \mathcal{C}}, \, \phi^F : F(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

be the data forming that structure. Since π is a strict monoidal functor and $P = \pi \cdot F$ as monoidal functors, we derive that for any objects X, Y of $\mathcal{C}$

$$\pi(m^F_{X,Y}) : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$$

is the tensor preserving isomorphism $m^P_{X,Y} : P(X \otimes_{\mathcal{D}} Y) \to P(X) \otimes_{\mathcal{C}} P(Y)$ of the monoidal functor P. By the same argument

$$\pi(\phi_F) : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$$

is the unit preserving isomorphism $\phi^P : P(I_{\mathcal{D}}) \to I_{\mathcal{C}(T)}$ of P. Thus we deduce that for any objects X, Y of $\mathcal{C}$ we have $m^F_{X,Y} = m^P_{X,Y}$ and $\phi^F = \phi^P$. This implies that there exists at most one monoidal functor F such that $P = \pi \cdot F$ as monoidal functors. On the other hand define $m^F_{X,Y} = m^P_{X,Y}$ for objects X, Y in $\mathcal{C}$ and $\phi^F = \phi^P$. We check now that F equipped with these data is a monoidal functor. Fix objects X, Y in $\mathcal{C}$. The square

$$\begin{tikzcd}
T(P(X \otimes_{\mathcal{D}} Y)) \arrow[r, "\tau_{X \otimes_{\mathcal{C}} Y}"] \arrow[d, "T\left(m^P_{X,Y}\right)"'] & P(X \otimes_{\mathcal{C}} Y) \arrow[d, "m^P_{X,Y}"] \\
T(P(X) \otimes_{\mathcal{C}} P(Y)) \arrow[r, "(\tau_X \otimes_{\mathcal{C}} \tau_Y) \cdot m_{P(X), P(Y)}"'] & P(X) \otimes_{\mathcal{C}} P(Y)
\end{tikzcd}$$

is commutative due to the fact that $\tau : T \cdot P \Rightarrow P$ is a monoidal natural isomorphisms. This implies that $m^F_{X,Y}$ is a morphism in $\mathcal{C}(T)$. It follows that $m^F_{X,Y}$ is a natural isomorphism and due to the definition of associativity in $\mathcal{C}(T)$, we derive its compatibility with $m^F_{X,Y}$. Similarly, since the square

$$\begin{tikzcd}
T(P(I_{\mathcal{D}})) \arrow[r, "\tau_{I_{\mathcal{D}}}"] \arrow[d, "T(\phi^P)"'] & P(I_{\mathcal{D}}) \arrow[d, "\phi^P"] \\
T(I_{\mathcal{C}}) \arrow[r, "\phi^T"'] & I_{\mathcal{C}}
\end{tikzcd}$$

is commutative, we deduce that ϕ^F is a morphism in $\mathcal{C}(T)$. By definition of left and right unit in $\mathcal{C}(T)$, we derive their compatibility with ϕ^F. This finishes the verification of the fact that F with $\{m^F_{X,Y}\}_{X,Y\in\mathcal{C}}$ and ϕ^F is a monoidal functor. Definitions of $\{m^F_{X,Y}\}_{X,Y\in\mathcal{C}}$ and ϕ^F show that the identities $P = \pi \cdot F$ holds on the level of monoidal structures. Since the 2-forgetful functor from 2-category of monoidal categories into 2-category of categories is faithful on 2-cells, the identity $\sigma_F = \tau$ of natural isomorphisms is also the identity of monoidal natural isomorphisms. □

Theorem 6.2.3. *Let $(\mathcal{C}, T)$ be a pair consiting of a category and its endofunctor $T : \mathcal{C} \to \mathcal{C}$. Assume that T preserves colimits. Then the following assertions hold.*

(1) *$\pi : \mathcal{C}(T) \to \mathcal{C}$ creates colimits.*

(2) *Suppose that $\mathcal{D}$ is a category, $P : \mathcal{D} \to \mathcal{C}$ a functor preserving small colimits and $\tau : T \cdot P \Rightarrow P$ a natural isomorphisms. Then the unique functor $F : \mathcal{D} \to \mathcal{C}(T)$ such that $P = \pi \cdot F$ and $\sigma_F = \tau$ preserves small colimits.*

Proof. Let I be a small category and $D : I \to \mathcal{C}(T)$ be a diagram such that the composition $\pi \cdot D : I \to \mathcal{C}$ admits a colimit given by the cocone $(X, \{g_i\}_{i\in I})$. Since T preserves colimits, we derive that $(T(X), \{T(u_i)\}_{i\in I})$ is a colimit of $T \cdot \pi \cdot D : I \to \mathcal{C}$. Now $\sigma_D : T \cdot \pi \cdot D \to \pi \cdot D$ is a natural isomorphism. Hence there exists a unique arrow $u : T(X) \to X$ such that $u \cdot T(g_i) = g_i \cdot \sigma_{D(i)}$ for $i \in I$. Clearly u is an isomorphism and hence (X, u) is an object of $\mathcal{C}(T)$. Moreover, the family $\{g_i\}_{i\in I}$ together with (X, u) is a colimiting cocone over D. This proves **(1)**. Now **(2)** is a consequence of **(1)**. □

Now we apply the results above to certain more general diagrams of categories.

Definition 6.2.4. A diagram

$$\ldots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \ldots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of categories and functors is called *a telescope of categories*.

Definition 6.2.5. Let

$$\ldots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \ldots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal (finitely) cocontinuous functors. Then *a 2-categorical limit of the telescope* consists of a monoidal category $\mathcal{C}$, a family of monoidal (finitely) cocontinuous functors $\{\pi_n : \mathcal{C} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ and a family of monoidal natural isomorphisms $\{\sigma_n : F_{n+1} \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n\in\mathbb{N}}$ such that the following universal property holds. For any monoidal category $\mathcal{D}$, family $\{P_n : \mathcal{D} \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ of (finitely) cocontinuous monoidal functors and a family $\{\tau_n : F_n P_{n+1} \Rightarrow P_n\}_{n\in\mathbb{N}}$ of monoidal natural isomorphisms there exists a unique monoidal (finitely) cocontinuous functor $F : \mathcal{D} \to \mathcal{C}$ satisfying $P_n = \pi_n \cdot F$ and $(\sigma_n)_F = \tau_n$ for every $n \in \mathbb{N}$.

Corollary 6.2.6. *Let*

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

be a telescope of monoidal categories and monoidal (finitely) cocontinuous functors. Then its 2*-limit exists.*

Proof. We decompose the task of constructing its 2-limit as follows. First note that one may form a product $\mathcal{C} = \prod_{n\in\mathbb{N}} \mathcal{C}_n$. Next the functors $\{F_n\}_{n\in\mathbb{N}}$ induce an endofunctor $T = \prod_{n\in\mathbb{N}} F_n \times t$, where $\mathbf{1}$ is the terminal category (it has single object and single identity arrow) and $t : \mathcal{C}_0 \to \mathbf{1}$ is the unique functor. Consider the category $\mathcal{C}(T)$. We define $\{\pi_n : \mathcal{C}(T) \to \mathcal{C}_n\}_{n\in\mathbb{N}}$ to be a family of functors given by coordinates of $\pi : \mathcal{C}(T) \to \mathcal{C}$ and $\{\sigma_n : F_n \cdot \pi_{n+1} \Rightarrow \pi_n\}_{n\in\mathbb{N}}$ to be a family of natural isomorphisms given by coordinates of $\sigma : \pi \cdot T \Rightarrow \pi$. Now this data form a 2-limit of the telescope by compilation of Theorem 6.2.2 and Theorem 6.2.3. □

It is worth to extract from previous results a more concrete description of the 2-limit of a telescopes of categories.

Remark 6.2.7 (2-limit of a telescope)**.** Consider a telescope

$$\dots \xrightarrow{F_{n+1}} \mathcal{C}_{n+1} \xrightarrow{F_n} \mathcal{C}_n \xrightarrow{F_{n-1}} \dots \xrightarrow{F_2} \mathcal{C}_2 \xrightarrow{F_1} \mathcal{C}_1 \xrightarrow{F_0} \mathcal{C}_0$$

of categories. Then its 2-limit is the category that can be described as follows. Its objects are pairs $(\{X_n\}_{n\in\mathbb{N}}, \{u_n\}_{n\in\mathbb{N}})$ consisting of a sequence $\{X_n\}_{n\in\mathbb{N}}$ such that X_n is an object of $\mathcal{C}_n$ for every $n \in \mathbb{N}$ and a sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n : F_n(X_{n+1}) \to X_n$ is an isomorphism in $\mathcal{C}_n$ for every $n \in \mathbb{N}$. Next if $(\{X_n\}_{n\in\mathbb{N}}, \{u_n\}_{n\in\mathbb{N}})$ and $(\{Y_n\}_{n\in\mathbb{N}}, \{w_n\}_{n\in\mathbb{N}})$ are two objects in the 2-limit, then a morphism between them consists of a sequence $\{f_n\}_{n\in\mathbb{N}}$ of morphisms such that $f_n : X_n \to Y_n$ is a morphism in $\mathcal{C}_n$ for every $n \in \mathbb{N}$ and squares

$$\begin{tikzcd} F_n(X_{n+1}) \arrow[r, "u_n"] \arrow[d, "F_n(f_{n+1})"'] & X_n \arrow[d, "f_n"] \\ F_n(Y_{n+1}) \arrow[r, "w_n"'] & Y_n \end{tikzcd}$$

that are commutative for every $n \in \mathbb{N}$.

6.3 Formal M-schemes

We introduce formal schemes equipped with actions of monoid k-schemes.

Definition 6.3.1. Let $\mathbf{M}$ be a monoid k-scheme. *A formal* $\mathbf{M}$*-scheme* consists of a sequence $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{M}$-schemes together with $\mathbf{M}$-equivariant closed immersions

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \dots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \dots$$

satisfying the following assertions.

(1) We have $Z_0 = Z_n^{\mathbf{M}}$ for every $n \in \mathbb{N}$.

(2) Let $\mathcal{I}_n$ be an ideal of $\mathcal{O}_{Z_n}$ defining Z_0. Then for every $m \le n$ the subscheme $Z_m \subset Z_n$ is defined by $\mathcal{I}_n^{m+1}$.

Example 6.3.2. Let $\mathbf{M}$ be a monoid k-scheme and let Z be an $\mathbf{M}$-scheme. Suppose that $Z^{\mathbf{M}}$ is a closed subscheme of Z. Consider its quasi-coherent ideal $\mathcal{I}$.. Then for every $n \in \mathbb{N}$ ideal $\mathcal{I}^n$ is quasi-coherent ideal and its vanishing scheme $V(\mathcal{I}^n)$ is an $\mathbf{M}$-stable closed subscheme of Z. Hence

$$V(\mathcal{I}) \hookrightarrow V(\mathcal{I}^2) \hookrightarrow \dots \hookrightarrow V(\mathcal{I}^n) \hookrightarrow \dots$$

is a formal $\mathbf{M}$-scheme. We denote it by $\widehat{Z}$.

Definition 6.3.3. Let $\mathbf{M}$ be a monoid k-scheme and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. We say that $\mathcal{Z}$ is *locally noetherian* if for all $n \in \mathbb{N}$ schemes Z_n are locally noetherian.

Definition 6.3.4. Let $\mathbf{M}$ be a monoid k-scheme. Suppose that $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ and $\mathcal{W} = \{W_n\}_{n\in\mathbb{N}}$ are formal $\mathbf{M}$-schemes. Then *a morphism $f : \mathcal{Z} \to \mathcal{W}$ of formal $\mathbf{M}$-schemes* consists of a family of $\mathbf{M}$-equivariant morphisms $f = \{f_n : Z_n \to W_n\}_{n\in\mathbb{N}}$ such that the diagram

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d, "f_0"'] & Z_1 \arrow[r, hook] \arrow[d, "f_1"'] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, "f_n"'] & Z_{n+1} \arrow[r, hook] \arrow[d, "f_{n+1}"'] & \dots \\
W_0 \arrow[r, hook] & W_1 \arrow[r, hook] & \dots \arrow[r, hook] & W_n \arrow[r, hook] & W_{n+1} \arrow[r, hook] & \dots
\end{tikzcd}$$

is commutative.

Remark 6.3.5. There is certain subtlety concerning pullback functors of coherent $\mathbf{G}$-sheaves and this is the right place to elaborate on it. Suppose that X, Y, Z are locally noetherian k-schemes on which group k-scheme $\mathbf{G}$ acts. Let $f : X \to Y$ and $g : Y \to Z$ be $\mathbf{G}$-equivariant morphisms. Then we have (Remark 3.12.5) three monoidal functors

$$(g \cdot f)^* : \mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(X),\ g^* : \mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(Y),\ f^* : \mathfrak{Coh}_{\mathbf{G}}(Y) \to \mathfrak{Coh}_{\mathbf{G}}(X)$$

It is not the case that $(g \cdot f)^* = f^* \cdot g^*$. It is rather the case that there is a canonical isomorphism $(g \cdot f)^* \simeq f^* \cdot g^*$ of monoidal functors.

Definition 6.3.6. Let $\mathbf{M}$ be a monoid k-scheme and let $\mathbf{G}$ be its group of units. Let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a locally noetherian formal $\mathbf{M}$-scheme. Then we have the corresponding telescope of monoidal categories

$$\dots \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_{n+1}) \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_n) \longrightarrow \dots \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_2) \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_1) \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_0)$$

and finitely cocontinuous monoidal functors given by restricting $\mathbf{G}$-equivariant coherent sheaves to closed $\mathbf{G}$-subschemes. Then we define *a category $\mathfrak{Coh}_{\mathbf{G}}(\mathcal{Z})$ of coherent $\mathbf{G}$-sheaves on* $\mathcal{Z}$ as a monoidal category which is a 2-limit of the telescope above. This category is defined uniquely up to a monoidal equivalence by Corollary 6.2.6.

Fix now a monoid k-scheme $\mathbf{M}$ and let $\mathbf{G}$ be its group of units. Let Z be a locally noetherian k-scheme with action of $\mathbf{M}$ and suppose that $Z^{\mathbf{M}}$ is a closed subscheme of Z. Let $\mathcal{I}$ be the ideal sheaf of $Z^{\mathbf{M}}$ in Z. We have a commutative diagram

$$\begin{tikzcd}
V(\mathcal{I}) \arrow[r, hook] \arrow[drr, hook, bend right] & V(\mathcal{I}^2) \arrow[r, hook] \arrow[dr, hook] & \dots \arrow[r, hook] & V(\mathcal{I}^n) \arrow[r, hook] \arrow[dl, hook] & \dots \\
 & & Z & &
\end{tikzcd}$$

in the category of k-schemes with $\mathbf{M}$-actions. Thus we have a diagram of finitely cocontinuous monoidal functors (Remark 3.12.5)

$$\begin{tikzcd}
\mathfrak{Coh}_{\mathbf{G}}(V(\mathcal{I})) & \mathfrak{Coh}_{\mathbf{G}}(V(\mathcal{I}^2)) \arrow[l] & \dots \arrow[l] & \mathfrak{Coh}_{\mathbf{G}}(V(\mathcal{I}^n)) \arrow[l] & \dots \arrow[l] \\
 & & \mathfrak{Coh}_{\mathbf{G}}(Z) \arrow[ull, bend left] \arrow[ul] \arrow[ur, bend right] & &
\end{tikzcd}$$

which by Remark 6.3.5 is commutative up to canonically defined isomorphisms of functors. By Corollary 6.2.6 this induces a unique finitely cocontinuous monoidal functor $\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$. The fact that the diagram above is commutative only up to a canonical isomorphisms makes 2-categorical limits (in the sense of the previous section) indispensable.

Definition 6.3.7. Let Z be a locally noetherian $\mathbf{M}$-scheme such that $Z^{\mathbf{M}}$ is a closed subscheme of Z. Let $\mathbf{G}$ be a group of units of $\mathbf{M}$. Then the finitely cocontinuous monoidal functor $\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$ defined above is called *the comparison functor*.

Corollary 6.3.8. *Let $\mathbf{M}$ be a monoid k-scheme and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Then Z_n is a locally linear $\mathbf{M}$-scheme for every $n \in \mathbb{N}$.*

Proof. Let $\mathcal{I}_n$ be an ideal defining Z_0 in Z_n. Since $\mathcal{Z}$ is a formal $\mathbf{M}$-scheme, we derive that $\mathcal{I}_n^{n+1} = 0$ and clearly Z_0 is a locally linear $\mathbf{M}$-scheme. Thus we apply Corollary 3.11.4 and derive that Z_n is a locally linear $\mathbf{M}$-scheme for every $n \in \mathbb{N}$. □

Corollary 6.3.9. *Let $\mathbf{M}$ be a Kempf monoid over k and let $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Then $\mathcal{Z}$ is a part of the commutative diagram*

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[drr, two heads, bend right, "r_0 = 1_{Z_0}"'] & Z_1 \arrow[r, hook] \arrow[dr, two heads, "r_1"] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[dl, two heads, bend left, "r_n"'] & \dots \\
 & & Z_0 & &
\end{tikzcd}$$

in which vertical morphisms $r_n : Z_n \twoheadrightarrow Z_0$ are affine $\mathbf{M}$-equivariant morphisms induced by multiplications by zero of $\mathbf{M}$ (Proposition 5.5.9) such that $r_{n|Z_0} = 1_{Z_0}$. Moreover, the following assertions hold.

(1) *If $\mathcal{Z}$ is locally noetherian, then every r_n is finite.*

(2) *If* $\mathbf{N}$ *is a submonoid* k*-scheme of* $\mathbf{M}$ *containing the zero of* $\mathbf{M}$*, then* $\mathcal{Z}$ *is a formal* $\mathbf{N}$*-scheme.*

Proof. This is an immediate consequence of Corollary 6.3.8 and Proposition 5.5.9. $\square$

6.4 Quasi-coherent G-sheaves on schemes affine over bases with trivial action of G

In order to address algebraization of formal $\mathbf{M}$-schemes we need to rephrase the notion of quasi-coherent $\mathbf{G}$-sheaf in the case of $\mathbf{G}$-schemes which are affine over schemes with trivial action of $\mathbf{G}$. This description enables to use representation theory of $\mathbf{G}$ in studying $\mathbf{G}$-sheaves.

Remark 6.4.1. Let $\mathbf{G}$ be an affine group k-scheme and let X be a k-scheme equipped with an action $a : \mathbf{G} \times_k X \to X$ of $\mathbf{G}$. Suppose that $r : X \to Y$ is a $\mathbf{G}$-equivariant morphism to a trivial $\mathbf{G}$-scheme. Assume that r is affine. Then $X = \mathrm{Spec}_Y \mathcal{A}$, where $\mathcal{A}$ is a quasi-coherent algebra on Y and the action a corresponds to the morphism $\mathcal{A} \to k[\mathbf{G}] \otimes_k \mathcal{A}$ of algebras over $\mathcal{O}_Y$ such that for every open affine subscheme V of Y its restriction

$$\mathcal{A}(V) \to k[\mathbf{G}] \otimes_k \mathcal{A}(V)$$

to sections over V is the coaction of $k[\mathbf{G}]$ on $\mathcal{A}(V)$. Now suppose that $\mathcal{F}$ is a quasi-coherent $\mathbf{G}$-sheaf on X with respect to $\gamma : \mathcal{F} \to a_* \pi^* \mathcal{F}$ (Remark 3.12.8), where $\pi : \mathbf{G} \times_k X \to X$ is the projection. Then $r_* \mathcal{F} = \mathcal{M}$ is a quasi-coherent sheaf on Y which is an $\mathcal{A}$-module and $r_* \gamma$ is the morphism $\mathcal{M} \to k[\mathbf{G}] \otimes_k \mathcal{M}$ of quasi-coherent sheaves on Y such that the following assertions hold.

(1) For every open affine subscheme V of Y the restriction

$$\mathcal{M}(V) \to k[\mathbf{G}] \otimes_k \mathcal{M}(V)$$

to sections over V is the coaction of $k[\mathbf{G}]$ on $\mathcal{M}(V)$.

(2) $\mathcal{M} \to k[\mathbf{G}] \otimes_k \mathcal{M}$ is the morphism of $\mathcal{A}$-modules, where $k[\mathbf{G}] \otimes_k \mathcal{M}$ carries the structure of an $\mathcal{A}$-module induced by the restriction of scalars $\mathcal{A} \to k[\mathbf{G}] \otimes_k \mathcal{A}$ along action a.

Let $\mathbf{G}$ be an affine monoid k-scheme and let X be a k-scheme equipped with the trivial action of $\mathbf{G}$. Fix λ in $\mathbf{Irr}(\mathbf{G})$ and a quasi-coherent $\mathbf{G}$-sheaf $\mathcal{F}$ on X. We define a quasi-coherent $\mathbf{G}$-subsheaf $\mathcal{F}[\lambda]$ of $\mathcal{F}$ by formula

$$U \mapsto \mathcal{F}(U)[\lambda]$$

for every open affine subscheme U of X. Here $\mathcal{F}(U)[\lambda]$ is the isotypic component (Definition 3.9.9) of the representation $\mathcal{F}(U)$ of $\mathbf{G}$ (Example 3.12.10) corresponding to λ. Fact 3.9.10 together with the fact that $\mathbf{G}$-acts trivially on X imply that this gives rise to a quasi-coherent subsheaf of $\mathcal{F}$.

Definition 6.4.2. We call $\mathcal{F}[\lambda]$ *the isotypic component of* $\mathcal{F}$ *corresponding to* λ*.*

Fact 6.4.3. *Let $\mathbf{G}$ be an affine group k-scheme and let X be a k-scheme equipped with the trivial action of $\mathbf{G}$. Suppose that $\mathcal{F}_1, \mathcal{F}_2$ are quasi-coherent $\mathbf{G}$-sheaves on X. Fix $\lambda_1, \lambda_2, \eta_1, ..., \eta_n$ in $\mathbf{Irr}(\mathbf{G})$ and assume that*

$$V_{\lambda_1} \otimes_k V_{\lambda_2} \simeq \bigoplus_{i=1}^{n} V_{\eta_i}$$

as $\mathbf{G}$-representations, where by V_λ we denote the irreducible representation in class $\lambda \in \mathbf{Irr}(\mathbf{G})$. Then

$$\big(\mathcal{F}[\lambda_1] \otimes_{\mathcal{O}_X} \mathcal{F}_2[\lambda_2]\big)[\lambda] = 0$$

for $\lambda \notin \{\eta_1, ..., \eta_n\}$.

Proof. Consider an open affine subscheme U of X. The canonical surjection

$$\Gamma(U, \mathcal{F}_1)[\lambda_1] \otimes_k \Gamma(U, \mathcal{F}_2)[\lambda_2] \longrightarrow\!\!\!\!\!\rightarrow \Gamma(U, \mathcal{F}_1)[\lambda_1] \otimes_{\mathcal{O}_X(U)} \Gamma(U, \mathcal{F}_2)[\lambda_2]$$

is a morphism of $\mathbf{G}$-representations. Since $V_{\lambda_1} \otimes_k V_{\lambda_2} \simeq \bigoplus_{i=1}^n V_{\eta_i}$, we derive by uniqueness of isotypic decomposition (Theorem 3.9.8) that

$$\big(\Gamma(U, \mathcal{F}_1)[\lambda_1] \otimes_k \Gamma(U, \mathcal{F}_2)[\lambda_2]\big)[\lambda] = 0$$

for $\lambda \neq \eta_i$ and $1 \leq i \leq n$. This implies that $\big(\Gamma(U, \mathcal{F}_1)[\lambda_1] \otimes_{\mathcal{O}_X(U)} \Gamma(U, \mathcal{F}_2)[\lambda_2]\big)[\lambda] = 0$ for $\lambda \notin \{\eta_1, ..., \eta_n\}$. Since U is an arbitrary affine open subscheme of X, we deduce that the statement holds. □

6.5 Algebraization of formal M-schemes over Kempf monoids

Now we are ready to prove results concerning *algebraizations* of formal $\mathbf{M}$-schemes for Kempf monoids.

Theorem 6.5.1. *Let $\mathbf{M}$ be a Kempf monoid and let $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$ be a formal $\mathbf{M}$-scheme. Then there exists a locally linear $\mathbf{M}$-scheme Z such that $\widehat{Z}$ is isomorphic to $\mathcal{Z}$. Moreover, we have that*

$$Z = \mathrm{colim}_{n \in \mathbb{N}} Z_n$$

in category of $\mathbf{M}$-schemes affine over Z_0.

Setup. The monoid $\mathbf{M}$ is affine and admits a zero $\mathbf{o}$. By Corollary 6.3.9 a formal $\mathbf{M}$-scheme $\mathcal{Z} = \{Z_n\}_{n \in \mathbb{N}}$ corresponds to a sequence of surjections

$$... \longrightarrow\!\!\!\!\!\rightarrow \mathcal{A}_{n+1} \longrightarrow\!\!\!\!\!\rightarrow \mathcal{A}_n \longrightarrow\!\!\!\!\!\rightarrow ... \longrightarrow\!\!\!\!\!\rightarrow \mathcal{A}_1 \longrightarrow\!\!\!\!\!\rightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$$

of quasi-coherent algebras on Z_0 such that the following assertions hold.

(1) For each $n \in \mathbb{N}$ we fix a morphism $\mathcal{A}_n \to k[\mathbf{M}] \otimes_k \mathcal{A}_n$ such that for every open affine neighborhood U of Z_0 its restriction

$$\mathcal{A}_n(U) \to k[\mathbf{M}] \otimes_k \mathcal{A}_n(U)$$

to sections on U is a coaction of $k[\mathbf{M}]$ on $\mathcal{A}_n(U)$.

(2) For every $n \in \mathbb{N}$ the epimorphism $\mathcal{A}_{n+1} \twoheadrightarrow \mathcal{A}_n$ preserves the coaction described in **(1)**.

(3) The morphism $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0$ is the surjection inducing $Z_n^{\mathbf{M}} \hookrightarrow Z_n$ for every $n \in \mathbb{N}$.

(4) $\mathcal{A}_0 \hookrightarrow \mathcal{A}_n \twoheadrightarrow \mathcal{A}_0$ is an isomorphism for every $n \in \mathbb{N}$.

(5) If $\mathcal{I}_n$ is the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0$ in $\mathcal{A}_n$, then $\mathcal{I}_n^{m+1}$ is the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$ for $m \leq n$ and $n \in \mathbb{N}$.

Since $\mathbf{M}$ is a Kempf monoid, there exists a closed subgroup T of the center $Z(\mathbf{G})$ of the unit group $\mathbf{G}$ of $\mathbf{M}$ such that T is a torus and by Proposition 5.5.6 the closure $\overline{T}$ of T with reduced structure is a closed toric submonoid of $\mathbf{M}$ with zero. We derive by Corollary 6.3.9 that $Z_n^{\overline{\mathbf{T}}} = Z_0 = Z_n^{\mathbf{M}}$ for every $n \in \mathbb{N}$. Let $\{V_\lambda\}_{\lambda \in \mathbf{Irr}(T)}$ be a set of irreducible representations of T such that V_λ is contained in λ.

Lemma 6.5.1.1. *Let λ be in $\mathbf{Irr}(T)$. Then there exists $n_\lambda \in \mathbb{N}$ such that for each $n > n_\lambda$ and any $\lambda_1, ..., \lambda_n \in \mathbf{Irr}(\overline{T}) \setminus \{\lambda_0\}$ the representation*

$$\bigotimes_{i=1}^{n} V_{\lambda_i}$$

has trivial isotypic component of type λ. We have $n_{\lambda_0} = 0$, where λ_0 is an isomorphism type of the trivial representation of T.

Proof of the lemma. Let K be an algebraically closed extension of k. Pick A_λ and f as in Theorem 5.4.4 and define

$$n_\lambda = \sup_{m \in A_\lambda} f(m)$$

Fix $n > n_\lambda$. We have

$$K \otimes_k V_{\lambda_1} \otimes_k ... \otimes_k V_{\lambda_n} = \bigoplus_{(m_1,...,m_n) \in A_{\lambda_1} \times_k ... \times_k A_{\lambda_n}} K \cdot \chi^{m_1 + ... + m_n}$$

and since $m_1, ...m_n \in A_{\lambda_1} \cup ... \cup A_{\lambda_n} \subseteq S \setminus \{0\}$ we derive that

$$f(m_1 + ... + m_n) = f(m_1) + ... + f(m_n) \geq n > n_\lambda = \sup_{m \in A_\lambda} f(m)$$

This implies that isotypic component of $V_{\lambda_1} \otimes_k ... \otimes_k V_{\lambda_n}$ corresponding to λ is trivial. $\square$

Lemma 6.5.1.2. *Fix λ in $\mathbf{Irr}(T)$. Then $\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$ is an isomorphism for $n \geq n_\lambda$.*

Proof of the lemma. For $\lambda \notin \mathbf{Irr}(\overline{T})$ we have $\mathcal{A}_{n+1}[\lambda] = \mathcal{A}_n[\lambda] = 0$. This follows from Theorem 3.8.3 and from the observation that by definition for every open affine subset U of Z_0 we have the coaction of $k[\overline{T}]$ on $\mathcal{A}_{n+1}(U)$ and $\mathcal{A}_n(U)$ induced by the coaction of $k[\mathbf{M}]$ on these algebras.
Fix $\lambda \in \mathbf{Irr}(\overline{T})$. By Lemma 6.5.1.1 and Fact 6.4.3 we derive that

$$\underbrace{\left(\mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} ... \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1}\right)}_{n+1 \text{ times}}[\lambda] = 0$$

for every $n \geq n_\lambda$. Next the multiplication

$$\underbrace{\left(\mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1} \otimes_{\mathcal{O}_{Z_0}} ... \otimes_{\mathcal{O}_{Z_0}} \mathcal{I}_{n+1}\right)}_{n+1 \text{ times}} \longrightarrow \mathcal{A}_{n+1}$$

is a morphism of quasi-coherent T-sheaves with image $\mathcal{I}_{n+1}^{n+1}$. Thus we derive that $\mathcal{I}_{n+1}^{n+1}[\lambda] = 0$ for $n \geq n_{\lambda}$. Hence the kernel of $\mathcal{A}_{n+1}[\lambda] \twoheadrightarrow \mathcal{A}_n[\lambda]$ is trivial. $\square$

Proof of Theorem 6.5.1. According to Proposition 3.9.11 and the fact that T is central in $\mathbf{M}$ we derive that $\mathcal{A}_n[\lambda](U)$ is a linear representation of $\mathbf{M}$ for every open affine U of Z_0. For $\lambda \in \mathbf{Irr}(T)$ we define

$$\mathcal{A}[\lambda] = \mathcal{A}_n[\lambda]$$

where $n \geq n_{\lambda}$ as in Lemma 6.5.1.2. Note that $\mathcal{A}[\lambda] = 0$ for $\lambda \notin \mathbf{Irr}(\overline{T})$. We set

$$\mathcal{A} = \bigoplus_{\lambda \in \mathbf{Irr}(\overline{T})} \mathcal{A}[\lambda]$$

Since $Z_n^{\overline{T}} = Z_n^{\mathbf{M}} = Z_0$ by Corollary 6.3.9, we deduce that $\mathcal{A}[\lambda_0] = \mathcal{A}_0 = \mathcal{O}_{Z_0}$ canonically (where λ_0 is the trivial T-representation). Note that $\mathcal{A}$ is a quasi-coherent sheaf on Z_0 with coaction of $k[\mathbf{M}]$ on each sections over affine open U of Z_0 (by definition $\mathcal{A}$ is a direct sum of such sheaves). Actually $\mathcal{A} = \lim_{n \in \mathbb{N}} \mathcal{A}_n$ in the category of quasi-coherent sheaves on Z_0 with coaction of $k[\mathbf{M}]$. We construct the $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$. For this pick $\lambda_1, \lambda_2 \in \mathbf{Irr}(\overline{T})$. Consider irreducible representations V_{λ_1} and V_{λ_1} in classes λ_1 and λ_2, respectively. Suppose that $\eta_1, ..., \eta_s$ are finitely many classes in $\mathbf{Irr}(\overline{T})$ such that $V_{\lambda_1} \otimes_k V_{\lambda_2}$ can be completely decomposed onto irreducible representation in these classes ($\overline{T}$ is linearly reductive by Corollary 5.4.2). According to Fact 6.4.3 we deduce that the image of the multiplication

$$\mathcal{A}_n[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}_n[\lambda_2] \longrightarrow \mathcal{A}_n$$

is contained in $\bigoplus_{i=1}^{s} \mathcal{A}_n[\eta_i]$. By Lemma 6.5.1.2 all these multiplications for

$$n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$$

can be identified. Now we define

$$\mathcal{A}[\lambda_1] \otimes_{\mathcal{O}_{Z_0}} \mathcal{A}[\lambda_2] \to \bigoplus_{i=1}^{s} \mathcal{A}[\eta_i] \subseteq \mathcal{A}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$. This gives an $\mathcal{O}_{Z_0}$-algebra structure on $\mathcal{A}$. So $\mathcal{A}$ is in fact the limit of $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ in the category of quasi-coherent algebras on Z_0 with coaction of $k[\mathbf{M}]$. This implies that

$$Z = \operatorname{Spec}_{Z_0} \mathcal{A} = \operatorname{colim}_{n \in \mathbb{N}} Z_n$$

in the category of schemes affine over Z_0 and equipped with an action of $\mathbf{M}$. Note that from the description of $\mathcal{A}$ it follows that for every $n \in \mathbb{N}$ we have a surjective morphism $p_n : \mathcal{A} \twoheadrightarrow \mathcal{A}_n$ of algebras. We denote its kernel by $\mathcal{J}_n$ and we put $\mathcal{J} = \mathcal{J}_0$. We have

$$\mathcal{J} = \bigoplus_{\lambda \in \mathbf{Irr}(\overline{T}) \setminus \{\lambda_0\}} \mathcal{A}[\lambda]$$

Recall that we denote by $\mathcal{I}_n$ the kernel of $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_0 = \mathcal{O}_{Z_0}$ for $n \in \mathbb{N}$. Then $\mathcal{I}_n = \mathcal{J}/\mathcal{J}_n$. Fix $m \in \mathbb{N}$ and consider $n \in \mathbb{N}$ such that $n \geq m$. Since $\mathcal{Z}$ is a formal $\mathbf{M}$-scheme, the sheaf $\mathcal{I}_n^{m+1}$ is the kernel of the morphism $\mathcal{A}_n \twoheadrightarrow \mathcal{A}_m$. Thus

$$\mathcal{J}_m/\mathcal{J}_n = \mathcal{I}_n^{m+1} = (\mathcal{J}^{m+1} + \mathcal{J}_n)/\mathcal{J}_n$$

Both $\mathcal{J}_m$ and $\mathcal{J}^{m+1}$ are $\mathbf{Irr}(\overline{T})$-graded by their isotypic $\overline{T}$-components and for given $\lambda \in \mathbf{Irr}(\overline{T})$ and for $n \geq n_\lambda$ the isotypic component $\mathcal{J}_n[\lambda]$ is zero by Lemma 6.5.1.2. Hence $\mathcal{J}_m = \mathcal{J}^{m+1}$ for every $m \in \mathbb{N}$. Thus $\widehat{Z} = \mathcal{Z}$ and, since the canonical affine morphism $Z \to Z_0$ is $\mathbf{M}$-equivariant and Z_0 is equipped with the trivial action of $\mathbf{M}$, we deduce that Z a locally linear $\mathbf{M}$-scheme. □

Theorem 6.5.2. *Let $\mathbf{M}$ be a Kempf monoid. Suppose that Z is a locally linear $\mathbf{M}$-scheme such that $\widehat{Z} = \{Z_n\}_{n\in\mathbb{N}}$ and*

$$Z = \operatorname{colim}_{n\in\mathbb{N}} Z_n$$

in the category of $\mathbf{M}$-schemes affine over $Z^{\mathbf{M}}$, where Z is affine over $Z^{\mathbf{M}}$ via the canonical affine retraction $r^Z : Z \to Z^{\mathbf{M}}$ (Proposition 5.5.9). If W is a locally linear $\mathbf{M}$-scheme and $\widehat{W}$ and $\widehat{Z}$ are isomorphic as formal $\mathbf{M}$-schemes, then W and Z are $\mathbf{M}$-equivariantly isomorphic.

Proof. Let $r^W : W \to W^{\mathbf{M}}$ be the affine retraction (Proposition 5.5.9). We also denote $\widehat{W} = \{W_n\}_{n\in\mathbb{N}}$. Note that we have an identification $W_n \simeq Z_n$ of $\mathbf{M}$-schemes for every $n \in \mathbb{N}$. By the universal property of colimits there exists an $\mathbf{M}$-equivariant morphism $f : Z \to W$ such that $r^W \cdot f = r^Z$ and $f_{|Z_n}$ is isomorphic to the closed immersion $W_n \hookrightarrow W$ for every $n \in \mathbb{N}$. We consider now Z and W as $\mathbf{M}$-schemes affine over the same base $Z^{\mathbf{M}} = W^{\mathbf{M}}$ equipped with the trivial $\mathbf{M}$-action. Then Z, W correspond to quasi-coherent algebras $\mathcal{A}, \mathcal{B}$ on $Z^{\mathbf{M}} = W^{\mathbf{M}}$, respectively, and moreover, there are quasi-coherent ideals $\mathcal{I} \subseteq \mathcal{A}$, $\mathcal{J} \subseteq \mathcal{B}$ such that

$$\mathcal{A}/\mathcal{I} = \mathcal{O}_{Z^{\mathbf{M}}} = \mathcal{O}_{W^{\mathbf{M}}} = \mathcal{B}/\mathcal{J}$$

Then f corresponds to a morphism $h : \mathcal{B} \to \mathcal{A}$ of quasi-coherent algebras such that $h(\mathcal{J}) \subseteq \mathcal{I}$ and for every $n \in \mathbb{N}$ morphism h induces an isomorphism

$$\mathcal{B}/\mathcal{J}^{n+1} \simeq \mathcal{A}/\mathcal{I}^{n+1}$$

of quasi-coherent algebras. Moreover, for every open affine subscheme U of $Z^{\mathbf{M}} = W^{\mathbf{M}}$ morphism h is a morphism of canonically defined $k[\mathbf{M}]$-comodules $\mathcal{A}(U) \to \mathcal{B}(U)$. Pick now an algebraically closed extension K of k and a zero preserving closed immersion $\mathbb{A}^1_K \hookrightarrow \operatorname{Spec} K \times_k \mathbf{M}$ of monoid K-schemes (Corollary 5.5.7). Then we have induced $\mathbb{N}$-gradings on

$$K \otimes_k \mathcal{A} = \mathcal{A}_K = \bigoplus_{i\in\mathbb{N}} \mathcal{A}_K[i],\ K \otimes_k \mathcal{B} = \mathcal{B}_K = \bigoplus_{i\in\mathbb{N}} \mathcal{B}_K[i]$$

and $h_K = 1_K \otimes_k h$ is a $\mathbb{N}$-graded homomorphism of algebras. Since the closed immersion of monoid K-schemes considered above is zero preserving and according to Proposition 5.5.9, we deduce that

$$\operatorname{Spec} K \times_k Z^{\mathbf{M}} = (\operatorname{Spec} K \times_k Z)^{\mathbf{M}_K} = (\operatorname{Spec} K \times_k Z)^{\mathbb{A}^1_K}$$

as K-schemes and hence

$$\mathcal{I}_K = K \otimes_k \mathcal{I} = \bigoplus_{i>0} \mathcal{A}_K[i],\ \mathcal{J}_K = K \otimes_k \mathcal{J} = \bigoplus_{i>0} \mathcal{B}_K[i]$$

Moreover, h_K induces isomorphisms of $\mathbb{N}$-graded algebras

$$\mathcal{B}_K/\mathcal{J}_K^{n+1} \simeq \mathcal{A}_K/\mathcal{I}_K^{n+1}$$

for every $n \in \mathbb{N}$. These imply that for every $i \in \mathbb{N}$ morphism $h_K[i] : \mathcal{B}_K[i] \to \mathcal{A}_K[i]$ is an isomorphism and hence h_K is an isomorphism. By faithfully flat descent we deduce that h is an isomorphism of quasi-coherent algebras on $Z^{\mathbf{M}} = W^{\mathbf{M}}$. Thus f is an $\mathbf{M}$-equivariant isomorphism. □

Corollary 6.5.3. *Let* $\mathbf{M}$ *be a Kempf monoid. Suppose that* Z *and* W *are locally linear* $\mathbf{M}$*-schemes such that* $\widehat{Z}$ *and* $\widehat{W}$ *are isomorphic as formal* $\mathbf{M}$*-schemes. Then* Z *and* W *are* $\mathbf{M}$*-equivariantly isomorphic.*

Proof. This is a consequence of Theorems 6.5.1 and 6.5.2. □

Example 6.5.4. Let $\mathbf{M}$ be a Kempf monoid and let Y be a k-scheme. We consider Y as an $\mathbf{M}$-scheme with the trivial $\mathbf{M}$-action. Since $\mathbf{M}$ is a Kempf monoid it admits the zero $\mathbf{o}$. For every $n \in \mathbb{N}$ let $\mathbf{M}_n$ be the n-th infinitesimal neighborhood of $\mathbf{o}$ in $\mathbf{M}$. Note that $\mathbf{M}_n$ is a closed $\mathbf{M}$-stable subscheme of $\mathbf{M}$ for every $n \in \mathbb{N}$. Hence we have a formal $\mathbf{M}$-scheme

$$\mathbf{M}_0 \hookrightarrow \mathbf{M}_1 \times_k Y \hookrightarrow \ldots \hookrightarrow \mathbf{M}_n \times_k Y \hookrightarrow \mathbf{M}_{n+1} \times_k Y \hookrightarrow \ldots$$

Observe that $\mathbf{M} \times_k Y$ is a locally linear $\mathbf{M}$-scheme and it is the unique locally linear $\mathbf{M}$-scheme such that $\widehat{\mathbf{M} \times_k Y} = \left\{\mathbf{M}_n \times_k Y\right\}_{n \in \mathbb{N}}$ by Corollary 6.5.3.

Previous results make it possible to prove a correspondence between $\mathbf{M}$-equivariant morphisms of locally linear $\mathbf{M}$-schemes and morphisms of corresponding formal $\mathbf{M}$-schemes.

Corollary 6.5.5. *Let* $\mathbf{M}$ *be a Kempf monoid and let* Z, W *be locally linear* $\mathbf{M}$*-schemes. Then the canonical map*

$$\mathrm{Mor}_{\mathbf{M}}\big(Z, W\big) \longrightarrow \mathrm{Mor}\big(\widehat{Z}, \widehat{W}\big)$$

is a bijection, where $\mathrm{Mor}_{\mathbf{M}}\big(Z, W\big)$ *is the class of* $\mathbf{M}$*-equivariant morphisms* $Z \to W$ *and* $\mathrm{Mor}\big(\widehat{Z}, \widehat{W}\big)$ *is the class of morphisms of formal* $\mathbf{M}$*-schemes.*

Proof. Suppose that $\widehat{Z} = \{Z_n\}_{n \in \mathbb{N}}$ and $\widehat{W} = \{W_n\}_{n \in \mathbb{N}}$. Consider a morphism $\{f_n : Z_n \to W_n\}_{n \in \mathbb{N}}$ of formal $\mathbf{M}$-schemes. Since the retraction onto fixed points in Proposition 5.5.9 is given by the multiplication by zero of $\mathbf{M}$, we derive that the square

$$\begin{tikzcd} Z_n \arrow[r, "f_n"] \arrow[d, two heads, "r_n^Z"'] & W_n \arrow[d, two heads, "r_n^W"] \\ Z_0 \arrow[r, "f_0"'] & W_0 \end{tikzcd}$$

is commutative for $n \in \mathbb{N}$, where r_n^W and r_n^Z are canonical retractions. Hence for every $n \in \mathbb{N}$ we have a diagram

$$\begin{tikzcd}
Z_n \arrow[r, "p_n"] \arrow[rr, bend left, "f_n"] \arrow[dr, "r_n^Z"'] & W_n \times_{W_0} Z_0 \arrow[r, "q_n"] \arrow[d, two heads, "r_n"'] & W_n \arrow[d, two heads, "r_n^W"'] \\
& Z_0 \arrow[r, "f_0"'] & W_0
\end{tikzcd}$$

v

in which the rightmost square is cartesian and $p_n : Z_n \to W_n \times_{W_0} Z_0$ is the unique morphism that makes the diagram commutative. Now since r_n^W is affine by **(3)** of Proposition 5.5.9, we derive that $r_n : W_n \times_{W_0} Z_0 \to Z_0$ is affine as its base change. Our goal is to show that there exists a unique morphism $p : Z \to W \times_{W_0} Z_0$ such that the square

$$\begin{tikzcd}
Z \arrow[r, dotted, "p"] & W \times_{W_0} Z_0 \\
Z_n \arrow[u, hook] \arrow[r, "p_n"'] & W_n \times_{W_0} Z_0 \arrow[u, hook]
\end{tikzcd}$$

is commutative. Corollary 6.5.3 and Theorem 6.5.1 imply that

$$Z = \mathrm{colim}_{n \in \mathbb{N}} Z_n$$

in the category of **M**-schemes which are affine over Z_0. Thus by the universal property of colimits we deduce that p exists. Now composing p with the morphism $q : W \times_{W_0} Z_0 \to W$ coming from the cartesian square

$$\begin{tikzcd}
W \times_{W_0} Z_0 \arrow[r, "q"] \arrow[d, two heads, "r"'] & W \arrow[d, two heads, "r^W"] \\
Z_0 \arrow[r, "f_0"'] & W_0
\end{tikzcd}$$

we obtain an **M**-equivariant morphism $f = q \cdot p : Z \to W$. By construction $f_{|Z_n}$ induces f_n for every $n \in \mathbb{N}$ and by uniqueness of p we infer that f is a unique **M**-equivariant morphism with this property. This completes the proof. □

Corollary 6.5.6. *Let* **M** *be a Kempf monoid over k. Then the functor*

$$\text{category of locally linear } \mathbf{M}\text{-schemes} \xrightarrow{Z \mapsto \widehat{Z}} \text{category of formal } \mathbf{M}\text{-schemes}$$

is an equivalence of categories.

Proof. The fact that the functor is essentially surjective follows from Theorem 6.5.1. It is also full and faithful by Corollary 6.5.5. □

Theorem 6.5.7. *Let* $\mathbf{M}$ *be a Kempf monoid and let* Z *be a locally linear* $\mathbf{M}$*-scheme. Suppose that* $r : Z \to Z^{\mathbf{M}}$ *is the canonical retraction. If the formal* $\mathbf{M}$*-scheme* $\widehat{Z}$ *is locally noetherian, then* r *is of finite type.*

Proof. Since r is affine (Proposition 5.5.9), we derive that $\mathcal{A} = r_*\mathcal{O}_Z$ is a quasi-coherent $\mathbf{M}$-algebra on $Z^{\mathbf{M}}$. We denote by $\mathcal{J}$ the ideal of $\mathcal{A}$ that corresponds to the closed immersion $Z^{\mathbf{M}} \hookrightarrow Z$. We know that the formal $\mathbf{M}$-scheme

$$Z^{\mathbf{M}} = \mathrm{Spec}_{Z^{\mathbf{M}}}\mathcal{A}/\mathcal{J} \hookrightarrow \ldots \hookrightarrow \mathrm{Spec}_{Z^{\mathbf{M}}}\mathcal{A}/\mathcal{J}^{n+1} \hookrightarrow \mathrm{Spec}_{Z^{\mathbf{M}}}\mathcal{A}/\mathcal{J}^{n+2} \hookrightarrow \ldots$$

is locally noetherian. Hence $\mathcal{J}/\mathcal{J}^{n+1}$ is $\mathcal{A}/\mathcal{J}^{n+1}$-module of finite type. Thus $\{\mathcal{J}^i/\mathcal{J}^{i+1}\}_{1\le i\le n}$ are finite type $\mathcal{A}/\mathcal{J}$-modules. Thus the filtration

$$0 \subseteq \mathcal{J}^n/\mathcal{J}^{n+1} \subseteq \ldots \subseteq \mathcal{J}/\mathcal{J}^{n+1} \subseteq \mathcal{A}/\mathcal{J}^{n+1}$$

has factors that are of finite type over $\mathcal{O}_{Z^{\mathbf{M}}} = \mathcal{A}/\mathcal{J}$. This implies that $\mathcal{A}/\mathcal{J}^{n+1}$ is a coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-algebra for every $n \in \mathbb{N}$. The claim that r is of finite type is local on $Z^{\mathbf{M}}$, hence we may assume that $Z^{\mathbf{M}}$ is quasi-compact. This reduces the question to the noetherian $Z^{\mathbf{M}}$. The sheaf $\mathcal{J}/\mathcal{J}^2 \subseteq \mathcal{A}/\mathcal{J}^2$ is coherent over $\mathcal{O}_{Z^{\mathbf{M}}}$. Since $Z^{\mathbf{M}}$ is noetherian, there exists coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-subsheaf $\mathcal{M} \subseteq \mathcal{J}$ such that the morphism $\mathcal{M} \twoheadrightarrow \mathcal{J}/\mathcal{J}^2$ is surjective. Fix an algebraically closed extension K of k and denote

$$\mathcal{A}_K = K \otimes_k \mathcal{A}, \mathcal{J}_K = K \otimes_k \mathcal{J}, \mathcal{M}_K = K \otimes_k \mathcal{M}$$

Since $\mathbf{M}$ is a Kempf monoid by Corollary 5.5.7 there exists a closed immersion $\mathbb{A}^1_K \hookrightarrow \mathbf{M}_K$ of monoid K-schemes that preserve zero. This implies that we have $\mathbb{N}$-grading $\mathcal{A}_K = \bigoplus_{i\ge 0}\mathcal{A}_K[i]$ that gives rise to the action of $\mathbb{A}^1_K$. Moreover, by Proposition 5.5.9 we deduce that

$$\mathrm{Spec}\, K \times_k Z^{\mathbf{M}} = (\mathrm{Spec}\, K \times_k Z)^{\mathbf{M}_K} = (\mathrm{Spec}\, K \times_k Z)^{\mathbb{A}^1_K}$$

as K-schemes. This shows that $\mathcal{J}_K = K \otimes_k \mathcal{J} = \bigoplus_{i\ge 1}\mathcal{A}_K[i]$ is an ideal with positive grading. We have surjection $\mathcal{M}_K \twoheadrightarrow \mathcal{J}_K/\mathcal{J}_K^2$. By graded version of Nakayama's lemma, the ideal $\mathcal{J}_K$ is generated by $\mathcal{M}_K$. Then by induction on degrees we deduce that $\mathcal{A}_K$ is generated by $\mathcal{M}_K$ as a $K \otimes_k \mathcal{O}_{Z^{\mathbf{M}}}$-algebra. Thus $1_{\mathrm{Spec}\, K} \times_k r$ is of finite type and by faitfully flat descent also r is of finite type. □

6.6 Coherent sheaves on locally noetherian formal M-schemes over Kempf monoids

We prove an $\mathbf{M}$-equivariant version of Grothendieck's existence theorem in formal geometry ([FGI05, Theorem 8.4.2]). In Definition 6.3.7 we introduced the comparison functor. Now we show that under noetherian hypothesis this functor is an equivalence.

Theorem 6.6.1. *Let* $\mathbf{M}$ *be a Kempf monoid with group of units* $\mathbf{G}$ *and let* Z *be a locally linear* $\mathbf{M}$*-scheme. Suppose that* $r : Z \to Z^{\mathbf{M}}$ *is the canonical retraction. If* Z *is locally noetherian, then the comparison functor*

$$\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$$

is an equivalence of monoidal categories.

Setup. Since $\mathbf{M}$ is a Kempf torus, there exists a central closed torus T in $\mathbf{G}$ such that the scheme-theoretic closure $\overline{T}$ of T in $\mathbf{M}$ contains the zero. By Proposition 5.5.9 morphism r is affine and we pick a quasi-coherent algebra $\mathcal{A} = r_*\mathcal{O}_Z$ on $Z^{\mathbf{M}}$. We denote by $\mathcal{J}$ the ideal of $\mathcal{A}$ that corresponds to the closed immersion $Z^{\mathbf{M}} \hookrightarrow Z$. Then $\mathcal{O}_{Z^{\mathbf{M}}} = \mathcal{A}/\mathcal{J}$ and since r is a retraction, we derive that $\mathcal{A} = \mathcal{O}_{Z^{\mathbf{M}}} \oplus \mathcal{J}$ as $\mathcal{O}_{Z^{\mathbf{M}}}$-modules. Next $\widehat{Z}$ is locally noetherian (this follows from the fact that Z is locally noetherian). By Remark 6.4.1 and Remark 6.2.7 an object of $\mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$ corresponds to a sequence of surjections

$$\ldots \twoheadrightarrow \mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n \twoheadrightarrow \ldots \twoheadrightarrow \mathcal{M}_1 \twoheadrightarrow \mathcal{M}_0$$

of coherent sheaves on $Z^{\mathbf{M}}$ such that the following assertions hold.

(1) $\mathcal{M}_n$ is a module over $\mathcal{A}/\mathcal{J}^{n+1}$ for every $n \in \mathbb{N}$.

(2) $\mathcal{J}^{n+1}\mathcal{M}_{n+1}$ is the kernel of the epimorphism $\mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n$ for every $n \in \mathbb{N}$.

(3) For each $n \in \mathbb{N}$ there exists a morphism $\mathcal{M}_n \to k[\mathbf{G}] \otimes_k \mathcal{M}_n$ such that for every open affine neighborhood U of $Z^{\mathbf{M}}$ its restriction

$$\mathcal{M}_n(U) \to k[\mathbf{G}] \otimes_k \mathcal{M}_n(U)$$

to sections on U is a coaction of $k[\mathbf{G}]$ on $\mathcal{M}_n(U)$.

(4) $\mathcal{M}_n \to k[\mathbf{G}] \otimes_k \mathcal{M}_n$ is the morphism of $\mathcal{A}$-modules, where $k[\mathbf{G}] \otimes_k \mathcal{M}_n$ carries the structure of an $\mathcal{A}$-module induced by the restriction of scalars along the morphism $\mathcal{A}/\mathcal{J}^{n+1} \to k[\mathbf{G}] \otimes_k \mathcal{A}/\mathcal{J}^{n+1}$ that corresponds to the action of $\mathbf{G}$ on Z_n.

(5) For every $n \in \mathbb{N}$ the epimorphism $\mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n$ preserves the coaction described in **(3)**.

We fix an algebraically closed field K containing k. By Theorem 5.4.4 there exists a closed immersion $\operatorname{Spec} K \times_k \mathbb{G}_m \hookrightarrow T_K$ of group K-schemes that induces zero preserving closed immersion $\mathbb{A}^1_K \hookrightarrow \overline{T}_K$ of monoid K-schemes. By Proposition 5.5.9 we have

$$\operatorname{Spec} K \times_k Z^{\mathbf{M}} = (\operatorname{Spec} K \times_k Z)^{\mathbf{M}_K} = (\operatorname{Spec} K \times_k Z)^{\overline{T}_K} = (\operatorname{Spec} K \times_k Z)^{\mathbb{A}^1_K}$$

This implies that

$$\mathcal{A}_K = K \otimes_k \mathcal{A} = \bigoplus_{i \geq 0} \mathcal{A}_K[i],\ \mathcal{J}_K = K \otimes_k \mathcal{J} = \bigoplus_{i \geq 1} \mathcal{A}_K[i]$$

where gradation is induced by the action of $\mathbb{A}^1_K$. For every $n \in \mathbb{N}$ the action of $\operatorname{Spec} K \times_k \mathbb{G}_m$ on $K \otimes_k \mathcal{M}_n$ induced by the closed immersion $\operatorname{Spec} K \times_k \mathbb{G}_m \hookrightarrow \overline{T}_K \hookrightarrow \mathbf{G}_K$ of group K-schemes gives rise to a gradation

$$K \otimes_k \mathcal{M}_n = \bigoplus_{i \in \mathbb{Z}} (K \otimes_k \mathcal{M}_n)[i]$$

Let $\{V_\lambda\}_{\lambda\in\mathbf{Irr}(T)}$ be a set of irreducible representations of T such that V_λ is contained in $\lambda \in \mathbf{Irr}(T)$. As above for each λ we denote by $\mathcal{M}_n[\lambda]$ the isotypic component of $\mathcal{M}_n$ corresponding to λ.

Lemma 6.6.1.1. *The following assertions hold.*

(1) *There exists $i_0 \in \mathbb{Z}$ such that for every $n \in \mathbb{N}$ we have $(K \otimes_k \mathcal{M}_n)[i] = 0$ for $i < i_0$.*

(2) *For every $i \in \mathbb{Z}$ there exists $n_i \in \mathbb{N}$ such that for all $n \geq n_i$ the surjection $(K \otimes_k \mathcal{M}_{n+1})[i] \twoheadrightarrow (K \otimes_k \mathcal{M}_n)[i]$ is an isomorphism.*

(3) *For every λ in $\mathbf{Irr}(T)$ there exists a finite subset $B_\lambda \subseteq \mathbb{Z}$ such that*

$$K \otimes_k V_\lambda = \bigoplus_{i\in B_\lambda} (K \otimes_k V_\lambda)[i]$$

Define $n_\lambda = \sup_{i\in B_\lambda} n_i \in \mathbb{N}$. Then for all $n \geq n_\lambda$ the surjection $\mathcal{M}_{n+1}[\lambda] \twoheadrightarrow \mathcal{M}_n[\lambda]$ is an isomorphisms.

Proof of the lemma. Fix $n \in \mathbb{N}$ and consider the decomposition $K \otimes_k \mathcal{M}_n = \bigoplus_{i\in\mathbb{Z}} (K \otimes_k \mathcal{M}_n)[i]$. Since $K \otimes_k \mathcal{M}_n$ is a coherent $K \otimes_k \mathcal{O}_{Z^{\mathbf{M}}}$-module and the decomposition consists of modules over $K \otimes_k \mathcal{O}_{Z^{\mathbf{M}}}$, we derive that there are only finitely many $i \in \mathbb{Z}$ such that $(K \otimes_k \mathcal{M}_n)[i] \neq 0$. Hence we may write $K \otimes_k \mathcal{M}_n = \bigoplus_{i\geq i_n} (K \otimes_k \mathcal{M}_n)[i]$ for some $i_n \in \mathbb{Z}$ such that $(K \otimes_k \mathcal{M}_n)[i_n] \neq 0$. Moreover, we know that the kernel of the surjection

$$K \otimes_k \mathcal{M}_{n+1} = \bigoplus_{i\geq i_{n+1}} (K \otimes_k \mathcal{M}_{n+1})[i] \twoheadrightarrow \bigoplus_{i\geq i_n} (K \otimes_k \mathcal{M}_n)[i] = K \otimes_k \mathcal{M}_n$$

is $\mathcal{J}_K^{n+1}(K \otimes_k \mathcal{M}_{n+1})$ and hence is contained in $\bigoplus_{i\geq(i_{n+1}+n+1)} (K \otimes_k \mathcal{M}_{n+1})[i]$. This implies that $(K \otimes_k \mathcal{M}_n)[i] = (K \otimes_k \mathcal{M}_{n+1})[i]$ for $i_{n+1} \leq i \leq i_{n+1} + n$. In particular, we have $(K \otimes_k \mathcal{M}_n)[i_{n+1}] = (K \otimes_k \mathcal{M}_{n+1})[i_{n+1}] \neq 0$ and thus $i_{n+1} \geq i_n$. This shows that $i_n \geq i_0$ for every $n \in \mathbb{N}$ and **(1)** is proved. Now the surjection

$$K \otimes_k \mathcal{M}_{n+1} = \bigoplus_{i\geq i_0} (K \otimes_k \mathcal{M}_{n+1})[i] \twoheadrightarrow \bigoplus_{i\geq i_0} (K \otimes_k \mathcal{M}_n)[i] = K \otimes_k \mathcal{M}_n$$

induces an isomorphism for i-th graded component, where $i_0 \leq i \leq i_0 + n$. Hence for fixed $i \in \mathbb{Z}$ there exists $n_i \in \mathbb{N}$ such that for all $n \geq n_i$ the surjection $(K \otimes_k \mathcal{M}_{n+1})[i] \twoheadrightarrow (K \otimes_k \mathcal{M}_n)[i]$ is an isomorphism. Thus we proved **(2)**.
Fix now λ in $\mathbf{Irr}(T)$ and let V_λ be an irreducible representation in class λ. Since $K \otimes_k V_\lambda$ is a finite dimensional vector space over K, there exists a finite subset $B_\lambda \subseteq \mathbb{Z}$ such that $(K \otimes_k V_\lambda)[i] \neq 0$ if $i \in B_\lambda$. Now define $n_\lambda = \sup_{i\in B_\lambda} n_i$. The surjection $K \otimes_k \mathcal{M}_{n+1} \twoheadrightarrow K \otimes_k \mathcal{M}_n$ induces an isomorphism $(K \otimes_k \mathcal{M}_{n+1})[i] \simeq (K \otimes_k \mathcal{M}_n)[i]$ for every i in B_λ. Thus for $n \geq n_\lambda$ the surjection $\mathcal{M}_{n+1} \twoheadrightarrow \mathcal{M}_n$ induces an isomorphism $\mathcal{M}_{n+1}[\lambda] \simeq \mathcal{M}_n[\lambda]$. This completes the proof of **(3)**. $\square$

Proof of Theorem 6.6.1. Fix a coherent $\mathbf{G}$-sheaf $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ on $\widehat{Z}$ described as in the setup above. For fixed λ in $\mathbf{Irr}(T)$ we define $\mathcal{M}[\lambda] = \mathcal{M}_n[\lambda]$ for any $n \geq n_\lambda$, where $n_\lambda \in \mathbb{N}$ is as in **(3)** of Lemma 6.6.1.1 (in particular, $\mathcal{M}[\lambda]$ does not depend on $n \geq n_\lambda$). Next we define

$$\mathcal{M} = \bigoplus_{\lambda\in\mathbf{Irr}(T)} \mathcal{M}[\lambda]$$

By Proposition 3.9.11 for every $n \in \mathbb{N}$ and $\lambda \in \mathbf{Irr}(T)$ sheaf $\mathcal{M}_n[\lambda]$ admits a structure of a $\mathbf{G}$-sheaf. Therefore, $\mathcal{M}$ is a quasi-coherent $\mathbf{G}$-sheaf of $\mathcal{O}_{Z^{\mathbf{M}}}$-modules. We now show that $\mathcal{M}$ admits a canonical structure of an $\mathcal{A}$-module. For this pick λ_1 and λ_2 in $\mathbf{Irr}(T)$. Consider the irreducible representations V_{λ_1} and V_{λ_1} in classes λ_1 and λ_2, respectively. Suppose that $\eta_1,...,\eta_s$ are finitely many classes in $\mathbf{Irr}(T)$ such that $V_{\lambda_1} \otimes_k V_{\lambda_2}$ can be completely decomposed into irreducible representations contained in classes $\eta_1,...,\eta_s$. According to Fact 6.4.3 the image of the multiplication $\mathcal{A}[\lambda_1] \otimes_{\mathcal{O}_{Z^{\mathbf{M}}}} \mathcal{M}_n[\lambda_2] \to \mathcal{M}_n$ is contained in $\bigoplus_{i=1}^{s} \mathcal{M}_n[\eta_i]$. By **(3)** of Lemma 6.6.1.1 all these multiplications for $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$ can be identified. Now we define

$$\mathcal{A}[\lambda_1] \otimes_{\mathcal{O}_{Z^{\mathbf{M}}}} \mathcal{M}[\lambda_2] \to \bigoplus_{i=1}^{s} \mathcal{M}[\eta_i] \subseteq \mathcal{M}$$

as a morphism induced by the multiplication morphism for any $n \geq \sup\{n_{\lambda_1}, n_{\lambda_2}, n_{\eta_1}, ..., n_{\eta_s}\}$. This gives an $\mathcal{A}$-module structure on $\mathcal{M}$. Denote $K \otimes_k \mathcal{M}$ by $\mathcal{M}_K$. Note that the combination of **(2)** and **(3)** of Lemma 6.6.1.1 show that

$$\mathcal{M}_K[i] = (K \otimes_k \mathcal{M}_n)\,[i]$$

for $n \geq n_i$ and $i \geq i_0$, where $i_0 \in \mathbb{Z}$ and $n_i \in \mathbb{N}$ are as in Lemma 6.6.1.1. By **(1)** of Lemma 6.6.1.1 we have

$$\bigoplus_{\lambda \in \mathbf{Irr}(T)} \mathcal{M}[\lambda]_K = \mathcal{M}_K = \bigoplus_{i \geq i_0} \mathcal{M}_K[i]$$

We show that $\mathcal{M}/\mathcal{J}^{n+1}\mathcal{M} = \mathcal{M}_n$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. By faithfully flat descent it suffices to show that

$$\left(\mathcal{M}_K/\mathcal{J}_K^{n+1}\mathcal{M}_K\right)[i] = (K \otimes_k \mathcal{M}_n)\,[i]$$

for every $i \in \mathbb{Z}$. Let us fix $i \in \mathbb{Z}$. Pick m greater than $\sup_{i_0 \leq j \leq i} n_j$ and n. Then

$$\mathcal{M}_K[j] = (K \otimes_k \mathcal{M}_m)\,[j],\ \left(\mathcal{J}_K^{n+1}\mathcal{M}_K\right)[j] = \left(\mathcal{J}_K^{n+1}\left(K \otimes_k \mathcal{M}_m\right)\right)[j]$$

for $i_0 \leq j \leq i$. Since $\mathcal{M}_m/\mathcal{J}^{n+1}\mathcal{M}_m = \mathcal{M}_n$ as $m \geq n$, we derive that

$$\left(\mathcal{M}_K/\mathcal{J}_K^{n+1}\mathcal{M}_K\right)[i] = \mathcal{M}_K[i]/\left(\mathcal{J}_K^{n+1}\mathcal{M}_K\right)[i] =$$

$$= (K \otimes_k \mathcal{M}_m)\,[i]/\left(\mathcal{J}_K^{n+1}\left(K \otimes_k \mathcal{M}_m\right)\right)[i] = (K \otimes_k \mathcal{M}_n)\,[i]$$

and this completes the proof of our claim. Next we prove that $\mathcal{M}$ is an $\mathcal{A}$-module of finite type. Since this question is local on $Z^{\mathbf{M}}$, we may assume that this scheme is noetherian. Clearly $\mathcal{M}[\lambda]$ is a coherent $\mathcal{O}_{Z^{\mathbf{M}}}$-module for each $\lambda \in \mathbf{Irr}(T)$. Now we may pick $\lambda_1, ..., \lambda_r$ in $\mathbf{Irr}(T)$ such that we have a surjection

$$\bigoplus_{j=1}^{r} \mathcal{M}[\lambda_j] \twoheadrightarrow \mathcal{M}/\mathcal{J}\mathcal{M} = \mathcal{M}_0$$

induced by the canonical surjection $\mathcal{M} \twoheadrightarrow \mathcal{M}/\mathcal{J}\mathcal{M} = \mathcal{M}_0$. Let

$$\mathcal{G} = \bigoplus_{j=1}^{r} \mathcal{M}[\lambda_j]$$

be a $\mathcal{O}_{Z\mathbf{M}}$-submodule of $\mathcal{M}$. Then $\mathcal{G}$ is a coherent $\mathcal{O}_{Z\mathbf{M}}$-module. We derive that

$$\mathcal{M} = \mathcal{G} + \mathcal{J}\mathcal{M}$$

Since $\mathcal{J}_K = \bigoplus_{i\geq 1} \mathcal{A}_K[i]$ and $\mathcal{M}_K = \bigoplus_{i\geq i_0} \mathcal{M}_K[i]$, graded Nakayama lemma proves that $\mathcal{M}_K = \sum_{j\geq 1} \mathcal{J}_K^j \cdot \mathcal{G}_K$. Thus $\mathcal{G}_K$ generates $\mathcal{M}_K$ as an $\mathcal{A}_K$-module. By faithfully flat descent we deduce that $\mathcal{G}$ generates $\mathcal{M}$ as an $\mathcal{A}$-module. Since $\mathcal{G}$ is a coherent $\mathcal{O}_{Z\mathbf{M}}$-module, we derive that $\mathcal{M}$ is an $\mathcal{A}$-module of finite type. All these facts imply that $\mathcal{M}$ corresponds to a coherent $\mathbf{G}$-sheaf on Z such that its image under the comparison functor $\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$ is a coherent $\mathbf{G}$-sheaf on $\widehat{Z}$ with $\mathbf{G}$-structure described by $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$. Hence the comparison functor is essentialy surjective. Note also that

$$\mathcal{M} = \lim_{n\in\mathbb{N}}\mathcal{M}_n$$

in the category of sheaves of $\mathcal{O}_{Z\mathbf{M}}$-modules. Now we are going to prove that $\mathfrak{Coh}_{\mathbf{G}}(Z) \to \mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$ is full and faithful. For this consider a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r, two heads] & \mathcal{M}_{n+1} \arrow[r, two heads] \arrow[d, "f_{n+1}"'] & \mathcal{M}_n \arrow[r, two heads] \arrow[d, "f_n"'] & \ldots \arrow[r, two heads] & \mathcal{M}_1 \arrow[r, two heads] \arrow[d, "f_1"'] & \mathcal{M}_0 \arrow[d, "f_0"'] \\
\ldots \arrow[r, two heads] & \mathcal{N}_{n+1} \arrow[r, two heads] & \mathcal{N}_n \arrow[r, two heads] & \ldots \arrow[r, two heads] & \mathcal{N}_1 \arrow[r, two heads] & \mathcal{N}_0
\end{tikzcd}$$

that represents the morphism in $\mathfrak{Coh}_{\mathbf{G}}(\widehat{Z})$ (Remark 6.2.7). This means that f_n is a morphism of $\mathcal{A}/\mathcal{J}^{n+1}$-modules and preserves the $k[\mathbf{G}]$-coactions for every $n \in \mathbb{N}$. Next suppose that $\mathcal{N}$ is an $\mathcal{A}$-module with $k[\mathbf{G}]$-coaction that corresponds to an object of $\mathfrak{Coh}_{\mathbf{G}}(Z)$ which image under the comparison functor yields $\{\mathcal{N}_n\}_{n\in\mathbb{N}}$. We define $f : \mathcal{M} \to \mathcal{N}$ as follows. We pick $\lambda \in \mathbf{Irr}(T)$ and set $f[\lambda] : \mathcal{M}[\lambda] \to \mathcal{N}[\lambda]$ to be $f_n[\lambda] : \mathcal{M}_n[\lambda] \to \mathcal{N}_n[\lambda]$ for sufficiently large $n \in \mathbb{N}$. By **(3)** of Lemma 6.6.1.1 this definition makes sense and by construction of an $\mathcal{A}$-module structure on $\mathcal{M}$ and $\mathcal{N}$ gives rise to a morphism of $\mathcal{A}$-modules that preserves the $k[\mathbf{G}]$-coactions. Moreover, we have

$$f = \lim_{n\in\mathbb{N}} f_n$$

in the category of sheaves of $\mathcal{O}_{Z\mathbf{M}}$-modules. Thus f is a unique morphism of sheaves of $\mathcal{O}_{Z\mathbf{M}}$-modules such that the square

$$\begin{tikzcd}
\mathcal{M} \arrow[r, two heads] \arrow[d, "f"'] & \mathcal{M}_n \arrow[d, "f_n"] \\
\mathcal{N} \arrow[r, two heads] & \mathcal{N}_n
\end{tikzcd}$$

is commutative for every $n \in \mathbb{N}$. Next denote $K \otimes_k f = f_K$ and fix $i \in \mathbb{Z}$. Then by **(2)** and **(3)** of Lemma 6.6.1.1 we have

$$f_K[i] = (1_K \otimes_k f_n)[i]$$

for sufficiently large $n \in \mathbb{N}$. Fix now $n \in \mathbb{N}$. According to **(1)** of Lemma 6.6.1.1 for $i \in \mathbb{Z}$ we may pick $m \geq n$ such that

$$f_K[j] = (1_K \otimes_k f_m)[j]$$

for all $j \leq i$. Thus

$$f_K[i] \operatorname{mod} \left(\mathcal{J}_K^{n+1}\mathcal{M}_K\right)[i] = (1_K \otimes_k f_m)[i] \operatorname{mod} \left(\mathcal{J}_K^{n+1}(K \otimes_k \mathcal{M}_m)\right)[i] = (1_K \otimes_k f_n)[i]$$

Since $i \in \mathbb{Z}$ is arbitrary, we derive that

$$f_K \operatorname{mod} \mathcal{J}_K^{n+1}\mathcal{M}_K = (1_K \otimes_k f_n)$$

By faithfully flat descent we deduce that $f_n = \left(1_{\mathcal{A}/\mathcal{J}^{n+1}} \otimes_{\mathcal{A}} f\right)$ for every $n \in \mathbb{N}$. Therefore, f is a unique morphism in $\mathfrak{Coh}_{\mathbf{G}}(Z)$ such that its image under the comparison functor is $\{f_n\}_{n \in \mathbb{N}}$. This completes the proof that the comparison functor is full and faithful. We proved that it is essentially surjective above. Thus the comparison functor is an equivalence of categories. According to the definition the comparison functor is monoidal. Hence it is an equivalence of monoidal categories. $\square$

Chapter 7

Białynicki-Birula Decompositions

7.1 Introduction

In this chapter we finally employ previously obtained results to establish Białynicki-Birula decomposition for general schemes locally of finite type over k. In the first four sections we introduce fibered categories and principal $\mathbf{G}$-bundles and then state beautiful theorem of Hall and Rydh ([HR19]) that enables to construct morphisms of stacks from finitely cocontinuous monoidal functors between their categories of coherent sheaves. The main result of these considerations is Theorem 7.5.5 which follows from the fact that the comparison functor $\mathfrak{Coh}_{\mathbf{G}}(\mathcal{Z}) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$ is an equivalence (Theorem 6.6.1) combined with tannakian formalism (Corollary 7.5.4). In eight section we employ results of Section 3.8 to prove the representability of the algebraic Białynicki-Birula functor in affine case and then using techniques of algebraization from the previous chapter we prove that the formal Białynicki-Birula functor is always representable for Kempf monoids. Then Theorem 7.5.5 implies that algebraic Białynicki-Birula decomposition exists for an arbitrary Kempf monoid $\mathbf{M}$ and a scheme X locally of finite type over k with an action of units of $\mathbf{M}$. The ninth section is devoted to smoothness of Białynicki-Birula decomposition. We close this chapter by discussing results of Jelisiejew and the author ([JS19], [JS20]) not covered in this thesis and by giving some application of the generalized Białynicki-Birula decompositions.

7.2 Fibered categories

Let $\mathcal{C}$ be a locally small category. In order to make our notation in this section clear we denote by $h^{\mathcal{C}} : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ the Yoneda embedding for $\mathcal{C}$. In particular, if X is an object of $\mathcal{C}$, then $h^{\mathcal{C}}_X$ denotes the presheaf representable by X.
We fix a functor $p : \mathcal{E} \to \mathcal{B}$. Consider a morphism $\phi : \xi \to \eta$ of $\mathcal{E}$ such that $p(\phi) = f$ and $f : X \to Y$. We depict this situation by the square diagram

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Note that to every such square there corresponds a commutative square

$$\begin{tikzcd} h^{\mathcal{E}}_{\xi} \arrow[r, "h^{\mathcal{E}}_{\phi}"] \arrow[d, "p_{\mathrm{mor}}"'] & h^{\mathcal{E}}_{\eta} \arrow[d, "p_{\mathrm{mor}}"] \\ h^{\mathcal{B}}_{X} \cdot p \arrow[r, "\left(h^{\mathcal{B}}_{f}\right)_{p}"'] & h^{\mathcal{B}}_{Y} \cdot p \end{tikzcd}$$

of presheaves on $\mathcal{E}$, where p_{mor} are maps induced by p on sets of morphisms.

Definition 7.2.1. Consider a square

$$\begin{tikzcd} \xi \arrow[r, "\phi"] \arrow[d, mapsto] & \eta \arrow[d, mapsto] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We call the square *cartesian* and ϕ *a cartesian morphism with respect to* p if the corresponding square of presheaves on $\mathcal{E}$ is cartesian in the category of presheaves.

One can rephrase definition above in terms of presheaves as follows. Morphism $\phi : \xi \to \eta$ is cartesian with respect to p if the square

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{E}}(\zeta, \xi) \arrow[r, "\mathrm{Mor}_{\mathcal{E}}\left(1_{\zeta}, \phi\right)"] \arrow[d, "p_{\mathrm{mor}}"'] & \mathrm{Mor}_{\mathcal{E}}(\zeta, \eta) \arrow[d, "p_{\mathrm{mor}}"] \\ \mathrm{Mor}_{\mathcal{B}}\big(p(\zeta), p(\xi)\big) \arrow[r, "\mathrm{Mor}_{\mathcal{B}}\left(1_{p(\zeta)}, p(\phi)\right)"'] & \mathrm{Mor}_{\mathcal{B}}\big(p(\zeta), p(\eta)\big) \end{tikzcd}$$

of sets is cartesian for every object ζ of $\mathcal{E}$.

Fact 7.2.2. *Let $p : \mathcal{E} \to \mathcal{B}$ be a functor, let $f : X \to Y$ be a morphism of $\mathcal{B}$ and let η be an object of $\mathcal{E}$. Suppose that $\phi_1 : \xi_1 \to \eta, \phi_2 : \xi_2 \to \eta$ are morphisms of $\mathcal{E}$ that are cartesian with respect to p and assume that $p(\phi_1) = p(\phi_2)$. Then there exists a unique morphism $\theta : \xi_1 \to \xi_2$ such that $\phi_1 = \phi_2 \cdot \theta$. Moreover, θ is an isomorphism.*

Proof. There exists a unique natural transformation $\sigma : h^{\mathcal{E}}_{\xi_1} \to h^{\mathcal{E}}_{\xi_2}$ such that $h^{\mathcal{E}}_{\phi_1} = h^{\mathcal{E}}_{\phi_2} \cdot \sigma$. Moreover, σ is a natural isomorphism. Since $h^{\mathcal{E}} : \mathcal{E} \to \widehat{\mathcal{E}}$ is full and faithful, we derive that there exists a unique morphism $\theta : \xi_1 \to \xi_2$ such that $h^{\mathcal{E}}_{\theta} = \sigma$. Then θ satisfies the assertion. $\square$

Definition 7.2.3. Let $p : \mathcal{E} \to \mathcal{B}$ be a functor, let $f : X \to Y$ be a morphism of $\mathcal{B}$ and let η be an object of $\mathcal{E}$ such that $p(\eta) = Y$. A pair (ξ, ϕ) such that ξ is an object of $\mathcal{E}$ and $\phi : \xi \to \eta$ is a morphism of $\mathcal{E}$ is called *a pullback of η along f* if the following conditions are satisfied.

(1) $p(\phi) = f$

(2) ϕ is cartesian morphism of p.

Note that Fact 7.2.2 implies that pullbacks are unique up to a unique isomorphism.

Definition 7.2.4. Let $p : \mathcal{E} \to \mathcal{B}$ be a functor. Then p is *a fibered category* if and only if for every morphism $f : X \to Y$ of $\mathcal{B}$ and every object η of $\mathcal{E}$ such that $p(\eta) = Y$ there exists a pullback of η along f. If $p : \mathcal{E} \to \mathcal{B}$ is a fibered category, then we say that $\mathcal{E}$ *is fibered over $\mathcal{B}$ with respect to p*.

Now we give some examples of fibered categories. The first is prototypical for the notion of a fibered category. It shows that any category $\mathcal{B}$ with fiber products gives rise in a canonical way to a fibered category over $\mathcal{B}$ with cartesian arrows as cartesian squares in $\mathcal{B}$.

Example 7.2.5 (the fibered category of arrows)**.** Let $\mathcal{B}$ be a category. We define the category $\mathrm{Arr}(\mathcal{B})$ of arrows of $\mathcal{B}$ as follows. Objects of $\mathrm{Arr}(\mathcal{B})$ are morphisms $\pi : \tilde{X} \to X$ of $\mathcal{B}$. Now if $\pi : \tilde{X} \to X$ and $\psi : \tilde{Y} \to Y$ are objects of $\mathrm{Arr}(\mathcal{B})$, then a morphism $\pi \to \psi$ is a pair (f, ϕ) such that $f : X \to Y$ and $\phi : \tilde{X} \to \tilde{Y}$ are morphisms in $\mathcal{B}$ making the square

$$\begin{tikzcd} \tilde{X} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \tilde{Y} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutative. There exists a functor $p_{\mathrm{Arr}(\mathcal{B})} : \mathrm{Arr}(\mathcal{B}) \to \mathcal{B}$ given by formula $p_{\mathrm{Arr}(\mathcal{B})}\big((f, \phi)\big) = f$. Suppose now that $f : X \to Y$ and $\psi : \tilde{Y} \to Y$ are morphisms of $\mathcal{B}$ and there exists a commutative square

$$\begin{tikzcd} \tilde{X} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \tilde{Y} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It is a direct consequence of the definition that (f, ϕ) is a cartesian morphisms of $p_{\mathrm{Arr}(\mathcal{B})}$ if and only if the square above is cartesian. Thus $p_{\mathrm{Arr}(\mathcal{B})}$ is a fibered category provided that $\mathcal{B}$ admits fiber products.

Definition 7.2.6. Suppose that $p_1 : \mathcal{E}_1 \to \mathcal{B}$ and $p_2 : \mathcal{E}_2 \to \mathcal{B}$ are fibered categories. Then a functor $F : \mathcal{E}_1 \to \mathcal{E}_2$ is *a morphism of fibered categories* if the following two assertions are satisfied.

(1) $p_1 = F \cdot p_2$ or in other words F is a functor over $\mathcal{B}$.

(2) Image under F of a cartesian morphism of p_1 is a cartesian morphism of p_2.

Next example is closely related to the previous one, but is of more topological flavour.

Example 7.2.7 (the fibered category vector bundles)**.** Recall that **Top** denotes the category of topological spaces. We define a category $\mathbf{VectBund}_{\mathbb{R}}$ of real vector bundles as follows. Objects of $\mathbf{VectBund}_{\mathbb{R}}$ are topological $\mathbb{R}$-vector bundles $\pi : \mathcal{V} \to X$. Now if $\pi : \mathcal{V} \to X$ and $\psi : \mathcal{W} \to Y$ are topological $\mathbb{R}$-vector bundles, then a morphism $\pi \to \psi$ is a pair (f, ϕ) such that $f : X \to Y$ and $\phi : \mathcal{V} \to \mathcal{W}$ are continuous maps making the square

$$\begin{tikzcd} \mathcal{V} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathcal{W} \arrow[d, "\psi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutative and moreover, ϕ induces an $\mathbb{R}$-linear map on fibers i.e. for each point x in X map ϕ induces an $\mathbb{R}$-linear map $\pi^{-1}(x) \to \psi^{-1}(f(x))$. We have the functor $\mathbf{VectBund}_{\mathbb{R}} \to \mathrm{Arr}(\mathbf{Top})$ that forgets about $\mathbb{R}$-vector bundle structure. Since topological vector bundles are stable under continuous change of base, we deduce (according to description of cartesian squares in Example 7.2.5) that the composition of this forgetful functor with $p_{\mathrm{Arr}(\mathbf{Top})} : \mathrm{Arr}(\mathbf{Top}) \to \mathbf{Top}$ is the fibered category. Thus we have a commutative triangle

$$\begin{tikzcd} \mathbf{VectBund}_{\mathbb{R}} \arrow[rr] \arrow[dr] & & \mathrm{Arr}(\mathbf{Top}) \arrow[dl, "p_{\mathrm{Arr}(\mathbf{Top})}"] \\ & \mathbf{Top} & \end{tikzcd}$$

and the functor $\mathbf{VectBund}_{\mathbb{R}} \to \mathrm{Arr}(\mathbf{Top})$ is a morphism of fibered categories.

Definition 7.2.8. Suppose that $p_1 : \mathcal{E}_1 \to \mathcal{B}, p_2 : \mathcal{E}_2 \to \mathcal{B}$ are fibered categories and assume that $F_1, F_2 : \mathcal{E}_1 \to \mathcal{E}_2$ are morphisms of fibered categories. A natural transformation $\tau : F_1 \to F_2$ such that $p_2\tau$ is the identity transformation of p_1 is called *a natural transformation of morphisms of fibered categories*.

7.3 Example: Principal G-Bundles

We devote this whole section to another class of examples of fibered categories. We fix a category with finite limits $\mathcal{B}$ and a group object $\mathbf{G}$ of $\mathcal{B}$. We denote by $\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and $e : \mathbf{1} \to \mathbf{G}$ the multiplication and unit of $\mathbf{G}$, respectively.

Definition 7.3.1. Let $\mathcal{P}$ be an object of $\mathcal{B}$ equipped with an action of $\mathbf{G}$, let T be an object of $\mathcal{B}$ with the trivial action of $\mathbf{G}$ and let $\pi : \mathcal{P} \to T$ be an $\mathbf{G}$-equivariant morphism with respect to these $\mathbf{G}$-actions. We say that a $\mathbf{G}$-equivariant morphism π is *a trivial principal* $\mathbf{G}$*-bundle on* T if there exists a $\mathbf{G}$-equivariant isomorphism $\phi : \mathcal{P} \to \mathbf{G} \times T$ such that $\mathbf{G} \times T$ is equipped with an action of $\mathbf{G}$ given by $\mu \times 1_T$ and the triangle

$$\begin{tikzcd} \mathcal{P} \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & \mathbf{G} \times T \arrow[dl, "\mathrm{pr}_T"] \\ & T & \end{tikzcd}$$

is commutative.

Definition 7.3.2. Let $\mathcal{P}$ be an object of $\mathcal{B}$ equipped with an action of $\mathbf{G}$, let T be an object of $\mathcal{B}$ with trivial action of $\mathbf{G}$ and let $\pi : \mathcal{P} \to T$ be a $\mathbf{G}$-equivariant morphism with respect to these $\mathbf{G}$-actions. Consider a sieve S ([MM94, page 37]) on T. For every arrow $h : \widetilde{T} \to T$ in S we construct a cartesian square

$$\begin{tikzcd} h^*\mathcal{P} \arrow[r] \arrow[d, "\pi_h"'] & \mathcal{P} \arrow[d, "\pi"] \\ \widetilde{T} \arrow[r, "h"'] & T \end{tikzcd}$$

in $\mathcal{B}$. We consider h as a $\mathbf{G}$-equivariant morphism with respect to trivial $\mathbf{G}$-actions on T and $\widetilde{T}$. Then there exists a unique action of $\mathbf{G}$ on $h^*\mathcal{P}$ which makes π_h into a $\mathbf{G}$-equivariant morphism in such a way that the square consists of objects of $\mathcal{B}$ with $\mathbf{G}$-actions and $\mathbf{G}$-equivariant morphisms. Suppose that $\mathbf{G}$-equivariant morphism π_h is a trivial principal $\mathbf{G}$-bundle on $\widetilde{T}$ for every h in S. Then we say that S *trivializes* π.

In the remaining part of this section we fix a Grothendieck topology $\mathcal{J}$ on $\mathcal{B}$ ([MM94, Chapter III, Section 2, Definition 1]), which is by definition a collection of families $\{\mathcal{J}(X)\}_{X \in \mathcal{B}}$ of sieves (called *covering sieves*) that satisfy certain conditions.

Definition 7.3.3. Let $\mathcal{P}$ be an object of $\mathcal{B}$ equipped with an action of $\mathbf{G}$, let T be an object of $\mathcal{B}$ with trivial action of $\mathbf{G}$ and let $\pi : \mathcal{P} \to T$ be a $\mathbf{G}$-equivariant morphism with respect to these $\mathbf{G}$-actions. Suppose that there exists a covering sieve S in $\mathcal{J}(T)$ that trivializes π. Then π is called *a principal* $\mathbf{G}$*-bundle with respect to* $\mathcal{J}$.

Now we define a category $\mathbb{B}\mathbf{G}$ that depends on the site $(\mathcal{B}, \mathcal{J})$. Its objects are principal $\mathbf{G}$-bundles with respect to $\mathcal{J}$ and if $\pi : \mathcal{P} \to T, \psi : Q \to Z$ are principal $\mathbf{G}$-bundles with respect to $\mathcal{J}$, then a morphism $\pi \to \psi$ is a pair (f, ϕ) such that $f : T \to Z$ and $\phi : \mathcal{P} \to Q$ are morphisms in $\mathcal{B}$ such that ϕ is $\mathbf{G}$-equivariant and the square

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\phi"] \arrow[d, "\pi"'] & Q \arrow[d, "\psi"] \\ T \arrow[r, "f"'] & Z \end{tikzcd}$$

is commutative. We have a functor $p_{\mathbb{B}\mathbf{G}} : \mathbb{B}\mathbf{G} \to \mathcal{B}$ given by $p_{\mathbb{B}\mathbf{G}}\big((f, \phi)\big) = f$. Let $\psi : Q \to Z$ be a principal $\mathbf{G}$-bundle with respect to $\mathcal{J}$ and let $f : T \to Z$ be a morphism. Consider the cartesian square

$$\begin{tikzcd}
f^*Q \arrow[r, "\phi"] \arrow[d, "\pi"'] & Q \arrow[d, "\psi"] \\
T \arrow[r, "f"'] & Z
\end{tikzcd}$$

in $\mathcal{B}$. Then there exists a unique action of $\mathbf{G}$ on f^*Q such that the square above consists of $\mathbf{G}$-equivariant morphisms (T, Z are equipped with trivial $\mathbf{G}$-actions). Moreover, with respect to this action $\psi : f^*Q \to T$ becomes a principal $\mathbf{G}$-bundle with respect to $\mathcal{J}$. Indeed, if S is in $\mathcal{J}(Z)$ and S trivializes ψ, then its pullback f^*S trivializes π and is an element of $\mathcal{J}(T)$ (by definition of Grothendieck topology). This shows that $p_{\mathbb{B}\mathbf{G}} : \mathbb{B}\mathbf{G} \to \mathcal{B}$ is a fibered category. Moreover, we have a functor $\mathbb{B}\mathbf{G} \to \mathrm{Arr}(\mathcal{B})$ that forgets about $\mathbf{G}$-actions. Hence there exists a commutative triangle

$$\begin{tikzcd}
\mathbb{B}\mathbf{G} \arrow[rr] \arrow[dr, "p_{\mathbb{B}\mathbf{G}}"'] & & \mathrm{Arr}(\mathcal{B}) \arrow[dl, "p_{\mathrm{Arr}(\mathcal{B})}"] \\
& \mathcal{B} &
\end{tikzcd}$$

According to Example 7.2.5 and description of cartesian morphisms of $p_{\mathbb{B}\mathbf{G}}$ the functor $\mathbb{B}\mathbf{G} \to \mathrm{Arr}(\mathcal{B})$ described above is a morphism of fibered categories.

Definition 7.3.4. $p_{\mathbb{B}\mathbf{G}} : \mathbb{B}\mathbf{G} \to \mathcal{B}$ is called *the fibered category of principal* $\mathbf{G}$*-bundles with respect to* $\mathcal{J}$.

Suppose that X is an object of $\mathcal{B}$ equipped with an action of $\mathbf{G}$. We define a category $[X/\mathbf{G}]$ depending on action of $\mathbf{G}$ on X and the topology $\mathcal{J}$ as follows. Its objects are pairs (π, α) such that π is a principal $\mathbf{G}$-bundle with respect to $\mathcal{J}$ and α is a $\mathbf{G}$-equivariant morphism. We depict such pairs by diagrams

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\alpha"] \arrow[d, "\pi"'] & X \\
T &
\end{tikzcd}$$

Suppose that $(\pi : \mathcal{P} \to T, \alpha : \mathcal{P} \to X)$ and $(\psi : Q \to Z, \beta : Q \to X)$ are two such objects. Then a morphism $(\pi, \alpha) \to (\psi, \beta)$ is a morphism $(f, \phi) : \pi \to \psi$ in $\mathbb{B}\mathbf{G}$ such that $\alpha = \beta \cdot \phi$. We have a functor $pr_{X,\mathbb{B}\mathbf{G}} : [X/\mathbf{G}] \to \mathbb{B}\mathbf{G}$ which sends (π, α) to π. We denote by $p_{[X/\mathbf{G}]} : [X/\mathbf{G}] \to \mathcal{B}$ the composition of this functor $pr_{X,\mathbb{B}\mathbf{G}} : [X/\mathbf{G}] \to \mathbb{B}\mathbf{G}$ with $p_{\mathbb{B}\mathbf{G}} : \mathbb{B}\mathbf{G} \to \mathcal{B}$. By description of cartesian morphisms of $p_{\mathbb{B}\mathbf{G}}$ we deduce that $p_{[X/\mathbf{G}]}$ is a fibered category. We have a commutative triangle

$$\begin{tikzcd}
{[X/\mathbf{G}]} \arrow[rr, "pr_{X,\mathbb{B}\mathbf{G}}"] \arrow[dr, "p_{[X/\mathbf{G}]}"'] & & \mathbb{B}\mathbf{G} \arrow[dl, "p_{\mathbb{B}\mathbf{G}}"] \\
& \mathcal{B} &
\end{tikzcd}$$

and the functor $pr_{X,\mathbb{B}\mathbf{G}} : [X/\mathbf{G}] \to \mathbb{B}\mathbf{G}$ described above is a morphism of fibered categories.

Note that if $\mathbf{1}$ is a terminal object of $\mathcal{B}$ equipped with trivial action of $\mathbf{G}$, then we have a canonical isomorphism $[\mathbf{1}/\mathbf{G}] \simeq \mathbb{B}\mathbf{G}$ of categories over $\mathcal{B}$.

Definition 7.3.5. $p_{[X/\mathbf{G}]} : [X/\mathbf{G}] \to \mathcal{B}$ is called *the quotient fibered category of X with respect to $\mathcal{J}$.*

7.4 2-fiber products of groupoids and quotient fibered categories

In this section we introduce the notion of a 2-fiber product. This notion plays an important role in subsequent section. Consider a diagram of categories and their functors

$$\begin{tikzcd} & \mathcal{C}_1 \arrow[d, "g_1"] \\ \mathcal{C}_2 \arrow[r, "g_2"'] & \mathcal{C} \end{tikzcd}$$

Then we define a category $\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2$ as follows. Objects of $\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2$ are triples (x, y, u) such that x is an object of $\mathcal{C}_1$, y is an object of $\mathcal{C}_2$ and $u : g_1(x) \to g_2(y)$ is an isomorphism in $\mathcal{C}$. Now if (x_1, y_1, u_1) and (x_2, y_2, u_2) are objects of $\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2$, then a morphism $(x_1, y_1, u_1) \to (x_2, y_2, u_2)$ in $\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2$ is a pair $(v : x_1 \to x_2, w : y_1 \to y_2)$ of morphisms in the ordinary (strict) categorical product $\mathcal{C}_1 \times \mathcal{C}_2$ such that the square

$$\begin{tikzcd} g_1(x_1) \arrow[r, "u_1"] \arrow[d, "g_1(v)"'] & g_2(y_1) \arrow[d, "g_2(w)"] \\ g_1(x_2) \arrow[r, "u_2"'] & g_2(y_2) \end{tikzcd}$$

is commutative in $\mathcal{C}$. The composition of morphisms is the same as in $\mathcal{C}_1 \times \mathcal{C}_2$. Moreover, there are functors

$$\pi_1 : \mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2 \to \mathcal{C}_1, \quad \pi_2 : \mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2 \to \mathcal{C}_2$$

given by

$$\pi_1((x, y, u)) = x, \quad \pi_1((v, w)) = v, \quad \pi_2((x, y, u)) = y, \quad \pi_2((v, w)) = w$$

There also exists a canonical natural isomorphism $\sigma : g_1 \cdot \pi_1 \to g_2 \cdot \pi_2$ given by $\sigma_{(x,y,u)} = u$. This makes the square

$$\begin{tikzcd} \mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & \mathcal{C}_1 \arrow[d, "g_1"] \\ \mathcal{C}_2 \arrow[r, "g_2"'] & \mathcal{C} \end{tikzcd}$$

commutative up to σ.

Definition 7.4.1. The category $\mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2$ together with data consisting of π_1, π_2, σ is called *the 2-fiber product of the diagram*

$$\begin{tikzcd} & \mathcal{C}_1 \arrow[d, "g_1"] \\ \mathcal{C}_2 \arrow[r, "g_2"'] & \mathcal{C} \end{tikzcd}$$

Definition 7.4.2. Consider the square with the 2-fiber product

$$\begin{tikzcd} \mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & \mathcal{C}_1 \arrow[d, "g_2"] \\ \mathcal{C}_2 \arrow[r, "g_1"'] & \mathcal{C} \end{tikzcd}$$

If $\Phi : \mathcal{C} \to \mathcal{C}_1 \times_{\mathcal{C}} \mathcal{C}_2$ is an equivalence of categories, then it induces the square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\pi_1 \cdot \Phi"] \arrow[d, "\pi_2 \cdot \Phi"'] & \mathcal{C}_1 \arrow[d, "g_2"] \\ \mathcal{C}_2 \arrow[r, "g_1"'] & \mathcal{C} \end{tikzcd}$$

of categories and functors, which is commutative up to a natural isomorphism. We call such squares *a* 2*-cartesian squares*.

Remark 7.4.3. It is crucial for the reasons which will be clear in the next section to note that 2-fiber products admits certain 2-universal property similar to the (one dimensional) universal property of the usual fiber product. For details see [Ols16, 3.4.9 and especially discussion at the beginning of page 82], where this universality is discussed for groupoids, but this restriction is not serious. There is no need to further discuss it here. It suffices to note that this 2-universal property is preserved by equivalences of categories and hence it holds for all 2-cartesian squares.

Remark 7.4.4. Note that in Section 6.2 we introduced another class of 2-categorical limits. They can be described in terms of 2-fibered products. Indeed, note that the category $\mathcal{C}(T)$ from the beginning of Section 6.2 can be described as the 2-fiber product of the diagram

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "1_{\mathcal{C}}"] \\ \mathcal{C} \arrow[r, "T"'] & \mathcal{C} \end{tikzcd}$$

In particular, this implies that 2-limits of telescopes have not only the one dimensional universal property described there, but they also admit certain 2-categorical universal property (Remark 7.4.3). This property is also preserved by equivalences of categories.

Now we come back to discussion of quotient fibered categories. We fix a category with finite limits $\mathcal{B}$ and a group object $\mathbf{G}$ of $\mathcal{B}$. We denote by $\mu : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and $e : \mathbf{1} \to \mathbf{G}$ the multiplication and unit of $\mathbf{G}$, respectively. In the remaining part of this section we fix a Grothendieck topology $\mathcal{J}$ on $\mathcal{B}$. Let X, Y be objects of $\mathcal{B}$ equipped with actions of $\mathbf{G}$. We denote by $\mathrm{Mor}\left([X/\mathbf{G}],[Y/\mathbf{G}]\right)$ the following groupoid. Its objects are morphisms $[X/\mathbf{G}] \to [Y/\mathbf{G}]$ of fibered categories over $\mathcal{B}$. Its arrows are isomorphisms of morphisms of fibered categories (Definition 7.2.8). Let $\star$ be the category with one object and the identity morphism. Consider now the 2-fiber product

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbb{B}\mathbf{G}}\left([X/\mathbf{G}],[Y/\mathbf{G}]\right) \arrow[r] \arrow[d] & \mathrm{Mor}([X/\mathbf{G}],[Y/\mathbf{G}]) \arrow[d, "\mathrm{Mor}(1_{[X/\mathbf{G}]},\, pr_{Y,\mathbb{B}\mathbf{G}})"] \\
\star \arrow[r, "\star \mapsto pr_{X,\mathbb{B}\mathbf{G}}"'] & \mathrm{Mor}([X/\mathbf{G}],\mathbb{B}\mathbf{G})
\end{tikzcd}$$

of groupoids, where

$$pr_{X,\mathbb{B}\mathbf{G}} : [X/\mathbf{G}] \to \mathbb{B}\mathbf{G}, pr_{X,\mathbb{B}\mathbf{G}} : [Y/\mathbf{G}] \to \mathbb{B}\mathbf{G}$$

are canonical functors. This defines the groupoid $\mathrm{Mor}_{\mathbb{B}\mathbf{G}}\left([X/\mathbf{G}],[Y/\mathbf{G}]\right)$.

Remark 7.4.5. The groupoid $\mathrm{Mor}_{\mathbb{B}\mathbf{G}}\left([X/\mathbf{G}],[Y/\mathbf{G}]\right)$ can be described explicitly as follows. Its objects are pairs (F,τ), where $F : [X/\mathbf{G}] \to [Y/\mathbf{G}]$ is a morphism of fibered categories and $\tau : pr_{X,\mathbf{G}} \to pr_{Y,\mathbf{G}} \cdot F$ is a natural isomorphism defined over $\mathcal{B}$ (see Definition 7.2.8). An arrow $(F_1,\tau_1) \to (F_2,\tau_2)$ in $\mathrm{Mor}_{\mathbb{B}\mathbf{G}}\left([X/\mathbf{G}],[Y/\mathbf{G}]\right)$ is a natural isomorphism $\sigma : F_1 \to F_2$ over $\mathcal{B}$ such that $pr_{Y,\mathbf{G}}\sigma = \tau_2 \cdot \tau_1^{-1}$.

After this formal introduction we show that under some mild assumptions on Grothendieck topology $\mathcal{J}$ fibered category $p_{[X/\mathbf{G}]} : [X/\mathbf{G}] \to \mathcal{B}$ encapsulates all essential information concerning the action of $\mathbf{G}$ on X. In the theorem below we denote the set of $\mathbf{G}$-equivariant morphisms $X \to Y$ by $\mathrm{Mor}_{\mathbf{G}}(X,Y)$ and consider it as a discrete groupoid.

Theorem 7.4.6. *Let $\mathcal{J}$ be a Grothendieck topology on $\mathcal{B}$ and assume that representable presheaves on $\mathcal{B}$ are separated with respect to $\mathcal{J}$. Let X, Y be objects of $\mathcal{B}$ equipped with $\mathbf{G}$-actions. Then there exists an equivalence*

$$\mathrm{Mor}_{\mathbf{G}}(X,Y) \simeq \mathrm{Mor}_{\mathbb{B}\mathbf{G}}\big([X/\mathbf{G}],[Y/\mathbf{G}]\big)$$

of groupoids that sends a $\mathbf{G}$-equivariant morphism f to a pair $\left([f/\mathbf{G}], 1_{pr_{X,\mathbf{G}}}\right)$, where the functor

$$[f/\mathbf{G}] : [X/\mathbf{G}] \to [Y/\mathbf{G}]$$

is given by

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "d"] \arrow[d, "\pi"'] & X & & \mathcal{P} \arrow[r, "f \cdot d"] \arrow[d, "\pi"'] & Y \\
T & & \arrow[r, mapsto, "F"] & T &
\end{tikzcd}$$

Proof. Consider first a morphism $F : [X/\mathbf{G}] \to [Y/\mathbf{G}]$ of fibered categories such that $pr_{Y,\mathbf{G}} \cdot F$ is equal to $pr_{X,\mathbf{G}}$. In other words F is strictly over $\mathbb{B}\mathbf{G}$. We show that for such F there exists a unique $\mathbf{G}$-equivariant $f : X \to Y$ such that $F = [f/\mathbf{G}]$. Denote by a_X, a_Y actions of $\mathbf{G}$ on X, Y, respectively. Moreover, for every object T of $\mathcal{B}$ we denote by $q_T : T \to \mathbf{1}$ the unique morphism into a terminal object of $\mathcal{B}$. We first describe certain object of $[X/\mathbf{G}]$. Observe that $(\mathbf{G}\times X, \mu\times 1_X)$ is an object of $\mathcal{B}$ equipped with an action of $\mathbf{G}$. Next the projection $\mathrm{pr}_X : \mathbf{G}\times X \to X$ can be considered as a $\mathbf{G}$-equivariant morphism from this $\mathbf{G}$-object to X with the trivial action of $\mathbf{G}$. Since the square

$$\begin{tikzcd}
\mathbf{G}\times\mathbf{G}\times X \arrow[r, "1_{\mathbf{G}}\times a_X"] \arrow[d, "\mu\times 1_X"'] & \mathbf{G}\times X \arrow[d, "a_X"] \\
\mathbf{G}\times X \arrow[r, "a_X"'] & X
\end{tikzcd}$$

is commutative, we derive that a_X is a $\mathbf{G}$-equivariant morphism $(\mathbf{G}\times X, \mu\times 1_X) \to (X, a_X)$. This gives (pr_X, a_X) the structure of an object of $[X/\mathbf{G}]$. The functor F sends $\big(\mathrm{pr}_X, a_X\big)$ to some object of $[Y/\mathbf{G}]$. This object is necessarily of the form (pr_X, α) for some $\mathbf{G}$-equivariant morphism $\alpha : (\mathbf{G}\times X, \mu\times 1_X) \to (Y, a_Y)$. Indeed, this follows from the fact that F is strictly over $\mathbb{B}\mathbf{G}$. Now if $F = [f/\mathbf{G}]$ for some $\mathbf{G}$-equivariant morphism f as it is described in the statement, then $\alpha = f\cdot a$ and hence $f = \alpha\cdot\langle e\cdot q_X, 1_X\rangle$ (recall that $e : \mathbf{1}\to\mathbf{G}$ is the unit morphism). This proves that f is unique. Our goal is to show that it exists. That is our goal is to show that a morphism $f = \alpha\cdot\langle e\cdot q_X, 1_X\rangle$ is $\mathbf{G}$-equivariant and determines F as it is described in the statement. First we fix some object T of $\mathcal{B}$ and the projection $\mathrm{pr}_T : \mathbf{G}\times T\to T$ considered as a trivial principal $\mathbf{G}$-bundle. Let (pr_T, c) be an object of $[X/\mathbf{G}]$. Then c is a $\mathbf{G}$-equivariant morphism $c : (\mathbf{G}\times T, \mu\times 1_T) \to (X, a_X)$. Functor F sends (pr_T, c) to some object (pr_T, γ). We claim that $\gamma = f\cdot c$. Let $\mathrm{pr}_{23} : \mathbf{G}\times\mathbf{G}\times T \to \mathbf{G}\times T$ be the projection on the last two factors. There are diagrams

$$\begin{tikzcd}
\mathbf{G}\times\mathbf{G}\times T \arrow[r, "\mu\times 1_T"] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbf{G}\times T \arrow[r, "c"] \arrow[d, "\mathrm{pr}_T"] & X \\
\mathbf{G}\times T \arrow[r, "\mathrm{pr}_T"'] & T &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{G}\times\mathbf{G}\times T \arrow[r, "1_{\mathbf{G}}\times c"] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbf{G}\times X \arrow[r, "a_X"] \arrow[d, "\mathrm{pr}_X"] & X \\
\mathbf{G}\times T \arrow[r, "c"'] & X &
\end{tikzcd}$$

representing morphisms

$$(\mathrm{pr}_T, \mu\times 1_T) : \big(\mathrm{pr}_{23}, c\cdot(\mu\times 1_T)\big) \to \big(\mathrm{pr}_T, c\big),\ (c, 1_{\mathbf{G}}\times c) : \big(\mathrm{pr}_{23}, a_X\cdot(1_{\mathbf{G}}\times c)\big) \to \big(\mathrm{pr}_X, a_X\big)$$

in $[X/\mathbf{G}]$. Moreover, c is $\mathbf{G}$-equivariant $(\mathbf{G}\times T, \mu\times 1_T) \to (X, a_X)$ and hence we derive that $c\cdot(\mu\times 1_T) = a_X\cdot(1_{\mathbf{G}}\times c)$. Thus the morphisms in $[X/\mathbf{G}]$ described above have common domain. Since F is strictly over $\mathbb{B}\mathbf{G}$, we derive that their images under F are

$$\begin{tikzcd}
\mathbf{G}\times\mathbf{G}\times T \arrow[r, "\mu\times 1_T"] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbf{G}\times T \arrow[r, "\gamma"] \arrow[d, "\mathrm{pr}_T"] & Y \\
\mathbf{G}\times T \arrow[r, "\mathrm{pr}_T"'] & T &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{G}\times\mathbf{G}\times T \arrow[r, "1_{\mathbf{G}}\times c"] \arrow[d, "\mathrm{pr}_{23}"'] & \mathbf{G}\times X \arrow[r, "\alpha"] \arrow[d, "\mathrm{pr}_X"] & Y \\
\mathbf{G}\times T \arrow[r, "c"'] & X &
\end{tikzcd}$$

This implies that $\gamma \cdot (\mu \times 1_T) = \alpha \cdot (1_{\mathbf{G}} \times c)$. We deduce that

$$\gamma = \gamma \cdot (\mu \times 1_T) \cdot \langle e \cdot q_{\mathbf{G} \times T}, 1_{\mathbf{G} \times T} \rangle = \alpha \cdot (1_{\mathbf{G}} \times c) \cdot \langle e \cdot q_{\mathbf{G} \times T}, 1_{\mathbf{G} \times T} \rangle = \alpha \cdot \langle e \cdot q_X, 1_X \rangle \cdot c = f \cdot c$$

and the claim is proved. We apply this to α to derive that $\alpha = f \cdot a_X$. Next recall that $\alpha \cdot (\mu \times 1_X) = a_Y \cdot (1_{\mathbf{G}} \times \alpha)$ because α is a $\mathbf{G}$-equivariant morphism $(\mathbf{G} \times X, \mu \times 1_X) \to (Y, a_Y)$. Thus

$$a_Y \cdot (1_{\mathbf{G}} \times f) = a_Y \cdot (1_{\mathbf{G}} \times \alpha) \cdot (1_{\mathbf{G}} \times \langle e \cdot q_X, 1_X \rangle) = \alpha \cdot (\mu \times 1_X) \cdot (1_{\mathbf{G}} \times \langle e \cdot q_X, 1_X \rangle) = \alpha$$

Hence $f \cdot a_X = \alpha = a_Y \cdot (1_{\mathbf{G}} \times f)$. Thus f is $\mathbf{G}$-equivariant and F is given as in the statement on the subcategory of $[X/\mathbf{G}]$ consisting of trivial principal $\mathbf{G}$-bundles. Now consider any principal $\mathbf{G}$-bundle $\pi : \mathcal{P} \to T$ with respect to $\mathcal{J}$ and let $d : \mathcal{P} \to X$ be a $\mathbf{G}$-equivariant morphism to (X, a_X). We know that F sends (π, d) to some object of $[Y/\mathbf{G}]$ of the form (π, δ). It suffices to prove that $\delta = f \cdot d$. For this consider a sieve S in $\mathcal{J}(T)$ such that S trivializes π. Pick $h : \widetilde{T} \to T$ in S and a cartesian square

$$\begin{tikzcd}
h^*\mathcal{P} \arrow[r, "h'"] \arrow[d, "\pi_h"'] & \mathcal{P} \arrow[d, "\pi"] \\
\widetilde{T} \arrow[r, "h"'] & T
\end{tikzcd}$$

Then $(\pi_h, d \cdot h')$ is an object of $[X/\mathbf{G}]$. Since F is strictly over $\mathbb{B}\mathbf{G}$, we derive that $F(\pi_h, d \cdot h') = (\pi_h, \delta \cdot h')$. By definition π_h is a trivial $\mathbf{G}$-bundle. Thus (from what we proved above) we have

$$\delta \cdot h' = f \cdot d \cdot h'$$

This holds for pullback h' of every h in S along π. These pullbacks $\{h'\}_{h \in S}$ generate the sieve $\pi^*\mathcal{S}$ on $\mathcal{P}$ and hence the formula

$$\delta \cdot m = f \cdot d \cdot m$$

holds for every morphism m in $\pi^*\mathcal{S}$. Moreover, $\pi^*\mathcal{S}$ is a covering sieve on the site $(\mathcal{B}, \mathcal{J})$. According to the assumption on $\mathcal{J}$ we infer that $h^{\mathcal{B}}_{\mathcal{P}} = \mathrm{Mor}_{\mathcal{B}}(-, \mathcal{P}) : \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ is a separated presheaf with respect to $\mathcal{J}$. Thus the formula

$$\delta \cdot m = f \cdot d \cdot m$$

which holds for every m in $\pi^*\mathcal{S}$ implies that $\delta = f \cdot d$. Therefore, if $F : [X/\mathbf{G}] \to [Y/\mathbf{G}]$ is a morphism of fibered categories such that $pr_{Y,\mathbf{G}} \cdot F = pr_{X,\mathbf{G}}$, then $F = [f/\mathbf{G}]$ for a unique $f \in \mathrm{Mor}_{\mathbf{G}}(X, Y)$.

Now suppose that (F, τ) is an arbitrary object of $\mathrm{Mor}_{\mathbb{B}\mathbf{G}}([X/\mathbf{G}], [Y/\mathbf{G}])$. Consider a principal $\mathbf{G}$-bundle $\pi : \mathcal{P} \to T$ with respect to $\mathcal{J}$ and let $d : \mathcal{P} \to X$ be a $\mathbf{G}$-equivariant morphism. Then F sends (π, d) to some $(\psi : Q \to T, \delta : Q \to Y)$ in $[Y/\mathbf{G}]$. Next we have a commutative triangle

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "\tau_{(\pi,d)}"] \arrow[dr, "\pi"'] & & Q \arrow[dl, "\psi"] \\
& T &
\end{tikzcd}$$

and $\tau_{(\pi,d)}$ is a $\mathbf{G}$-equivariant isomorphism. We construct a morphism $F_{\text{strict}} : [X/\mathbf{G}] \to [Y/\mathbf{G}]$ of fibered categories such that $pr_{Y,\mathbf{G}} \cdot F_{\text{strict}}$ is equal to $pr_{X,\mathbf{G}}$. We define it by formula

$$\begin{tikzcd} \mathcal{P} \arrow[r, "d"] \arrow[d, "\pi"'] & X \\ T & \end{tikzcd} \quad \xmapsto{F_{\text{strict}}} \quad \begin{tikzcd} \mathcal{P} \arrow[r, "\delta \cdot \tau_{(\pi,d)}"] \arrow[d, "\pi"'] & Y \\ T & \end{tikzcd}$$

This is well defined according to the fact that F is a morphism of fibered categories and τ is an isomorphism $pr_{X,\mathbf{G}} \to pr_{Y,\mathbf{G}} \cdot F$ defined over $\mathcal{B}$. It follows from the definition of F_{strict} that it is defined strictly over $\mathbb{B}\mathbf{G}$. Moreover, τ induces an isomorphism of objects $(F_{\text{strict}}, 1_{pr_{X,\mathbf{G}}})$ and (F, τ) of $\mathrm{Mor}_{\mathbb{B}\mathbf{G}}([X/\mathbf{G}], [Y/\mathbf{G}])$. Combining this with the first part of the our argument we deduce that every object (F, τ) of $\mathrm{Mor}_{\mathbb{B}\mathbf{G}}([X/\mathbf{G}], [Y/\mathbf{G}])$ is isomorphic to an object of the form $([f/\mathbf{G}], 1_{pr_{X,\mathbf{G}}})$ for some $\mathbf{G}$-equivariant morphism $f : X \to Y$. Hence the functor

$$\mathrm{Mor}_{\mathbf{G}}(X, Y) \to \mathrm{Mor}_{\mathbb{B}\mathbf{G}}\big([X/\mathbf{G}], [Y/\mathbf{G}]\big)$$

in the statement is essentially surjective.
It remains to prove that if $f_1, f_2 : X \to Y$ are $\mathbf{G}$-equivariant and $([f_1/\mathbf{G}], 1_{pr_{X,\mathbf{G}}}) \simeq ([f_1/\mathbf{G}], 1_{pr_{X,\mathbf{G}}})$ as objects of $\mathrm{Mor}_{\mathbb{B}\mathbf{G}}\big([X/\mathbf{G}], [Y/\mathbf{G}]\big)$, then $f_1 = f_2$. For this observe that an isomorphism $\sigma : [f_1/\mathbf{G}] \to [f_2/\mathbf{G}]$ of morphisms of fibered categories such that $pr_{Y,\mathbf{G}} \cdot \sigma = 1_{pr_{X,\mathbf{G}}}$ is the identity. Hence $[f_1/\mathbf{G}] = [f_2/\mathbf{G}]$ and thus $f_1 = f_2$. □

7.5 Tannakian formalism for quotient stacks

In this section we discuss an application of the main result of [HR19]. For this we need to briefly discuss *algebraic stacks*, although for our purposes there is no need to use seriously any technicalities of this language. We refer the interested reader to the excellent exposition [Ols16] of this subject. We note the following facts.

(1) *An algebraic stack* is a category fibered over $\mathbf{Sch}_k$ satisfying certain extra conditions described in [Ols16, Definition 4.6.1] and [Ols16, Definition 8.1.4]. By [Ols16, Definition 8.2.1, Example 8.2.3] there are well defined notions of *locally noetherian, noetherian* and *excellent algebraic stacks*.

(2) *A morphism of algebraic stacks* is a morphism of fibered categories over $\mathbf{Sch}_k$. If $\mathcal{X}$ and $\mathcal{Y}$ are algebraic stacks, then we denote by $\mathrm{Mor}_k(\mathcal{X}, \mathcal{Y})$ the corresponding category of morphisms.

(3) For every locally noetherian algebraic stack $\mathcal{X}$ there exists an abelian monoidal category $\mathfrak{Coh}(\mathcal{X})$ of coherent sheaves on $\mathcal{X}$ ([Ols16, Definition 9.1.14]). If $\mathcal{X}$ and $\mathcal{Y}$ are locally noetherian algebraic stacks, then we denote by $\mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}(\mathcal{X}), \mathfrak{Coh}(\mathcal{Y}))$ the groupoid of right exact, monoidal functors $\mathfrak{Coh}(\mathcal{X}) \to \mathfrak{Coh}(\mathcal{Y})$ with monoidal natural isomorphisms as morphisms.

(4) If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of locally noetherian algebraic stacks, then f induces the functor $f^* : \mathfrak{Coh}(\mathcal{Y}) \to \mathfrak{Coh}(\mathcal{X})$ such that $f^* \in \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}(\mathcal{Y}), \mathfrak{Coh}(\mathcal{X}))$.

(5) Let $\mathbf{G}$ be a smooth algebraic group over k and let X be a k-scheme equipped with an action of $\mathbf{G}$. We consider $\mathbf{Sch}_k$ as a Grothendieck site with respect to *étale topology*

([Ols16, Example 2.1.13]). The quotient fibered category $[X/\mathbf{G}]$ with respect to this topology is an algebraic stack by [Ols16, Example 8.1.12].

(6) In **(5)** if the k-scheme X is locally noetherian (noetherian, excellent), then $[X/\mathbf{G}]$ is a locally noetherian (noetherian, excellent) by [Ols16, Definition 8.2.1, Example 8.2.3] and [Ols16, Example 8.1.12].

(7) In **(5)** if the k-scheme X is locally noetherian, then there exists an equivalence of monoidal categories $\mathfrak{Coh}([X/\mathbf{G}]) \simeq \mathfrak{Coh}_{\mathbf{G}}(X)$ ([Ols16, Exercise 9.H]) induced by the pullback along the canonical morphism $X \to [X/\mathbf{G}]$. Moreover, this equivalence is functorial with respect to $\mathbf{G}$-equivariant morphisms. That is if Y is another locally noetherian k-scheme with action of $\mathbf{G}$ and $f : X \to Y$ is a $\mathbf{G}$-equivariant morphism, then f induces a morphism $[f/\mathbf{G}] : [X/\mathbf{G}] \to [Y/\mathbf{G}]$ as in Theorem 7.4.6 and the square

$$\begin{tikzcd} \mathfrak{Coh}([Y/\mathbf{G}]) \arrow[r, "{[f/\mathbf{G}]^*}"] \arrow[d, "\simeq"'] & \mathfrak{Coh}([X/\mathbf{G}]) \arrow[d, "\simeq"] \\ \mathfrak{Coh}_{\mathbf{G}}(Y) \arrow[r, "f^*"'] & \mathfrak{Coh}_{\mathbf{G}}(X) \end{tikzcd}$$

of categories and functors is commutative.

(8) According to [HR19, paragraph after Theorem 1.1 on page 2] an algebraic stack $\mathcal{X}$ has *affine stabilizers* if the diagonal of $\mathcal{X}$ has affine fibers. If $\mathbf{G}$ is smooth and affine over k, then $[X/\mathbf{G}]$ has affine stabilizers according to discussion in [Ols16, Example 8.1.12].

Let us state the main result of [HR19].

Theorem 7.5.1 ([HR19, Theorem 1.1])**.** *Let $\mathcal{X}$ be a noetherian algebraic stack with affine stabilizers. For every locally excellent algebraic stack $\mathcal{T}$ the functor*

$$\mathrm{Mor}\,(\mathcal{X},\mathcal{T}) \xrightarrow{f \mapsto f^*} \mathrm{Hom}_{r,\otimes,\simeq}\,(\mathfrak{Coh}(\mathcal{T}),\mathfrak{Coh}(\mathcal{X}))$$

is an equivalence of categories.

Keeping our previous remarks in mind we deduce the following result.

Corollary 7.5.2. *Let $\mathbf{G}$ be a smooth and affine algebraic group over k and let X, Z be k-schemes equipped with an action of $\mathbf{G}$. Suppose that Z is noetherian and X is locally of finite type over k. Then*

$$\mathrm{Mor}\,([Z/\mathbf{G}],[X/\mathbf{G}]) \xrightarrow{f \mapsto f^*} \mathrm{Hom}_{r,\otimes,\simeq}\Big(\mathfrak{Coh}\big([X/\mathbf{G}]\big),\mathfrak{Coh}\big([Z/\mathbf{G}]\big)\Big)$$

is an equivalence of categories.

Proof. Note that $[Z/\mathbf{G}]$ is a noetherian algebraic stack according to **(5)** and **(6)**. It has affine stabilizers according to **(8)**. Similarly by **(5)** the stack $[X/\mathbf{G}]$ is an algebraic stack. Moreover, it is locally excellent according to the fact that X is locally excellent (it is locally of finite type over k and k is a field) and **(6)**. Then by Theorem 7.5.1 we derive that the functor in the statement is an equivalence of categories. $\square$

Remark 7.5.3. Let $\operatorname{Spec} k$ be a point equipped with the trivial action of a smooth and affine group $\mathbf{G}$. Then **(7)** together with Example 3.12.9 imply that $\mathfrak{Coh}([\operatorname{Spec} k/\mathbf{G}])$ can be identified with the category $\mathbf{Repf}(\mathbf{G})$ of finite dimensional representations of $\mathbf{G}$.

Corollary 7.5.4. *Let $\mathbf{G}$ be a geometrically integral, affine algebraic group over k and let X, Z be k-schemes equipped with an action of $\mathbf{G}$. Suppose that Z is noetherian and X is locally of finite type over k. We denote by $p_X^* : \mathbf{Repf}(\mathbf{G}) \to \mathfrak{Coh}_{\mathbf{G}}(X)$ and $p_Z^* : \mathbf{Repf}(\mathbf{G}) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$ the functors induced by $\mathbf{G}$-equivariant morphisms $p_X : X \to \operatorname{Spec} k$ and $p_Z : Z \to \operatorname{Spec} k$, respectively (see Remark 7.5.3). Then the square*

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbf{G}}(Z,X) \arrow[r, "f \mapsto f^*"] \arrow[d] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) \arrow[d, "\mathrm{Hom}_{r,\otimes,\simeq}(p_X^*, 1_{\mathfrak{Coh}_{\mathbf{G}}(Z)})"] \\
\star \arrow[r, "\star \mapsto p_Z^*"'] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathbf{Repf}(\mathbf{G}), \mathfrak{Coh}_{\mathbf{G}}(Z))
\end{tikzcd}$$

of groupoids and their functors, which is commutative up to canonical natural isomorphism given by $f^* \cdot p_X^* \simeq p_Z^*$ for every $\mathbf{G}$-equivariant morphism $f : Z \to X$, is a 2-cartesian square.

Proof. $\mathbf{G}$ is smooth as it is geometrically integral algebraic group. Since in étale topology every representable presheaf is a sheaf, Theorem 7.4.6 shows that the square

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbf{G}}(Z,X) \arrow[r, "f \mapsto {[f/\mathbf{G}]}"] \arrow[d] & \mathrm{Mor}([Z/\mathbf{G}], [X/\mathbf{G}]) \arrow[d, "\mathrm{Mor}(1_{[Z/\mathbf{G}]}, pr_{X,\mathbb{B}\mathbf{G}})"] \\
\star \arrow[r, "\star \mapsto pr_{Z,\mathbb{B}\mathbf{G}}"'] & \mathrm{Mor}([Z/\mathbf{G}], \mathbb{B}\mathbf{G})
\end{tikzcd}$$

is a 2-cartesian square. Next Corollary 7.5.2 and Remark 7.4.3 imply that the square

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbf{G}}(Z,X) \arrow[r, "f \mapsto {[f/\mathbf{G}]^*}"] \arrow[d] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}([X/\mathbf{G}]), \mathfrak{Coh}([Z/\mathbf{G}])) \arrow[d, "\mathrm{Hom}_{r,\otimes,\simeq}({[p_X/\mathbf{G}]^*}, 1_{\mathfrak{Coh}([Z/\mathbf{G}])})"] \\
\star \arrow[r, "\star \mapsto {[p_Z/\mathbf{G}]^*}"'] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}([\operatorname{Spec} k/\mathbf{G}]), \mathfrak{Coh}([Z/\mathbf{G}]))
\end{tikzcd}$$

is 2-cartesian. Next **(7)** combined with Remarks 7.5.3 and 7.4.3 show that

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbf{G}}(Z,X) \arrow[r, "f \mapsto f^*"] \arrow[d] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) \arrow[d, "\mathrm{Hom}_{r,\otimes,\simeq}(p_X^*, 1_{\mathfrak{Coh}_{\mathbf{G}}(Z)})"] \\
\star \arrow[r, "\star \mapsto p_Z^*"'] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathbf{Repf}(\mathbf{G}), \mathfrak{Coh}_{\mathbf{G}}(Z))
\end{tikzcd}$$

is 2-cartesian. □

Finally we are ready to state a consequence of Tannakian formalism and Theorem 6.6.1 that is essential to our proof of existence of Białynicki-Birula decomposition.

Theorem 7.5.5. *Let $\mathbf{M}$ be a Kempf monoid with the group $\mathbf{G}$ of units. Consider a noetherian, locally linear $\mathbf{M}$-scheme Z and let X be a scheme locally of finite type over k equipped with an action of $\mathbf{G}$. Denote by $\{Z_n\}_{n\in\mathbb{N}}$ the formal $\mathbf{M}$-scheme $\widehat{Z}$ (Example 6.3.2). Then the canonical map of sets*

$$\mathrm{Mor}_{\mathbf{G}}(Z,X) \to \lim_{n\in\mathbb{N}} \mathrm{Mor}_{\mathbf{G}}(Z_n, X)$$

is bijective.

Proof. Theorem 6.6.1, Definition 6.3.6 and Remark 7.4.4 imply that $\mathfrak{Coh}_{\mathbf{G}}(Z)$ is a 2-limit of the telescope

$$\dots \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_{n+1}) \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_n) \longrightarrow \dots \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_2) \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_1) \longrightarrow \mathfrak{Coh}_{\mathbf{G}}(Z_0)$$

and this 2-limit has a 2-categorical universal property, which implies that

$$\mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) = \lim_{n\in\mathbb{N}} \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z_n))$$

and

$$\mathrm{Hom}_{r,\otimes,\simeq}(\mathbf{Repf}(\mathbf{G}), \mathfrak{Coh}_{\mathbf{G}}(Z) = \lim_{n\in\mathbb{N}} \mathrm{Hom}_{r,\otimes,\simeq}(\mathbf{Repf}(\mathbf{G}), \mathfrak{Coh}_{\mathbf{G}}(Z_n))$$

where lim stands for 2-limit. By Corollary 7.5.4 we have a 2-cartesian square

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbf{G}}(Z,X) \arrow[r, "f \mapsto f^*"] \arrow[d] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) \arrow[d, "\mathrm{Hom}_{r,\otimes,\simeq}(p_X^*, 1_{\mathfrak{Coh}_{\mathbf{G}}(Z)})"] \\
\star \arrow[r, "\star \mapsto p_Z^*"'] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathbf{Repf}(\mathbf{G}), \mathfrak{Coh}_{\mathbf{G}}(Z))
\end{tikzcd}$$

and for each $n \in \mathbb{N}$ the square

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbf{G}}\left(Z_n, X\right) \arrow[r, "f \mapsto f^*"] \arrow[d] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z_n)) \arrow[d, "\mathrm{Hom}_{r,\otimes,\simeq}(p_X^*, 1_{\mathfrak{Coh}_{\mathbf{G}}(Z_n)})"] \\
\star \arrow[r, "\star \mapsto p_{Z_n}^*"'] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathbf{Repf}(\mathbf{G}), \mathfrak{Coh}_{\mathbf{G}}(Z_n))
\end{tikzcd}$$

is also 2-cartesian. Moreover, there are morphisms of these squares induced by closed $\mathbf{G}$-equivariant immersions $Z_n \hookrightarrow Z$ for $n \in \mathbb{N}$. Since 2-limits commute with each other as their one-dimensional cousins, we derive that 2-limit of the sequence of 2-cartesian squares

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbf{G}}\left(Z_n, X\right) \arrow[r, "f \mapsto f^*"] \arrow[d] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z_n)) \arrow[d, "\mathrm{Hom}_{r,\otimes,\simeq}(p_X^*, 1_{\mathfrak{Coh}_{\mathbf{G}}(Z_n)})"] \\
\star \arrow[r, "\star \mapsto p_{Z_n}^*"'] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathbf{Repf}(\mathbf{G}), \mathfrak{Coh}_{\mathbf{G}}(Z_n))
\end{tikzcd}$$

is the 2-cartesian square

$$\begin{tikzcd}
\mathrm{Mor}_{\mathbf{G}}\left(Z, X\right) \arrow[r, "f \mapsto f^*"] \arrow[d] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathfrak{Coh}_{\mathbf{G}}(X), \mathfrak{Coh}_{\mathbf{G}}(Z)) \arrow[d, "\mathrm{Hom}_{r,\otimes,\simeq}(p_X^*, 1_{\mathfrak{Coh}_{\mathbf{G}}(Z)})"] \\
\star \arrow[r, "\star \mapsto p_{Z}^*"'] & \mathrm{Hom}_{r,\otimes,\simeq}(\mathbf{Repf}(\mathbf{G}), \mathfrak{Coh}_{\mathbf{G}}(Z))
\end{tikzcd}$$

Thus the canonical functor

$$\mathrm{Mor}_{\mathbf{G}}(Z, X) \to \lim_{n \in \mathbb{N}} \mathrm{Mor}_{\mathbf{G}}(Z_n, X)$$

is an equivalence of categories. Since the categories on both sides are discrete (are sets), we derive that this functor is actually a bijective map. $\square$

The result above is also a consequence of [AHR20, Proposition 5.19] combined with Theorem 6.6.1. Actually the proof presented above is essentially the original proof of [AHR20, Proposition 5.19]. We present it for self-containment of this work.

7.6 Białynicki-Birula functors

In this section we fix a group k-scheme $\mathbf{G}$. Let $\mathbf{M}$ be a monoid k-scheme with zero $\mathbf{o}$ such that $\mathbf{G}$ is its group of units.

Definition 7.6.1. Let X be a k-scheme equipped with an action of $\mathbf{G}$. For every k-scheme Y (considered as $\mathbf{G}$-scheme with the trivial $\mathbf{G}$-action) we define

$$\mathcal{D}_X(Y) = \left\{\gamma : \mathbf{M} \times_k Y \to X \,\middle|\, \gamma \text{ is } \mathbf{G}\text{-equivariant}\right\}$$

This gives gives rise to a subpresheaf $\mathcal{D}_X$ of $\mathrm{Mor}_k(\mathbf{M}\times_k(-),X) : \mathbf{Sch}_k^{\mathrm{op}} \to \mathbf{Set}$. We call it *the Białynicki-Birula functor of X*.

Fact 7.6.2. *Let X be a scheme equipped with an action of* $\mathbf{G}$. *Then $\mathcal{D}_X$ is a Zariski sheaf.*

Proof. $\mathrm{Mor}_k(\mathbf{M}\times_k(-),X)$ is a Zariski sheaf and if we glue $\mathbf{G}$-equivariant morphisms, then the result is $\mathbf{G}$-equivariant. This shows that $\mathcal{D}_X$ is a Zariski subsheaf of $\mathrm{Mor}_k(\mathbf{M}\times_k(-),X)$. □

Remark 7.6.3. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then there are canonical morphisms of functors

$$\begin{tikzcd} \mathcal{D}_X \arrow[r, "i_X"] \arrow[d, "r_X"] & X \\ X^{\mathbf{G}} \arrow[u, bend left, "s_X"] & \end{tikzcd}$$

which we define now. For this let $\gamma \in \mathcal{D}_X(Y)$ for some k-scheme Y and we denote by $q : Y \to \operatorname{Spec} k$ the unique morphism. We define

$$i_X(\gamma) = \gamma_{|\{e\}\times_k Y} = \gamma\cdot\langle e\cdot q, 1_Y\rangle$$

and

$$r_X(\gamma) = \gamma_{|\{\mathbf{o}\}\times_k Y} = \gamma\cdot\langle \mathbf{o}\cdot q, 1_Y\rangle$$

where $e : \operatorname{Spec} k \to \mathbf{M}$ is the unit of $\mathbf{M}$ and $\mathbf{o} : \operatorname{Spec} k \to \mathbf{M}$ is the zero. Next if $f : Y \to X$ is a morphism in $X^{\mathbf{G}}(Y)$, then we define

$$s_X(f) = f\cdot pr_Y$$

where $pr_Y : \mathbf{M}\times_k Y \to Y$ is the projection. Finally note that $r_X\cdot s_X = 1_{X^{\mathbf{G}}}$.

Remark 7.6.4. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then $\mathbf{M}$ acts on $\mathcal{D}_X$. Indeed, fix k-scheme Y, $\gamma\in\mathcal{D}_X(Y)$ and $m\in\mathbf{M}(Y)$. Then we define the product

$$m\gamma = \gamma\cdot\langle m, 1_Y\rangle$$

and this determines an action of $\mathbf{M}$ on $\mathcal{D}_X$. Moreover, with respect to this action i_X is $\mathbf{G}$-equivariant and r_X, s_X are $\mathbf{M}$-equivariant ($X^{\mathbf{G}}$ is equipped with the trivial action of $\mathbf{M}$).

Remark 7.6.5. Let X, Y be k-schemes equipped with actions of $\mathbf{G}$ and let $f : X \to Y$ be a $\mathbf{G}$-equivariant morphism, then there exists a morphism of functors $\mathcal{D}_f : \mathcal{D}_X \to \mathcal{D}_Y$ given by

$$\mathcal{D}_f(\gamma) = f\cdot\gamma$$

for every element γ of the functor $\mathcal{D}_X$. Moreover, $\mathcal{D}_f$ preserves the action of $\mathbf{M}$ described in Remark 7.6.4 above.

Let X be a k-scheme equipped with an action of $\mathbf{G}$. It is useful to discuss subsheaves of $\mathcal{D}_X$ defined by closed $\mathbf{G}$-stable subschemes of X.

Theorem 7.6.6. *Let X be a k-scheme equipped with an action of the affine group k-scheme $\mathbf{G}$. Suppose that $\mathbf{G}$ is open and schematically dense in an affine monoid k-scheme $\mathbf{M}$. If $j : Z \hookrightarrow X$ is a closed $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd} \mathcal{D}_Z \arrow[r, "\mathcal{D}_j"] \arrow[d, "i_Z"'] & \mathcal{D}_X \arrow[d, "i_X"] \\ Z \arrow[r, "j"'] & X \end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

Proof. The fact that the square is commutative follows by examination of definitions in Remarks 7.6.3 and 7.6.5. Pick a k-scheme Y, $f : Y \to Z$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j \cdot f = i_X(\gamma)$. This is depicted in the diagram

$$\begin{tikzcd} & \gamma \arrow[d, mapsto, "i_X"] \\ f \arrow[r, mapsto, "j"'] & j \cdot f = \gamma_{|\{e\} \times_k Y} \end{tikzcd}$$

Our goal is to show that there exists a unique $\mathbf{G}$-equivariant morphism $\eta : \mathbf{M} \times_k Y \to Z$ such that $\mathcal{D}_j(\eta) = \gamma$ and $i_Z(\eta) = f$. This is depicted by the diagram

$$\begin{tikzcd} \eta \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \gamma = j \cdot \eta \\ f = \eta_{|\{e\} \times_k Y} & \end{tikzcd}$$

It suffices to prove that γ factors through j. First note that the assumption $\gamma_{|\{e\} \times_k Y} = j \cdot f$ implies that

$$\gamma_{|\mathbf{G} \times_k Y} = j \cdot a_Z \cdot (1_{\mathbf{G}} \times_k f)$$

where $a_Z : \mathbf{G} \times_k Z \to Z$ is the action. This implies that $\gamma_{|\mathbf{G} \times_k Y}$ factors through j. Consider scheme-theoretic preimage $\gamma^{-1}(Z)$. Then $\gamma^{-1}(Z)$ is a closed $\mathbf{G}$-stable (as an inverse image of a $\mathbf{G}$-stable closed subscheme under the $\mathbf{G}$-equivariant morphism) subscheme of $\mathbf{M} \times_k Y$, which contains $\mathbf{G} \times_k Y$. Since $\mathbf{G}$ is open, schematically dense in $\mathbf{M}$ and k is a field, we derive that $\mathbf{G} \times_k Y$ is open and schematically dense in $\mathbf{M} \times_k Y$. Thus $\gamma^{-1}(Z) = \mathbf{M} \times_k Y$ and hence γ factors through j. □

In order to prove any interesting results in the spirit of Theorem 7.6.6 which concerns open $\mathbf{G}$-stable subschemes, we need to assume that $\mathbf{M}$ is a Kempf monoid.

Theorem 7.6.7. *Let X be a k-scheme equipped with an action of the group $\mathbf{G}$ of units of a Kempf monoid $\mathbf{M}$. If $j : U \hookrightarrow X$ is an open $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd} \mathcal{D}_U \arrow[r, "\mathcal{D}_j"] \arrow[d, "r_U"'] & \mathcal{D}_X \arrow[d, "r_X"] \\ U^{\mathbf{G}} \arrow[r, "j^{\mathbf{G}}"'] & X^{\mathbf{G}} \end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

Proof. The fact that the square is commutative follows by examination of definitions in Remarks 7.6.3 and 7.6.5. Pick a k-scheme Y, $f \in U^{\mathbf{G}}(Y)$ and $\gamma \in \mathcal{D}_X(Y)$ such that $j^{\mathbf{G}}(f) = r_X(\gamma)$. This is depicted in the diagram

$$\begin{tikzcd} & \gamma \arrow[d, mapsto, "r_X"] \\ f \arrow[r, mapsto, "j^{\mathbf{G}}"'] & j \cdot f = \gamma_{|\{\mathbf{o}\}\times_k Y} \end{tikzcd}$$

Our goal is to show that there exists a unique $\mathbf{G}$-equivariant morphism $\eta : \mathbf{M} \times_k Y \to U$ such that $\mathcal{D}_j(\eta) = \gamma$ and $r_U(\eta) = f$. This is depicted by the diagram

$$\begin{tikzcd} \eta \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \gamma = j \cdot \eta \\ f = \eta_{|\{\mathbf{o}\}\times_k Y} & \end{tikzcd}$$

For this it suffices to prove that γ factors through j. Consider $W = \gamma^{-1}(U)$. Note that W is an open $\mathbf{G}$-stable (as an inverse image of a $\mathbf{G}$-stable open subscheme under the $\mathbf{G}$-equivariant morphism) subscheme of $\mathbf{M} \times_k Y$, which contains $\{\mathbf{o}\} \times_k Y$. Theorem 5.5.8 asserts that for every geometric point $\overline{y}$ of Y we have $W_{\overline{y}} = \mathbf{M}_{k(\overline{y})}$, where $W_{\overline{y}}$ is the fiber over $\overline{y}$ of the projection $\mathbf{M} \times_k Y \to Y$ restricted to W. Since W is an open subscheme of $\mathbf{M} \times_k Y$, this implies that $W = \mathbf{M} \times_k Y$ and hence γ factors through j. $\square$

As we shall see below both theorems are extremely useful properties of Białynicki-Birula functors.

7.7 Formal Białynicki-Birula functors

We introduce a formal version of the Białynicki-Birula functor, which enables us to apply formal geometry. We fix a group k-scheme $\mathbf{G}$. Let $\mathbf{M}$ be a monoid k-scheme with zero $\mathbf{o}$ such that $\mathbf{G}$ is its group of units.

Definition 7.7.1. For every $n \in \mathbb{N}$ let $\mathbf{M}_n \hookrightarrow \mathbf{M}$ be an n-th infinitesimal neighborhood of $\mathbf{o}$ in $\mathbf{M}$. Let X be a k-scheme equipped with an action of $\mathbf{G}$. For every k-scheme Y (considered as a $\mathbf{G}$-scheme with the trivial $\mathbf{G}$-action) we define

$$\widehat{\mathcal{D}}_X(Y) = \Big\{\{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \,\Big|\, \forall_{n\in\mathbb{N}}\, \gamma_n \text{ is } \mathbf{G}\text{-equivariant and } \gamma_{n+1|\mathbf{M}_n\times_k Y} = \gamma_n\Big\}$$

This gives gives rise to a functor $\widehat{\mathcal{D}}_X$. We call it *the formal Białynicki-Birula functor of X*.

Remark 7.7.2. Let X, Y be k-schemes equipped with actions of $\mathbf{G}$ and let $f : X \to Y$ be a $\mathbf{G}$-equivariant morphism, then there exists a morphism of functors $\widehat{\mathcal{D}}_f : \widehat{\mathcal{D}}_X \to \widehat{\mathcal{D}}_Y$ given by

$$\widehat{\mathcal{D}}_f\big(\{\gamma_n\}_{n\in\mathbb{N}}\big) = \{f \cdot \gamma_n\}_{n\in\mathbb{N}}$$

for every element γ of the functor $\widehat{\mathcal{D}}_X$.

Remark 7.7.3. Let X be a k-scheme equipped with an action of $\mathbf{G}$. Then there exists a canonical morphism of functors $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ given by

$$\gamma \mapsto \{\gamma_{|\mathbf{M}_n\times_k Y}\}_{n\in\mathbb{N}}$$

for every $\gamma \in \mathcal{D}_X(Y)$ and every k-scheme Y.

Remark 7.7.4. Let X be a k-scheme equipped with an action of $\mathbf{G}$. We define a morphism $\widehat{r}_X : \widehat{\mathcal{D}}_X \to X^{\mathbf{G}}$ by formula

$$\widehat{\mathcal{D}}_X(Y) \ni \{\gamma_n\}_{n\in\mathbb{N}} \mapsto \gamma_0 \in X^{\mathbf{G}}(Y)$$

for every k-scheme Y. This morphism fits into a commutative triangle

$$\begin{tikzcd} \mathcal{D}_X \arrow[rr] \arrow[dr, "r_X"'] & & \widehat{\mathcal{D}}_X \arrow[dl, "\widehat{r}_X"] \\ & X^{\mathbf{G}} & \end{tikzcd}$$

where the horizontal morphism is described in Remark 7.7.3.

The next result is analogous to Theorem 7.6.7, although its proof is much simpler.

Proposition 7.7.5. *Let X be a k-scheme equipped with an action of the group $\mathbf{G}$. If $j : U \hookrightarrow X$ is an open $\mathbf{G}$-stable subscheme of X, then the square*

$$\begin{tikzcd} \widehat{\mathcal{D}}_U \arrow[r, "\mathcal{D}_j"] \arrow[d, "\widehat{r}_U"'] & \widehat{\mathcal{D}}_X \arrow[d, "\widehat{r}_X"] \\ U^{\mathbf{G}} \arrow[r, "j^{\mathbf{G}}"'] & X^{\mathbf{G}} \end{tikzcd}$$

is cartesian in the category of presheaves on $\mathbf{Sch}_k$.

Proof. The fact that the square is commutative follows by examination of definitions in Remark 7.7.2. Pick a k-scheme Y, $f \in U^{\mathbf{G}}(Y)$ and $\{\gamma_n\}_{n\in\mathbb{N}} \in \widehat{\mathcal{D}}_X(Y)$ such that $j^{\mathbf{G}}(f) = \widehat{r}_X\big(\{\gamma_n\}_{n\in\mathbb{N}}\big)$. This is depicted in the diagram

$$\begin{tikzcd} & \{\gamma_n\}_{n\in\mathbb{N}} \arrow[d, mapsto, "\widehat{r}_X"] \\ f \arrow[r, mapsto, "j^{\mathbf{G}}"'] & j\cdot f = \gamma_0 \end{tikzcd}$$

Our goal is to show that there exists a unique family of $\mathbf{G}$-equivariant morphism $\eta_n : \mathbf{M}_n \times_k Y \to U$ for $n \in \mathbb{N}$ such that $\widehat{\mathcal{D}}_j\big(\{\eta_n\}_{n\in\mathbb{N}}\big) = \{\gamma_n\}_{n\in\mathbb{N}}$ and $\widehat{r}_U\big(\{\eta_n\}_{n\in\mathbb{N}}\big) = f$. This is depicted by the diagram

$$\begin{tikzcd} \{\eta_n\}_{n\in\mathbb{N}} \arrow[r, mapsto, "\mathcal{D}_j"] \arrow[d, mapsto, "r_U"'] & \{\gamma_n\}_{n\in\mathbb{N}} = \{j\cdot\eta_n\}_{n\in\mathbb{N}} \\ f = \eta_0 & \end{tikzcd}$$

For this it suffices to prove that γ_n factors through j for every $n \in \mathbb{N}$. Note that all maps $\{\gamma_n\}_{n\in\mathbb{N}}$ are equal set-theoretically and $\gamma_0 = j\cdot f$ factors through j. Thus γ_n factors through j for every $n \in \mathbb{N}$. $\square$

Theorem 7.7.6. *Let* $\mathbf{G}$ *be a group* k*-scheme and* $\mathbf{M}$ *be a Kempf monoid having* $\mathbf{G}$ *as a group of units. Suppose that* X *is a* k*-scheme equipped with an action of* $\mathbf{G}$*. Then the canonical morphism* $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ *is a monomorphism of presheaves.*

For the proof it is useful to make the following observation (essentially the same observation was made in the proof of Theorem 7.6.6).

Lemma 7.7.6.1. *Let* X *be a* k*-scheme equipped with an action of a monoid* k*-scheme* $\mathbf{M}$*. Suppose that* $j : Z \hookrightarrow X$ *is a closed* $\mathbf{G}$*-equivariant immersion, where* $\mathbf{G}$ *is a group of units of* $\mathbf{M}$*. If* $\mathbf{G}$ *is schematically dense in* $\mathbf{M}$*, then the action of* $\mathbf{G}$ *on* Z *extends to the action of* $\mathbf{M}$ *in such a way that* j *becomes* $\mathbf{M}$*-equivariant.*

Proof of the lemma. Let $a : \mathbf{M} \times_k X \to X$ be the action of $\mathbf{M}$ on X. Since j is $\mathbf{G}$-equivariant, we derive that $\mathbf{G} \times_k Z \subseteq a^{-1}(Z)$. Moreover, $\mathbf{G} \times_k Z$ is open and schematically dense in $\mathbf{M} \times_k Z$. Hence $\mathbf{M} \times_k Z \subseteq a^{-1}(Z)$ and thus $a_{|\mathbf{M}\times_k Z}$ factors through $j : Z \hookrightarrow X$. $\square$

Proof of Theorem 7.7.6. Let Y be a k-scheme and let $\gamma, \eta : \mathbf{M} \times_k Y \to X$ be $\mathbf{G}$-equivariant morphisms. Suppose that $\gamma_{|\mathbf{M}_n\times_k Y} = \eta_{|\mathbf{M}_n\times_k Y}$ for every $n \in \mathbb{N}$. Consider the kernel (equalizer) $j : E \hookrightarrow \mathbf{M} \times_k Y$ of the pair (γ, η). Then E admits an action of $\mathbf{G}$ such that j is $\mathbf{G}$-equivariant locally closed immersion and $\mathbf{M}_n \times_k Y \subseteq E$ for every $n \in \mathbb{N}$. Fix a point y in Y. Let $\mathbf{M}_y$ and E_y be fibers of the projection $\mathrm{pr} : \mathbf{M} \times_k Y \to Y$ and $\mathrm{pr}\cdot j$, respectively. Then $E_y \subseteq \mathbf{M}_y$ is a locally closed $\mathbf{G}_y$-equivariant subscheme, where $\mathbf{G}_y = \mathbf{G} \times_k \operatorname{Spec} k(y)$. Since $\mathbf{M}_y = \mathbf{M} \times_k \operatorname{Spec} k(y)$ is a Kempf monoid over $k(y)$ with group of units $\mathbf{G}_y$ and moreover, E_y contains all infinitesimal neighborhoods of the zero in $\mathbf{M}_y$, we deduce by Theorem 5.5.8 that $E_y = \mathbf{M}_y$. This implies that a locally closed immersion $j : E \hookrightarrow \mathbf{M} \times_k Y$ is bijective. Hence it is a closed immersion. Now Lemma 7.7.6.1 implies that E is a locally linear $\mathbf{M}$-scheme and j is $\mathbf{M}$-equivariant. Note that j induces an isomorphism $\widehat{E} \simeq \widehat{\mathbf{M} \times_k Y}$ of formal $\mathbf{M}$-schemes. Hence according to Corollary 6.5.6 we infer that j is an isomorphism. This proves that $\gamma = \eta$. Therefore, the map

$$\mathcal{D}_X(Y) \to \widehat{\mathcal{D}}_X(Y)$$

is injective. As Y is arbitrary we infer that the canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ of Remark 7.7.3 is a monomorphism of presheaves. □

7.8 Representability of Białynicki-Birula functor for Kempf monoids

In this section we prove that Białynicki-Birula functors are representable under mild assumptions (Theorem A). We start with proving representability for affine $\mathbf{G}$-schemes.

Theorem 7.8.1. *Let* $\mathbf{M}$ *be an affine monoid k-scheme with open and schematically dense group of units* $\mathbf{G}$. *Suppose that X is an affine k-scheme equipped with an ation of* $\mathbf{G}$. *Then $\mathcal{D}_X$ is representable and i_X is a closed immersion of k-schemes.*

Proof. Since X is an affine k-scheme, the action of $\mathbf{G}$ on X corresponds to the coaction of $k[\mathbf{G}]$ by $c : \Gamma(X, \mathcal{O}_X) \to k[\mathbf{G}] \otimes_k \Gamma(X, \mathcal{O}_X)$. Note that c is a morphism of k-algebras. By Theorem 3.8.6 there exists a universal morphism $q : \Gamma(X, \mathcal{O}_X) \twoheadrightarrow W$ of $\mathbf{G}$-representations into an $\mathbf{M}$-representation. Let $I \subseteq \Gamma(X, \mathcal{O}_X)$ be the ideal generated by $\ker(q)$. Fix f in I. Then

$$f = \sum_{i=1}^{n} g_i \cdot f_i$$

where $g_i \in k[\mathbf{G}]$ and $f_i \in \ker(q)$ for $1 \leq i \leq n$. Then

$$c(f) = c\left(\sum_{i=1}^{n} g_i \cdot f_i\right) = \sum_{i=1}^{n} c(g_i) \cdot c(f_i) \subseteq \Big(k[\mathbf{G}] \otimes_k \Gamma(X, \mathcal{O}_X)\Big) \cdot \Big(k[\mathbf{G}] \otimes_k \ker(q)\Big) \subseteq k[\mathbf{G}] \otimes_k I$$

Thus $c(I) \subseteq k[\mathbf{G}] \otimes_k I$ and hence I is a $\mathbf{G}$-representation. Consider

$$X^+ = V(I) = \operatorname{Spec} \Gamma(X, \mathcal{O}_X)/I \hookrightarrow X$$

Since $\Gamma(X, \mathcal{O}_X)/I$ is the quotient $\mathbf{G}$-representation of W, we deduce by Theorem 3.8.4 that $\Gamma(X, \mathcal{O}_X)/I$ is an $\mathbf{M}$-representation. Hence X^+ is a k-scheme equipped with an action of $\mathbf{M}$ and $X^+ \hookrightarrow X$ is $\mathbf{G}$-equivariant. Suppose now that Y is an affine k-scheme. Then $\mathbf{M} \times_k Y$ is an $\mathbf{M}$-scheme with respect to the left-hand side action of $\mathbf{M}$ and hence $\Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M} \times_k Y})$ is an $\mathbf{M}$-representation. Now Theorem 3.8.6 implies that if $\gamma : \mathbf{M} \times_k Y \to X$ is a $\mathbf{G}$-equivariant morphism, then a morphism $\gamma^{\#} : \Gamma(X, \mathcal{O}_X) \to \Gamma(\mathbf{M} \times_k Y, \mathcal{O}_{\mathbf{M}} \times_k Y)$ of k-algebras and $\mathbf{G}$-representations factors through $q : \Gamma(X, \mathcal{O}_X) \twoheadrightarrow W$ and thus by construction of I we have

$$\begin{tikzcd}
\Gamma(X,\mathcal{O}_X) \arrow[r, two heads] \arrow[rr, bend left, "\gamma^{\#}"] & \Gamma(X,\mathcal{O}_X)/I \arrow[r, "f"] & \Gamma(\mathbf{M}\times_k Y, \mathcal{O}_{\mathbf{M}}\times_k Y)
\end{tikzcd}$$

for some morphism f of k-algebras and $\mathbf{G}$-representations. Since both $\Gamma(X,\mathcal{O}_X)/I$ and $\Gamma(\mathbf{M}\times_k Y, \mathcal{O}_{\mathbf{M}}\times_k Y)$ are $\mathbf{M}$-representations and by Theorem 3.8.4 the subcategory $\mathbf{Rep}(\mathbf{M}) \subseteq \mathbf{Rep}(\mathbf{G})$ is full, we derive that f is a morphism of $\mathbf{M}$-representations. Thus f corresponds to a unique $\mathbf{M}$-equivariant morphism $\eta : \mathbf{M}\times_k Y \to X^+$ such that the diagram

$$\begin{tikzcd}
 & X^+ \arrow[d, hook] \\
\mathbf{M}\times_k Y \arrow[ur, bend left, "\eta"] \arrow[r, "\gamma"'] & X
\end{tikzcd}$$

is commutative. Now this result can be extended to an arbitrary k-scheme Y, since $\mathrm{Mor}_k(\mathbf{M}\times_k (-), X^+)$ is a Zariski sheaf and a morphism that is $\mathbf{M}$-equivariant locally on the domain is $\mathbf{M}$-equivariant. Thus for every k-scheme Y we have a bijection

$$\mathcal{D}_X(Y) \ni \gamma \mapsto \eta \in \left\{\mathbf{M}\text{-equivariant morphisms } \mathbf{M}\times_k Y \to X^+\right\}$$

Since we also have a bijection

$$\left\{\mathbf{M}\text{-equivariant morphisms } \mathbf{M}\times_k Y \to X^+\right\} \ni \eta \mapsto \eta_{|\{e\}\times_k X^+} \in \mathrm{Mor}_k(Y, X^+)$$

and both this bijections are natural, we derive that $\mathcal{D}_X$ is represented by X^+ and moreover, $i_X : \mathcal{D}_X \to X$ is a closed immersion $X^+ \hookrightarrow X$. □

Corollary 7.8.2. *Let $\mathbf{G}$ be a group k-scheme and let $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a k-scheme equipped with an action of $\mathbf{G}$ such that there exists a family $\mathcal{U}$ of open affine $\mathbf{G}$-stable open subschemes of X such that functors $\{U^{\mathbf{G}}\}_{U\in\mathcal{U}}$ form an open cover of $X^{\mathbf{G}}$. Then $\mathcal{D}_X$ is representable.*

Proof. Note that $\mathbf{G}$ is affine group k-scheme as a unit group of an affine monoid $\mathbf{M}$ (Proposition 3.2.6). Moreover, $\mathbf{M}$ is a Kempf monoid and hence $\mathbf{G}$ is open and schematically dense in $\mathbf{M}$ by Corollary 5.2.5. By Theorem 7.8.1 each $\mathcal{D}_U$ is representable by a k-scheme. On the other hand by Theorem 7.6.7 for each $U \in \mathcal{U}$ we have a cartesian square

$$\begin{tikzcd}
\mathcal{D}_U \arrow[r] \arrow[d, "r_U"'] & \mathcal{D}_X \arrow[d, "r_X"] \\
U^{\mathbf{G}} \arrow[r] & X^{\mathbf{G}}
\end{tikzcd}$$

of functors. This implies that $\{\mathcal{D}_U \hookrightarrow \mathcal{D}_X\}_{U\in\mathcal{U}}$ is an open cover of $\mathcal{D}_X$ as a pullback of an open cover $\{U^{\mathbf{G}} \hookrightarrow X^{\mathbf{G}}\}_{U\in\mathcal{U}}$. Hence Fact 7.6.2 and Theorem 2.4.9 imply that $\mathcal{D}_X$ is representable. □

Corollary 7.8.3. *Let* $\mathbf{G}$ *be group* k*-scheme and let* $\mathbf{M}$ *be a Kempf monoid having* $\mathbf{G}$ *as a group of units. Suppose that* X *is a locally linear* $\mathbf{G}$*-scheme. Then* $\mathcal{D}_X$ *is representable.*

Proof. This is a consequence of Corollary 7.8.2. Indeed, X admits a cover $\mathcal{U}$ by open $\mathbf{G}$-stable affine subschemes. Then $\{U^{\mathbf{G}}\}_{U\in\mathcal{U}}$ is an open cover of $X^{\mathbf{G}}$. □

Now we prove representability of formal Białynicki-Birula functor for arbitrary schemes with action of $\mathbf{G}$ and by Tannakian formalism we obtain general result concerning representability of Białynicki-Birula functor.

Theorem 7.8.4. *Let* $\mathbf{G}$ *be a group* k*-scheme and let* $\mathbf{M}$ *be a Kempf monoid having* $\mathbf{G}$ *as a group of units. Suppose that* X *is a* k*-scheme equipped with an action of* $\mathbf{G}$*. Then the following results hold.*

- **(1)** $\widehat{\mathcal{D}}_X$ *is representable. Moreover, the morphism* $\widehat{r}_X : \widehat{\mathcal{D}}_X \to X^{\mathbf{G}}$ *is affine and if* X *is locally noetherian, then it is of finite type.*
- **(2)** *If* X *is of finite type over* k*, then the canonical morphism* $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ *is an isomorphism of functors.*

Proof. Consider the ideal $\mathcal{I}$ in $\mathcal{O}_X$ corresponding to a closed subscheme $X^{\mathbf{G}}$ of X. We define X_n as a closed subscheme of X determined by the ideal $\mathcal{I}^n$ and we denote by $\mathcal{I}_n$ the ideal of X_0 in X_n. Then $\widehat{X} = \{X_n\}_{n\in\mathbb{N}}$ is a formal $\mathbf{G}$-scheme. Moreover, by Corollary 6.3.8 each X_n is a locally linear $\mathbf{G}$-scheme and hence by Corollary 7.8.3 there exists a k-scheme Z_n equipped with $\mathbf{M}$-action that represents $\mathcal{D}_{X_n}$. Note that the square

$$\begin{tikzcd} Z_n \arrow[r, hook] \arrow[d, "i_n"'] & Z_{n+1} \arrow[d, "i_{n+1}"] \\ X_n \arrow[r, hook] & X_{n+1} \end{tikzcd}$$

is cartesian according to Theorem 7.6.6 for each $n \in \mathbb{N}$. This implies that the subscheme of zeros of an ideal $i_{n+1}^{-1}\mathcal{I}_{n+1}^n \cdot \mathcal{O}_{Z_{n+1}}$ in Z_{n+1} is Z_n. Since the square

$$\begin{tikzcd} Z_0 \arrow[r, hook] \arrow[d, "i_0"'] & Z_{n+1} \arrow[d, "i_{n+1}"] \\ X_0 \arrow[r, hook] & X_{n+1} \end{tikzcd}$$

is cartesian as a combination of cartesian squares, we derive that the vanishing closed subscheme of $i_{n+1}^{-1}\mathcal{I}_{n+1}\cdot\mathcal{O}_{Z_{n+1}}$ in Z_{n+1} is Z_0. Note that

$$\left(i_{n+1}\mathcal{I}_{n+1}\cdot\mathcal{O}_{Z_{n+1}}\right)^n = i_{n+1}^{-1}\mathcal{I}_{n+1}^n\mathcal{O}_{Z_{n+1}}$$

Thus $\mathcal{Z} = \{Z_n\}_{n\in\mathbb{N}}$ is a formal scheme. According to Remarks 7.6.4 and 7.6.5, we derive that it is a formal $\mathbf{M}$-scheme. Now the commutative diagram

$$\begin{tikzcd}
Z_0 \arrow[r, hook] \arrow[d, "i_0"'] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] \arrow[d, "i_n"'] & Z_{n+1} \arrow[r, hook] \arrow[d, "i_{n+1}"'] & \dots & \\
X_0 \arrow[r, hook] & \dots \arrow[r, hook] & X_n \arrow[r, hook] & X_{n+1} \arrow[r, hook] & \dots \arrow[r, hook] & X
\end{tikzcd}$$

shows that $\{i_n\}_{n\in\mathbb{N}}$ is a morphism of formal $\mathbf{G}$-schemes. Since $\mathbf{M}$ is a Kempf monoid, Corollary 6.5.6 implies that there exists a locally linear $\mathbf{M}$-scheme Z such that $\widehat{Z} = \{Z_n\}_{n\in\mathbb{N}}$. Here our argument ramifies. We first provide the proof of **(1)** and later deal with **(2)**.

(1) Consider a k-scheme Y and a family $\{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \in \widehat{\mathcal{D}}_X(Y)$. Note that γ_n uniquely factors through X_n and hence there exists a unique $\mathbf{M}$-equivariant morphism $\delta_n : \mathbf{M}_n \times_k Y \to Z_n$. Hence the family $\{\delta_n\}_{n\in\mathbb{N}}$ is a morphism

$$\begin{tikzcd}
\mathbf{M}_0 \times_k Y \arrow[r, hook] \arrow[d, "\delta_0"'] & \dots \arrow[r, hook] & \mathbf{M}_n \times_k Y \arrow[r, hook] \arrow[d, "\delta_n"'] & \mathbf{M}_{n+1} \times_k Y \arrow[r, hook] \arrow[d, "\delta_{n+1}"'] & \dots \\
Z_0 \arrow[r, hook] & \dots \arrow[r, hook] & Z_n \arrow[r, hook] & Z_{n+1} \arrow[r, hook] & \dots
\end{tikzcd}$$

of a formal $\mathbf{M}$-schemes. According to Corollary 6.5.6 there exists a unique $\mathbf{M}$-equivariant morphism $\delta : \mathbf{M} \times_k Y \to Z$ such that $\delta_{|\mathbf{M}_n \times_k Y}$ induces $\delta_n : \mathbf{M}_n \times_k Y \to Z_n$ for every $n \in \mathbb{N}$. Note that δ as an $\mathbf{M}$-equivariant morphism is uniquely determined by a morphism $\eta = \delta_{|\{e\}\times_k Y}$ of k-schemes, where $e : \operatorname{Spec} k \to \mathbf{M}$ is the unit of $\mathbf{M}$. This proves that

$$\widehat{\mathcal{D}}_X(Y) \ni \{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \mapsto \eta \in \operatorname{Mor}_k(Y, Z)$$

is a bijection natural in Y. Thus $\widehat{\mathcal{D}}_X$ is represented by Z. Note that $\widehat{r}_X : \widehat{\mathcal{D}}_X \to X^{\mathbf{G}}$ is represented by the canonical retraction $r_Z : Z \to Z^{\mathbf{M}} = X^{\mathbf{G}}$ (Proposition 5.5.9). Hence $\widehat{r}_X$ is affine (Proposition 5.5.9). If X is locally noetherian, then $\widehat{Z} = \mathcal{Z}$ is a locally noetherian formal $\mathbf{M}$-scheme and hence by Theorem 6.5.7 we derive that $\widehat{r}_X$ is of finite type.

(2) Assume that X is of finite type over k. Then $\mathcal{Z}$ is noetherian formal $\mathbf{M}$-scheme and Theorem 6.5.7 implies that the canonical retraction (Proposition 5.5.9) $r : Z \to Z^{\mathbf{M}} = X^{\mathbf{G}}$ is of finite type. Since $X^{\mathbf{G}}$ is closed subscheme of X, we derive that Z is of finite type over k. Theorem 7.5.5 implies that there exists a unique $\mathbf{G}$-equivariant morphism $f : Z \to X$ such that for every $n \in \mathbb{N}$ we have a commutative square

$$\begin{tikzcd} Z_n \arrow[r, hook] \arrow[d, "i_n"'] & Z \arrow[d, "f"] \\ X_n \arrow[r, hook] & X \end{tikzcd}$$

Consider a k-scheme Y and a family $\{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}} \in \widehat{\mathcal{D}}_X(Y)$. Then γ_n factors through the composition of $i_n : Z_n \to X_n$ and the closed immersion $X_n \hookrightarrow X$ for every $n \in \mathbb{N}$. Thus a family $\{\gamma_n\}_{n\in\mathbb{N}}$ determines and is determined by a unique family $\{\delta_n : \mathbf{M}_n \times_k Y \to Z_n\}_{n\in\mathbb{N}}$ of $\mathbf{M}$-equivariant morphisms. As above Corollary 6.5.6 shows that there is an $\mathbf{M}$-equivariant morphism $\delta : \mathbf{M} \times_k Y \to Z$ such that $\delta_{|\mathbf{M}_n\times_k Y}$ induces δ_n for every $n \in \mathbb{N}$. Define $\gamma = f \cdot \delta$. Then $\gamma : \mathbf{M} \times_k Y \to X$ is a $\mathbf{G}$-equivariant morphism and $\gamma_{|\mathbf{M}_n\times_k Y} = \gamma_n$ for every $n \in \mathbb{N}$. This shows that the map

$$\mathcal{D}_X(Y) \to \widehat{\mathcal{D}}_X(Y)$$

is surjective for every k-scheme Y. By Theorem 7.7.6 we derive that it is injective and hence the canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism.

□

It is easy to strengthen **(2)** in Theorem 7.8.4.

Corollary 7.8.5. *Let* $\mathbf{G}$ *be a group* k*-scheme and* $\mathbf{M}$ *be a Kempf monoid having* $\mathbf{G}$ *as a group of units. Suppose that* X *is a scheme locally of finite type over* k *equipped with an action of* $\mathbf{G}$*. Then the canonical morphism* $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ *is an isomorphism. In particular,* $\mathcal{D}_X$ *is representable and* $r_X : \mathcal{D}_X \to X^{\mathbf{G}}$ *is affine and of finite type.*

Proof. Let $a : \mathbf{G} \times_k X \to X$ be an action of $\mathbf{G}$ on X. Consider an open affine subscheme V of X. Then a induces a surjective morphism $a_V : \mathbf{G} \times_k V \twoheadrightarrow \mathbf{G} \cdot V$ (Corollary 4.4.4). Since $\mathbf{G} \times_k V$ is affine k-scheme, it is quasi-compact. The image of a quasi-compact topological space under a continuous map is quasi-compact. Thus $\mathbf{G}$-stable hull $\mathbf{G} \cdot V$ of V is quasi-compact. Since X is locally of finite type over k, we derive that $\mathbf{G} \cdot V$ is of finite type over k. This proves that X admits an open cover $\mathcal{U}$ by an open $\mathbf{G}$-stable subschemes of finite type over k. By Remark 7.7.4 we have a commutative triangle

$$\begin{tikzcd} \mathcal{D}_X \arrow[rr] \arrow[dr, "r_X"'] & & \widehat{\mathcal{D}}_X \arrow[dl, "\widehat{r}_X"] \\ & X^{\mathbf{G}} & \end{tikzcd}$$

and according to Theorem 7.6.7 and Proposition 7.7.5 for every U in $\mathcal{U}$ base change of the triangle above along open immersion $U^{\mathbf{G}} \hookrightarrow X^{\mathbf{G}}$ yields a triangle

$$\begin{tikzcd}
\mathcal{D}_U \arrow[rr] \arrow[dr, "r_U"'] & & \widehat{\mathcal{D}}_U \arrow[dl, "\widehat{r}_U"] \\
& U^{\mathbf{G}} &
\end{tikzcd}$$

in which the horizontal morphism $\mathcal{D}_U \to \widehat{\mathcal{D}}_U$ is an isomorphism by **(2)** in Theorem 7.8.4 and the fact that U is a $\mathbf{G}$-scheme of finite type over k. Since $\widehat{\mathcal{D}}_X$ is representable by **(1)** in Theorem 7.8.4, it follows that $\mathcal{D}_X$ is representable and the canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism of functors. Thus r_X and $\widehat{r}_X$ are isomorphic and this completes the proof. □

7.9 Smoothness of Białynicki-Birula decomposition

The result below, which is interesting in its own sake, plays crucial role in our proof of the smoothness of Białynicki-Birula decomposition.

Theorem 7.9.1. *Let $\mathbf{M}$ be a Kempf monoid over k and let $f : X \to Y$ be an $\mathbf{M}$-equivariant morphism between locally linear $\mathbf{M}$-schemes. Assume that Y is locally noetherian. Suppose that the following assertions hold.*

(1) *The morphism $f^{\mathbf{M}} : X^{\mathbf{M}} \to Y^{\mathbf{M}}$ is an isomorphism.*

(2) *The commutative square*

$$\begin{tikzcd}
X^{\mathbf{M}} \arrow[r, hook] \arrow[d, "f^{\mathbf{M}}"'] & X \arrow[d, "f"] \\
Y^{\mathbf{M}} \arrow[r, hook] & Y
\end{tikzcd}$$

is cartesian.

(3) *$X^{\mathbf{M}}$ is contained in the étale locus of f.*

Then f is an isomorphism.

Proof. Let $\{f_n : X_n \to Y_n\}_{n\in\mathbb{N}}$ be the morphism $\widehat{X} \to \widehat{Y}$ induced by f on formal $\mathbf{M}$-schemes. By Corollary 6.5.6 it suffices to prove that $\{f_n\}_{n\in\mathbb{N}}$ is an isomorphism. For this it suffices to prove that each f_n is an isomorphism. Since f is étale at each point of $X_0 = X^{\mathbf{M}}$, it is also étale on some open neighborhood U of X_0. Étale morphisms are open and hence $f(U) = V$ is an open subscheme of Y. Let $f' : U \to V$ be the restriction of f. Note that $X_n \subseteq U$ and $Y_n \subseteq V$ for every $n \in \mathbb{N}$. The assumption **(2)** implies that each square

$$\begin{tikzcd}
X_n \arrow[r, hook] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\
Y_n \arrow[r, hook] & Y
\end{tikzcd}$$

is cartesian. Hence also the square

$$
\begin{tikzcd}
X_n \arrow[r, hook] \arrow[d, "f_n"'] & U \arrow[d, "f'"] \\
Y_n \arrow[r, hook] & V
\end{tikzcd}
$$

is cartesian for every $n \in \mathbb{N}$. Thus f_n is étale for every $n \in \mathbb{N}$. Finally, by assumption **(1)** morphism f_0 is an isomorphism. Now fix $n \in \mathbb{N}$ and consider a diagram

$$
\begin{tikzcd}
Y_0 \arrow[r, "f_0^{-1}"] \arrow[d, hook] & X_0 \arrow[r, hook] & X_n \arrow[d, "f_n"] \\
Y_n \arrow[rru, dotted, "s_n"] \arrow[rr, "1_{Y_n}"'] & & Y_n
\end{tikzcd}
$$

Since f_n is étale, there exists a unique morphism $s_n : Y_n \to X_n$ that makes the diagram commutative. Thus f_n admits a section s_n. Section of every morphism of schemes is locally closed and hence s_n is locally closed. Moreover, f_n induces the isomorphism f_0 and hence it is a homeomorphism on underlying topological spaces. Thus s_n is a homeomorphism. This implies that s_n is a surjective closed immersion of k-schemes. For every point x in X_n the étale morphism f_n induces an isomorphism $\widehat{f_n^\#} : \widehat{\mathcal{O}_{X_n,x}} \to \widehat{\mathcal{O}_{Y_n,f_n(x)}}$. Clearly $\widehat{s_n^\#} : \widehat{\mathcal{O}_{Y_n,f_n(x)}} \to \widehat{\mathcal{O}_{X_n,x}}$ is its inverse and thus s_n induces isomorphism on completions of local rings. Note that for every $y \in Y_n$ we have a commutative diagram

$$
\begin{tikzcd}
\widehat{\mathcal{O}_{Y_n,y}} \arrow[r, "\widehat{s_n^\#}"] & \widehat{\mathcal{O}_{X_n,s_n(y)}} \\
\mathcal{O}_{Y_n,y} \arrow[u, hook] \arrow[r, two heads, "s_n^\#"'] & \mathcal{O}_{X_n,s_n(y)} \arrow[u, hook]
\end{tikzcd}
$$

in which vertical arrows are injective due to the fact that X_n and Y_n are locally noetherian (this follows from the assumption that Y is locally noetherian). Since $\widehat{s_n^\#}$ is an isomorphism, we infer that $s_n^\#$ is injective. Therefore, it is both injective and surjective (s_n is a closed immersion). Thus $s_n^\# : \mathcal{O}_{Y_n,y} \to \mathcal{O}_{X_n,s_n(y)}$ is an isomorphism. This holds for every $y \in Y_n$. Hence s_n is an isomorphism of k-schemes and this implies that f_n is an isomorphism of k-schemes. □

In the remaining part of this section we fix a Kempf monoid $\mathbf{M}$ over k with the group $\mathbf{G}$ of units. We also fix a k-scheme X with an action of $\mathbf{G}$. If $\mathcal{D}_X$ is representable, then we denote by X^+ a unique (up to an isomorphism) k-scheme that represents it. According to Remark 7.6.3 we have a commutative diagram

$$\begin{tikzcd} X^+ \arrow[r, "i_X"] \arrow[d, "r_X"] & X \\ X^{\mathbf{G}} \arrow[u, bend left, "s_X"] \end{tikzcd}$$

where $r_X \cdot s_X = 1_{X^{\mathbf{G}}}$.

Corollary 7.9.2. *Assume that X is locally of finite type over k. Then X^+ exists and $r_X : X^+ \to X^{\mathbf{G}}$ is an affine morphism of finite type.*

Proof. This follows from Corollary 7.8.5. □

Theorem 7.9.3. *Let X be a scheme locally of finite type over k. Suppose that x is a point of $X^{\mathbf{G}}$ such that the morphism $r_X : X^+ \to X^{\mathbf{G}}$ is smooth at $s_X(x)$. Then there exist an open neighborhood V of x in $X^{\mathbf{G}}$ and an isomorphism $\phi : r_X^{-1}(V) \to \mathbb{A}^n_V$ of k-schemes such that the triangle*

$$\begin{tikzcd} r_X^{-1}(V) \arrow[rr, "\phi"] \arrow[dr, "\text{the restriction of } r_X"'] & & \mathbb{A}^n_V \arrow[dl, "\mathrm{pr}_V"] \\ & V & \end{tikzcd}$$

is commutative, where pr_V is the projection. Moreover, if $\mathbf{G}$ is linearly reductive, then one can choose ϕ to be $\mathbf{M}$-equivariant with respect to some action of $\mathbf{M}$ on $\mathbb{A}^n_V$.

For the proof we need the following result.

Lemma 7.9.3.1. *Consider a commutative diagram of k-schemes*

$$\begin{tikzcd} & V \arrow[dl, "s_1"'] \arrow[dr, "s_2"] & \\ X_1 \arrow[rr, "f"] \arrow[dr, "r_1"'] & & X_2 \arrow[dl, "r_2"] \\ & V & \end{tikzcd}$$

in which $r_1 \cdot s_1 = 1_V$, $r_2 \cdot s_2 = 1_V$ and s_1, s_2 are closed immersions. Suppose that r_1 is smooth at each point in the image of s_1. Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be ideals corresponding to s_1, s_2 respectively. Assume that f induces an isomorphism $\mathcal{I}_2/\mathcal{I}_2^2 \simeq \mathcal{I}_1/\mathcal{I}_1^2$ of quasi-coherent sheaves on V. Then f is étale at each point in the image of s_1.

Proof of the lemma. Let $s_i(V)$ be the closed subscheme of X_i given by the image of s_i for $i = 1, 2$. Note that f is a morphism of schemes over V as $r_1 = r_2 \cdot f$ and hence we may consider the cotangent morphism $f^*\Omega_{X_2/V} \to \Omega_{X_1/V}$ associated with f. Since we have the canonical identification of $s_1^* f^*$ with s_2^*, we derive that there exists a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_2/\mathcal{I}_2^2 \arrow[r] \arrow[dd, "\textbf{induced by } f^\#"'] & s_2^*\Omega_{X_2/V} \arrow[r] \arrow[d, "="] & \Omega_{s_2(V)/V} \arrow[r] & 0 \\
& & s_1^* f^*\Omega_{X_2/V} \arrow[d, "s_1^*(\textbf{cotangent map of } f)"] & & \\
0 \arrow[r] & \mathcal{I}_1/\mathcal{I}_1^2 \arrow[r] & s_1^*\Omega_{X_1/V} \arrow[r] & \Omega_{s_1(V)/V} \arrow[r] & 0
\end{tikzcd}$$

in which rows are conormal sequences corresponding to closed immersions s_2 and s_1, respectively. Note that for $i = 1,2$ these conormal sequences are exact due to the fact that $r_i \cdot s_i = 1_V$ is a smooth morphism ([Gro67, Proposition 17.2.5]). Moreover, $\Omega_{s_i(V)/V} = 0$ for $i = 1,2$ again by the fact that $r_i \cdot s_i = 1_V$. Thus the diagram above reduces to the square

$$\begin{tikzcd}
\mathcal{I}_2/\mathcal{I}_2^2 \arrow[r, "\simeq"] \arrow[dd, "\textbf{induced by } f^\#"'] & s_2^*\Omega_{X_2/V} \arrow[d, "="] \\
& s_1^* f^*\Omega_{X_2/V} \arrow[d, "s_1^*(\textbf{cotangent map of } f)"] \\
\mathcal{I}_1/\mathcal{I}_1^2 \arrow[r, "\simeq"] & s_1^*\Omega_{X_1/V}
\end{tikzcd}$$

in which both horizontal arrows are isomorphisms. We claim that this square is commutative. For this fix an open affine subscheme U of X_2 and a regular function $a \in \mathcal{O}_{X_2}(U)$. Then the application of top horizontal arrow and then two consecutive rightmost vertical arrows in the square to $a \bmod \mathcal{I}_2^2(U) \in \mathcal{I}_2(U)/\mathcal{I}_2(U)^2$ is depicted in the following diagram.

$$\begin{tikzcd}
a \bmod \mathcal{I}_2^2(U) \arrow[r, mapsto] & s_2^* d_{X_2/V}(a) \arrow[d, mapsto] \\
& s_1^* f^* d_{X_2/V}(a) \arrow[d, mapsto] \\
& s_1^* d_{X_1/V}(f^\#(a))
\end{tikzcd}$$

Here $d_{X_2/V}(a)$ is the relative differential over V of a function a and $d_{X_1/V}(f^\#(a))$ is a relative differential over V of a function $f^\#(a)$. On the other hand the application of leftmost vertical arrow and then bottom horizontal arrow in the square to $a \bmod \mathcal{I}_2^2(U) \in \mathcal{I}_2(U)/\mathcal{I}_2(U)^2$ is depicted in the following diagram.

$$\begin{tikzcd} a \bmod \mathcal{I}_2^2(U) \arrow[d, mapsto] & \\ f^{\#}(a) \bmod \mathcal{I}_1^2(U) \arrow[r, mapsto] & s_1^* d_{X_1/V}(f^{\#}(a)) \end{tikzcd}$$

Hence results are equal and the square is commutative. The leftmost vertical arrow in the square is a morphism $\mathcal{I}_2/\mathcal{I}_2^2 \to \mathcal{I}_1/\mathcal{I}_1^2$ induced by f and hence it is an isomorphism by assumption. Since (as we noticed) both horizontal sides of the square are isomorphism, we derive that all morphisms in the square are isomorphisms. Therefore, pullback $s_1^* f^* \Omega_{X_2/V} \to s_1^* \Omega_{X_1/V}$ of the cotangent map $f^* \Omega_{X_2/V} \to s_1^* \Omega_{X_1/V}$ along s_1 is an isomorphism. Since r_1 is smooth at each point in the image of s_1, we derive by [Gro67, Corollaire 17.11.2] that f is étale at every point of $s_1(V)$. □

Proof of Theorem 7.9.3. Since r_X is smooth at $s_X(x)$ and its smooth locus is open, there exists an open affine neighborhood V of x in $X^{\mathbf{G}}$ such that r_X is smooth at every point of $s_X(V)$. By Corollary 7.9.2 morphism $r_X^{-1}(V) \to V$ is affine and of finite type. Hence it corresponds to a morphism of k-algebras $A \to B$ of finite type, where $A = \Gamma(V, \mathcal{O}_{X^{\mathbf{G}}})$ and $B = \Gamma(r_X^{-1}(V), \mathcal{O}_{X^+})$. In these terms s_X restricted to a morphism $V \to r_X^{-1}(V)$ corresponds to a surjection $B \twoheadrightarrow A$. We denote its kernel by $\mathfrak{b}$. Since $r_X \cdot s_X$ restricted to V is 1_V, the conormal sequence

$$0 \longrightarrow \mathfrak{b}/\mathfrak{b}^2 \longrightarrow A \otimes_B \Omega_{B/A} \longrightarrow \Omega_{A/A} \longrightarrow 0$$

is split exact ([Gro67, Proposition 17.2.5]). Moreover, r_X is smooth at every point in $s_X(V)$. Thus $A \otimes_B \Omega_{B/A}$ is a finitely generated projective A-module. Hence $\mathfrak{b}/\mathfrak{b}^2$ is projective and finitely generated A-module. Shrinking V we may assume that $\mathfrak{b}/\mathfrak{b}^2$ is a free A-module of rank $n \in \mathbb{N}$. Consider central closed subtorus T of $\mathbf{G}$ such that the zero of $\mathbf{M}$ is contained in the toric monoid $\overline{T}$. Clearly the action of $\mathbf{M}$ on $r_X^{-1}(V)$ induces an action of $\overline{T}$. Moreover, by Proposition 5.5.9 we derive $r_X^{-1}(V)^{\overline{T}} = V = r_X^{-1}(V)^{\mathbf{M}}$. Hence B and $\mathfrak{b}$ are representations of $\overline{T}$. Hence also $\mathfrak{b}/\mathfrak{b}^2$ is a representation of $\overline{T}$. Since T is linearly reductive, we deduce by Corollary 3.9.14 that $\mathbf{Rep}(\overline{T})$ is semisimple. Thus the canonical surjection $\mathfrak{b} \twoheadrightarrow \mathfrak{b}/\mathfrak{b}^2$ of representations of $\overline{T}$ admits a right inverse $s : \mathfrak{b}/\mathfrak{b}^2 \to \mathfrak{b}$ in category $\mathbf{Rep}(\overline{T})$. Let $\mathrm{Sym}(\mathfrak{b}/\mathfrak{b}^2)$ be the symmetric A-algebra of $\mathfrak{b}/\mathfrak{b}^2$. Then s induces a morphism $\mathrm{Sym}(\mathfrak{b}/\mathfrak{b}^2) \to B$ of A-algebras. Since $\mathfrak{b}/\mathfrak{b}^2$ is a representation of $\overline{T}$, $\mathfrak{b}/\mathfrak{b}^2$ is a free A-module of rank n and s is a morphism of $\overline{T}$-representations, we derive that $\mathbb{A}^n_V = \operatorname{Spec}\mathrm{Sym}(\mathfrak{b}/\mathfrak{b}^2)$ admits an action of $\overline{T}$ and the morphism $\phi : r_X^{-1}(V) \to \mathbb{A}^n_V$ induced by s is $\overline{T}$-equivariant. Note that the isotypic component of $\mathfrak{b}/\mathfrak{b}^2$ corresponding to the trivial representation of $\overline{T}$ is zero and hence this holds for the direct sum $\mathrm{Sym}(\mathfrak{b}/\mathfrak{b}^2)_{>0}$ of all positive symmetric powers. Therefore, the canonical monomorphism $\left(\mathbb{A}^n_V\right)^{\overline{T}} \hookrightarrow \mathbb{A}^n_V$ is isomorphic to the zero section $V \hookrightarrow \mathbb{A}^n_V$ of $\mathbb{A}^n_V$ as the trivial vector bundle over V. Thus $\phi^{\overline{T}}$ is an isomorphism and moreover, the square

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "\phi^{\overline{T}}"'] & r_X^{-1}(V) \arrow[d, "\phi"] \\ V \arrow[r, hook] & \mathbb{A}^n_V \end{tikzcd}$$

in which the top horizontal arrow is the restriction of s_X is commutative. By construction we derive that ϕ induces an isomorphism between $\mathfrak{b}/\mathfrak{b}^2$ and $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal of the zero section $V \hookrightarrow \mathbb{A}^n_V$. Since $r_X^{-1}(V) \to V$ is smooth at each point of $s_X(V)$, we derive by Lemma 7.9.3.1 that ϕ is étale at every point in $s_X(V)$. Thus $\overline{T}$-equivariant morphism $\phi : r_X^{-1}(V) \to \mathbb{A}^n_V$ between locally linear $\overline{T}$-schemes satisfies the assumptions of Theorem 7.9.1. Since $\mathbb{A}^n_V$ is locally noetherian (V is of finite type over k), Theorem 7.9.1 implies that ϕ is an isomorphism.
If $\mathbf{G}$ is linearly reductive, then $\mathbf{Rep}(\mathbf{M})$ is semisimple by Corollary 3.9.14. Thus $s : \mathfrak{b}/\mathfrak{b}^2 \to \mathfrak{b}$ can be chosen in category $\mathbf{Rep}(\mathbf{M})$. Since ϕ is induced by s, it becomes $\mathbf{M}$-equivariant with respect to some action of $\mathbf{M}$ on $\mathbb{A}^n_V$. □

7.10 Summary of other results concerning Białynicki-Birula decomposition

In this section we give an overview of other results and comment on proofs and techniques used so far. Our goal is to present a complete survey of the current state of knowledge concerning generalized Białynicki-Birula decompositions.
We start by discussing Kempf monoids. As we noted in Corollary 5.5.4 each reductive monoid with zero is a Kempf monoid. We learned from Brion that this important result is due to Rittatore [Rit98]. Concerning representability and other properties of Białynicki-Birula functors we state here the following theorem, which shows that at least for normal linearly reductive monoids (this class contains all normal reductive monoids in characteristic zero) one can always restrict attention to linearly monoids with zero.

Theorem 7.10.1 ([JS19, Theorem 3.22]). *Let $\mathbf{M}$ be a linearly reductive and normal monoid over k with the group of units $\mathbf{G}$. Let $\overline{\mathbf{F}}$ be the intersection of all closed $\mathbf{G}$-stable subschemes of $\mathbf{M}$. Assume that $\overline{\mathbf{F}}$ admits a k-point (this is automatically satisfied if k is perfect). Let $\mathbf{N}$ be the stabilizer of a k-point in $\overline{\mathbf{F}}$ and let $\overline{\mathbf{N}}$ be its closure in $\mathbf{M}$. Then the following assertions hold.*

- **(1)** *The group $\mathbf{N}$ is linearly reductive and geometrically connected.*
- **(2)** *$\overline{\mathbf{N}}$ is a linearly reductive monoid with zero and with $\mathbf{N}$ as the group of units. It is normal.*
- **(3)** *If X is a k-scheme with action of $\mathbf{G}$, then the map*

$$\mathcal{D}_X^{\mathbf{M}}(Y) \ni \gamma \mapsto \gamma_{\overline{\mathbf{N}} \times_k X} \in \mathcal{D}_X^{\overline{\mathbf{N}}}(Y)$$

defined for every k-scheme Y is an isomorphism of presheaves; where $\mathcal{D}_X^{\mathbf{M}}, \mathcal{D}_X^{\overline{\mathbf{N}}}$ denote Białynicki-Birula functors for $\mathbf{M}$ and $\overline{\mathbf{N}}$, respectively.

When it comes to representability of Białynicki-Birula functors, we have one comment concerning our proof of Theorem 7.8.4. Note that our proof relies on the beautiful Theorem 7.5.1

due to Hall and Rydh. Inspired by this results and by earlier results of Brandenburg and Chirvasitu [BC14] Jelisiejew and the author obtain the following:

Theorem 7.10.2 ([JS20, Theorem A.1]). *Let X be a quasi-compact quasi-separated $\mathbf{G}$-scheme. Let Y be a $\mathbf{G}$-scheme. Let $F : \mathfrak{Qcoh}_{\mathbf{G}}(X) \to \mathfrak{Qcoh}_{\mathbf{G}}(Y)$ be a cocontinuous tensor functor and let $\alpha : F \cdot p_X^* \to p_Y^*$ be an isomorphism of functors $\mathbf{Rep}(\mathbf{G}) \to \mathfrak{Qcoh}_{\mathbf{G}}(Y)$. Then there exists a unique $\mathbf{G}$-equivariant morphism $f\colon Y \to X$ such that $(F, \alpha) \simeq (f^*, \alpha_f)$.*

Using this (unpublished) theorem one can strengthen Theorem 7.8.4 by replacing the assumption that X is of finite type over k with the assumption that it is noetherian. This propagates to Corollary 7.8.5, which then holds under the assumption that X is locally noetherian. For linearly reductive monoids $\mathbf{M}$ with zero Jelisiejew and the author were able to prove representability of Białynicki-Birula for algebraic spaces locally of finite type.

Theorem 7.10.3 ([JS19, Theorem 6.17]). *Let $\mathbf{M}$ be a linearly reductive monoid with zero and with the group of units $\mathbf{G}$. Let X be an algebraic space locally of finite type over k equipped with an action of $\mathbf{G}$. Then the canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism and both presheaves are representable.*

Now we discuss smoothness of the decomposition. Theorem 7.9.3 is an interesting result, although the assumption that r_X is smooth at $s_X(x)$ might be problematic to verify in practice. In linearly reductive case there is a remedy for this.

Theorem 7.10.4 ([JS19, Corollary 7.3]). *Let $\mathbf{M}$ be a linearly reductive monoid with zero and with the group of units $\mathbf{G}$. Let X be a smooth variety with action of $\mathbf{G}$. Then X^+, $X^{\mathbf{G}}$ are smooth and r_X is a smooth morphism at the image of s_X.*

The theorem above can be obtained by rather elementary means. In the article [JS19] it was a consequence of a far more general result.

Theorem 7.10.5 ([JS19, Theorem 7.1]). *Let $\mathbf{M}$ be a linearly reductive monoid with zero and with $\mathbf{G}$ as its group of units. Assume that X is a $\mathbf{G}$-scheme locally of finite type over k and let $f : X \to Y$ be a $\mathbf{G}$-equivariant smooth morphism. Then $\mathcal{D}_f : \mathcal{D}_X \to \mathcal{D}_Y$ is smooth.*

Crucial step in the proof to this general result depends on the following theorem presented in the beautiful work of Alper, Hall and Rydh.

Theorem 7.10.6 ([AHR20, Theorem 2.6]). *Let X be a quasi-separated algebraic space, locally of finite type over k, and with an action of an affine algebraic group $\mathbf{G}$ over k. Let x be a k-point of X with linearly reductive stabilizer $\mathbf{G}_x$. Then there exists an affine scheme W with an action of $\mathbf{G}$ and a $\mathbf{G}$-equivariant étale neighborhood $W \to X$ of x.*

Finally in the classical Białynicki-Birula decomposition morphism $i_X : X^+ \to X$ is a locally closed immersion on each irreducible component. The following result addresses this feature.

Theorem 7.10.7 ([JS19, Proposition 7.6]). *Let X be a geometrically normal variety over k and let $\mathbf{M}$ be a reductive monoid with zero. Suppose that Z is an irreducible component of X^+ considered as a closed subscheme with reduced structure. Then the restriction $i_{X|Z} : Z \to X$ is a locally closed immersion.*

Note that the proof of this theorem in [JS19] is based on a result due to Trautman [Tra92], which holds for general reductive group. Thus the theorem above holds for reductive monoids. Finally we give an application of generalized Białynicki-Birula decompositions discussed in this work.

Example 7.10.8. Let X be a smooth, quasi-projective scheme over k. Suppose that $\mathbb{G}_m$ acts on X with finitely many fixed points. Then Theorems 7.9.3 and 7.10.4 imply that X^+ (defined for $\mathbb{G}_m \subseteq \mathbb{A}^1_k$) is a disjoint sum of finitely many affine spaces (often called *cells*) corresponding to fixed points of the action. This also follows from classical Białynicki-Birula result.
Assume now that X is equipped with two commuting actions of $\mathbb{G}_m$. Denote them by a_i for $i = 1, 2$. Suppose that a_1 and a_2 admit the same fixed point locus $F \subseteq X$ which consists of finitely many points. Let x_0 be a point of F. Next suppose that W_i for $i = 1, 2$ is a cell over x_0 of the Białynicki-Birula decomposition (defined for $\mathbb{G}_m \subseteq \mathbb{A}^1_k$) with respect to a_i. Then as we noted above W_i is isomorphic (as k-scheme) with an affine space. According to Theorem 7.10.7 we may view W_i as a locally closed subscheme of X. We are going to prove that the intersection $W_1 \cap W_2$ is also isomorphic to an affine space. For this consider the action $a : \mathbb{G}_m \times_k \mathbb{G}_m \to X$ induced by a_1, a_2 and apply the Białynicki-Birula decomposition with respect to the monoid $\mathbb{A}^2_k$ which contains (in a canonical way) $\mathbb{G}_m \times_k \mathbb{G}_m$ as its group of units. Let W be a cell of this Białynicki-Birula decomposition corresponding to fixed point x_0 of $\mathbb{G}_m \times_k \mathbb{G}_m$. Similarly as above, from Theorems 7.9.3 and 7.10.4 we deduce that W is an affine space. Therefore, it suffices to prove that

$$W = W_1 \cap W_2$$

By definition W represents the functor

$$\mathcal{D}_1(Y) = \left\{\gamma : \mathbb{A}^2 \times_k Y \to X \,|\, \gamma \text{ is } \mathbb{G}_m \times_k \mathbb{G}_m\text{-equivariant and } \gamma\left(\{(0,0)\} \times_k Y\right) = \{x_0\}\right\}$$

Moreover note that $W_1 \cap W_2 = W_1 \times_X W_2$ and hence $W_1 \cap W_2$ represents the functor

$$\mathcal{D}_2(Y) = \left\{\gamma : \left(\mathbb{A}^2 \smallsetminus \{(0,0)\}\right) \times_k Y \to X \;\middle|\; \begin{array}{l} \gamma \text{ is } \mathbb{G}_m \times_k \mathbb{G}_m\text{-equivariant and} \\ \gamma(\{(0,1)\} \times_k Y) = \{x_0\} = \gamma(\{(1,0)\} \times_k Y) \end{array}\right\}$$

Consider $\gamma \in \mathcal{D}_2(Y)$. Let U be an open affine and $\mathbb{G}_m \times_k \mathbb{G}_m$-stable neighborhood of x_0 (it exists according to the classical result of Sumihiro [CLS11, Theorem 3.1.7]). Since $\gamma(\{0\} \times_k \mathbb{A}^1_k \times_k Y) = \{x_0\} = \gamma(\mathbb{A}^1_k \times_k \{0\} \times_k Y)$, we deduce as in the proof of Theorem 7.6.7 that γ factors through U. Next we have a cocartesian (pushout) square

$$\begin{tikzcd}
\mathbb{G}_m \times_k \mathbb{G}_m \arrow[r, hook] \arrow[d, hook] & \mathbb{A}^1_k \times_k \mathbb{G}_m \arrow[d, hook] \\
\mathbb{G}_m \times_k \mathbb{A}^1_k \arrow[r, hook] & \mathbb{A}^2_k
\end{tikzcd}$$

in the category of affine k-schemes with actions of $\mathbb{G}_m \times_k \mathbb{G}_m$. Hence we can extend γ uniquely to a morphism $\overline{\gamma} : \mathbb{A}^2_k \times_k Y \to U$. Thus there exists the unique morphism $\tilde{\gamma} : \mathbb{A}^2_k \times_k Y \to X$ which extends γ. This proves that functors $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic over X. Thus $W = W_1 \cap W_2$.
Note that the assumption that the cells W_1 and W_2 correspond to the same fixed point is

essential. Indeed, consider the projective line $\mathbb{P}^1_k$ with two actions of $\mathbb{G}_m$ given by formulas $a_1\big(t,[x_0,x_1]\big) = [tx_0,x_1]$ and $a_2\big(t,[x_0,x_1]\big) = [t^{-1}x_0,x_1]$. These two actions commute and their schemes of fixed points coincide. Then the cells for a_1 are

$$\mathbb{A}^1_k \simeq \big\{[x_0,x_1] \in \mathbb{P}^1_k \,|\, x_1 \neq 0\big\},\ \big\{[1,0]\big\}$$

and the cells for a_2 are

$$\mathbb{A}^1_k \simeq \big\{[x_0,x_1] \in \mathbb{P}^1_k \,|\, x_0 \neq 0\big\},\ \big\{[0,1]\big\}$$

The intersection of an a_1-cell $\big\{[x_0,x_1] \in \mathbb{P}^1_k \,|\, x_1 \neq 0\big\}$ corresponding to $[0,1]$ and an a_2-cell $\big\{[x_0,x_1] \in \mathbb{P}^1_k \,|\, x_0 \neq 0\big\}$ corresponding to $[1,0]$ is isomorphic to $\mathbb{G}_m$ as a k-scheme. Hence it is not an affine space.

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