GENERALIZED BIAŁYNICKI-BIRULA DECOMPOSITIONS
EXTENDED ABSTRACT

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Symmetries of mathematical objects were studied since antiquity. In the nineteenth century Galois et al. formalized symmetries by introducing the concept of group actions. Since then rigorous study of actions of groups proved to be extremely effective in algebra, combinatorics, geometry, topology and in other branches of mathematics. In this thesis we are interested in symmetries given by polynomial mappings over some field k. Such symmetries are encapsulated by the notion of *an action of an algebraic group*. An algebraic group is an algebraic variety equipped with the group structure such that its multiplication and inverse are morphisms of varieties. Structural properties of algebraic groups were studied in detail ([DG70], [Jan03], [Mil17]). If $\mathbf{G}$ is an algebraic group and X is a variety (or a scheme) over k, then an action of $\mathbf{G}$ on X is a morphism $a : \mathbf{G} \times_k X \to X$ such that $a(1, x) = x$ and

$$a(g_1, a(g_2, x)) = a(g_1 g_2, x)$$

for any points g_1, g_2 of $\mathbf{G}$ and every point x of X. This makes a variety X into a space on which polynomial symmetries described by $\mathbf{G}$ act. The subfield of algebraic geometry which concerns properties of actions of algebraic groups on varieties (and schemes) is a vast and classical domain of research ([MFK94]). Similarly to algebraic groups one can consider *algebraic monoids* and their actions. The structure and properties of algebraic monoids were extensively studied since 1980s ([Ren06], [CLSW14]).
Consider an algebraic group $\mathbf{G}$ and suppose that $\mathbf{M}$ is an algebraic monoid which contains $\mathbf{G}$ as its group of invertible elements. Suppose that X is a k-scheme with an action of $\mathbf{G}$. In this dissertation we are concerned with constructing a space X^+ which consists of points x in X such that the canonical orbit map

$$\mathbf{G} \ni g \mapsto gx \in X$$

can be extended to a $\mathbf{G}$-equivariant map

$$\mathbf{M} \to X$$

For $\mathbf{G} = \mathbb{G}_m$, $\mathbf{M} = \mathbb{A}^1_k$ and a smooth variety X the space X^+ was studied in the celebrated paper of Białynicki-Birula [BB73, Theorem 4.3]. In this thesis we generalize his results.

1. Historical background and main results

1.1. **Classical Białynicki-Birula decomposition.** We explain the Białynicki-Birula results over complex number field $\mathbb{C}$ for simplicity (Białynicki-Birula worked over arbitrary algebraically closed field). Consider a complex smooth projective variety X with an action of $\mathbb{C}^*$. We may view X as a projective manifold and for each x in X we define

$$x_0 = \lim_{t \to 0} t \cdot x$$

Note that this limit exists for every point x in X according to the fact that X is projective. Moreover, x_0 is a fixed point of the $\mathbb{C}^*$-action. Classically the fixed point locus $X^{\mathbb{C}^*}$ of X is a disjoint union $F_1, F_2, ..., F_n$ of smooth, closed subvarieties of X. For each i we define

$$X_i^+ = \left\{ x \in X \,\middle|\, \lim_{t \to 0} t \cdot x \in F_i \right\}$$

Białynicki-Birula proved the following result.

Theorem. *In the situation described above the following assertions hold.*

(1) *$X_i^+ \cap X_j^+ = \varnothing$ for $i \neq j$.*

(2) *The map $X_i^+ \hookrightarrow X$ is a locally closed immersion of algebraic varieties for every i.*

(3) *The canonical map*

$$X_i^+ \ni x \mapsto \lim_{t \to 0} t \cdot x \in F_i$$

is a morphism of algebraic varieties and moreover, it is a Zariski locally trivial fibration with fiber $\mathbb{C}^{n_i}$ for some $n_i \in \mathbb{N}$. This holds for every i.

The Theorem above (and its generalizations to singular varieties) has profound applications in algebraic geometry. [BBCM13, II, 4.2] contains a survey of classical applications to Betti numbers and homology. Here we give a sample of recent developments which were based on this result. Brosnan ([Bro05]) applied Białynicki-Birula decomposition to obtain decomposition of motives of isotropic smooth homogeneous projective varieties. Results due to Jelisiejew on Hilbert schemes ([Jel19a], [Jel19b]) used generalized version of the decomposition as their main tool. There are applications to cell decompositions of quiver varieties ([RW19], [Sau17]), localization formulas in equivariant cohomology ([Web17]) and mirror theorem for toric varieties ([Iri17]).

1.2. **Drinfeld's result.** In [Dri13] Drinfeld proposed the following functorial generalization of the classical Białynicki-Birula result. Let k be a field and let X be an arbitrary algebraic space over k with an action of $\mathbb{G}_m$. Consider the functor $\mathcal{D}_X$ on the category of k-schemes defined by the formula

$$\mathbf{Sch}_k \ni Y \mapsto \left\{ \gamma : \mathbb{A}^1_k \times_k Y \to X \,\middle|\, \gamma \text{ is } \mathbb{G}_m\text{-equivariant} \right\} \in \mathbf{Set}$$

There are canonical morphisms of functors

$$\begin{array}{ccc} \mathcal{D}_X & \xrightarrow{\ i_X\ } & X \\ {\scriptstyle s_X}\uparrow\downarrow{\scriptstyle r_X} & & \\ X^{\mathbb{G}_m} & & \end{array}$$

which we define now. For this let $\gamma \in \mathcal{D}_X(Y)$ for some k-scheme Y. We define

$$i_X(\gamma) = \gamma_{|\{1\} \times_k Y}, \qquad r_X(\gamma) = \gamma_{|\{0\} \times_k Y}$$

where $1 : \operatorname{Spec} k \to \mathbb{A}^1_k$ is the inclusion of 1 and $0 : \operatorname{Spec} k \to \mathbb{A}^1_k$ is the inclusion of the zero. Next if $f : Y \to X$ is a morphism which factors through $X^{\mathbb{G}_m}$, then we define

$$s_X(f) = f \cdot pr_Y$$

where $pr_Y : \mathbb{A}^1_k \times_k Y \to Y$ is the projection. The definition of $\mathcal{D}_X$ is a functorial reformulation of the limiting procedure discussed above. In order to provide intuitive justification of this claim let us make some observations.

- Consider a k-scheme Y and let $f : Y \to X$ be a morphism. Then f is a Y-point of X and the morphism
$$\mathbb{G}_m \times_k Y \ni (t, y) \mapsto t \cdot f(y) \in X$$
is the orbit of Y-point f with respect to the $\mathbb{G}_m$-action. A limiting procedure may be interpreted as the existence of the extension of the morphism above to a $\mathbb{G}_m$-equivariant morphism $\mathbb{A}^1_k \times_k Y \to X$. This is the motivation for the definition of $\mathcal{D}_X$.
- Under this interpretation one may view r_X as the morphism sending each Y-point to its limit Y-point provided that the latter exists.
- Similarly i_X can be considered as the inclusion of the space of points that admit limit into X and s_X can be considered as the inclusion of fixed points into the space of points that have limits.

The following theorem is one of main results of Drinfeld's article [Dri13, Theorem 1.4.3].

Theorem. *Let X be an algebraic space of finite type over k with an action of $\mathbb{G}_m$. Then*

(1) *$\mathcal{D}_X$ is representable by an algebraic space of finite type over k.*

(2) *The morphism r_X is affine.*

1.3. **The research questions.** Note that the scheme $\mathbb{A}^1_k$ is a monoid k-scheme with respect to the canonical operation that makes the set of its k-points into the abstract monoid $k^{\times}$. Then $0 \in k$ defines the zero of the monoid k-scheme $\mathbb{A}^1_k$. Moreover, the group of units of this monoid k-scheme can be identified with $\mathbb{G}_m$ via canonical open immersion $\mathbb{G}_m \hookrightarrow \mathbb{A}^1_k$. This suggests that one can generalize Drinfeld's functorial formulation as follows. Consider a monoid k-scheme $\mathbf{M}$ with a zero $\mathbf{o}$. Let $\mathbf{G}$ be its group of units. Then $\mathbf{G}$ is a group k-scheme. For every k-scheme (or algebraic space) X with an action of $\mathbf{G}$ define the functor $\mathcal{D}_X$ by the formula

$$\mathbf{Sch}_k \ni Y \mapsto \left\{\gamma : \mathbf{M} \times_k Y \to X \,\middle|\, \gamma \text{ is } \mathbf{G}\text{-equivariant}\right\} \in \mathbf{Set}$$

on the category of k-schemes. Clearly one can define morphisms r_X, s_X and i_X of functors as above. The goal of this work is to provide answers to the following questions.

Question. *Is $\mathcal{D}_X$ representable?*

Question. *Suppose that $\mathcal{D}_X$ is representable and smooth over $X^{\mathbf{G}}$. Is r_X a locally trivial fibration with affine spaces as fibers?*

1.4. **The results.** Originally Jelisiejew and the author were interested in answering these questions for *reductive monoids*. It turns out that both our questions have affirmative answers if X is a scheme locally of finite type over k and $\mathbf{M}$ is a reductive monoid over an arbitrary field k. There is even wider class of *Kempf monoids* for which this is the case. Precisely the following two theorems are the main results of this thesis.

Theorem A. *Let $\mathbf{G}$ be a group k-scheme and let $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Suppose that X is a scheme locally of finite type over k with an action of $\mathbf{G}$. Then $\mathcal{D}_X$ is representable by a scheme X^+ and $r_X : X^+ \to X^{\mathbf{G}}$ is affine and of finite type.*

Theorem B. *Let $\mathbf{G}$ be a group k-scheme and let $\mathbf{M}$ be a Kempf monoid having $\mathbf{G}$ as a group of units. Let X be a scheme locally of finite type over k with an action of $\mathbf{G}$. Suppose that x is a point of $X^{\mathbf{G}}$ such that the morphism $r_X : X^+ \to X^{\mathbf{G}}$ is smooth at $s_X(x)$. Then there exist an open neighborhood V of x in $X^{\mathbf{G}}$ and an isomorphism $\phi : r_X^{-1}(V) \to \mathbb{A}^n_V$ of k-schemes such that the triangle*

$$\begin{array}{ccc} r_X^{-1}(V) & \xrightarrow{\ \phi\ } & \mathbb{A}^n_V \\ & {\scriptstyle \text{the restriction of } r_X} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_V} & \\ & V & \end{array}$$

is commutative, where pr_V is the projection. Moreover, if $\mathbf{G}$ is linearly reductive, then one can choose ϕ to be $\mathbf{M}$-equivariant with respect to some action of $\mathbf{M}$ on $\mathbb{A}^n_V$.

Note that Theorem A is a generalization of the Drinfeld's result mentioned above (Subsection 1.2). Theorem B shows that the essential feature of the classical Białynicki-Birula decomposition – that is the fact that the canonical morphism $X^+ \to X^{\mathbf{G}}$ is a Zariski locally trivial fibration with affine spaces as fibers – holds also for this much more general setup.

2. Main ideas involved in the proof of Theorem A

In this and next section we give sketches of proofs of Theorems A and B.
In this section we start by discussing algebraic monoids. Next we introduce formal version of the Białynicki-Birula functor $\widehat{\mathcal{D}}_X$ and explain briefly the proof of its representability. In the last subsection we outline how representability of this functor combined with *coherent completeness* and *tannakian formalism* imply that the canonically defined morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism. Theorem A (i.e. representability of $\mathcal{D}_X$) is a consequence of the fact that $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is an isomorphism and representability of $\widehat{\mathcal{D}}_X$.

2.1. **Kempf monoids.** Let us first delve a little into the theory of algebraic monoids. The category of algebraic monoids is a rich and beautiful extension of the category of algebraic groups. There are whole monographs ([Ren06], [CLSW14]) devoted to this subject. In particular, (similarly to the case of algebraic groups) researchers and pioneers in the field of algebraic monoids concentrate they efforts on studying *reductive monoids*. An algebraic monoid $\mathbf{M}$ over k is *reductive* if the group $\mathbf{G}$ of units of $\mathbf{M}$ is a reductive algebraic group. Renner in [Ren06, Theorem 5.4] classifies normal reductive monoids over algebraically closed fields in terms of pairs $(\mathbf{G}, \overline{T}_{\max})$ consisting of a reductive group $\mathbf{G}$ and a normal toric monoid $\overline{T}_{\max}$ with maximal torus $T_{\max}$ of $\mathbf{G}$ as the group of units. He proves that if the action of the Weyl group of $T_{\max} \hookrightarrow \mathbf{G}$ extends to $\overline{T}_{\max}$, then there exists a unique (up to an isomorphism) normal reductive monoid $\mathbf{M}$ with $\mathbf{G}$ as the group of units such that the closure of $T_{\max}$ in $\mathbf{M}$ is $\overline{T}_{\max}$. Moreover, if $\overline{T}_{\max}$ is a monoid with zero, then also $\mathbf{M}$ is a monoid with zero.
It turns out, and this is the result due to Rittatore in [Rit98], that the class of reductive monoids with zero is contained in a larger class of *Kempf monoids*. By definition a geometrically integral algebraic monoid $\mathbf{M}$ with zero $\mathbf{o}$ is *a Kempf monoid* if there exists a central torus T inside the group of units of $\mathbf{M}$ such that its closure $\mathbf{cl}(T)$ in $\mathbf{M}$ contains $\mathbf{o}$. Representations of $\mathbf{M}$ are more tractable due to existence of the central torus T, which is linearly reductive and hence admits semisimple category of representations. Moreover, $\mathbf{M}$ is determined by the formal neighborhood of its zero.
In the remaining part of this section and in the next section we fix a Kempf monoid $\mathbf{M}$ and its group of units $\mathbf{G}$. For every $n \in \mathbb{N}$ let $\mathbf{M}_n \hookrightarrow \mathbf{M}$ be an n-th infinitesimal neighborhood of the zero $\mathbf{o}$ in $\mathbf{M}$.

2.2. **Formal Białynicki-Birula functor.** Let X be a k-scheme equipped with an action of $\mathbf{G}$. For every k-scheme Y we define

$$\widehat{\mathcal{D}}_X(Y) = \left\{ \{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n \in \mathbb{N}} \,\middle|\, \forall_{n\in\mathbb{N}}\, \gamma_n \text{ is } \mathbf{G}\text{-equivariant and } \gamma_{n+1|\mathbf{M}_n \times_k Y} = \gamma_n \right\}$$

This gives rise to a functor $\widehat{\mathcal{D}}_X$, which may be intuitively viewed as a formal-geometric version of $\mathcal{D}_X$. It turns out that the representability of $\widehat{\mathcal{D}}_X$ reduces easily to the algebraization in formal $\mathbf{M}$-equivariant geometry. Namely we consider formal $\mathbf{M}$-schemes, i.e., formal schemes ([FGI05, 8.1.3.2])

$$Z_0 \hookrightarrow Z_1 \hookrightarrow \ldots \hookrightarrow Z_n \hookrightarrow Z_{n+1} \hookrightarrow \ldots$$

such that each Z_n is equipped with action of a monoid k-scheme $\mathbf{M}$, all closed immersions $Z_n \hookrightarrow Z_{n+1}$ are $\mathbf{M}$-equivariant and $Z_n^{\mathbf{M}} = Z_0$ for every $n \in \mathbb{N}$. For every k-scheme Z with an action of $\mathbf{M}$ the sequence of infinitesimal neighborhoods $\widehat{Z}$ of fixed points $Z^{\mathbf{M}}$ in Z is an example of a formal $\mathbf{M}$-scheme. It turns out that every formal $\mathbf{M}$-scheme is of this form. This result takes form of an equivalence of categories and is a consequence of the fact mentioned above that $\mathbf{M}$ is determined by $\{\mathbf{M}_n\}_{n\in\mathbb{N}}$. As a consequence we obtain that $\widehat{\mathcal{D}}_X$ is representable and affine over $X^{\mathbf{G}}$.

2.3. **Coherent completeness and tannakian formalism.** The functors $\mathcal{D}_X$ and $\widehat{\mathcal{D}}_X$ are related by a canonical morphism $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$. It is not difficult to prove that this map is a monomorphism of functors. However, its surjectivity turn out to be a more subtle problem, since it is not clear how to recover topologically an element $\mathcal{D}_X$ out of a given element of $\widehat{\mathcal{D}}_X$. In order to explain this let us inspect the surjectivity of $\mathcal{D}_X(\operatorname{Spec} k) \to \widehat{\mathcal{D}}_X(\operatorname{Spec} k)$. A k-point of $\widehat{\mathcal{D}}_X$ is a sequence of morphisms $\{\mathbf{M}_n \to X\}_{n\in\mathbb{N}}$. All these morphisms have their images contained in the infinitesimal neighborhood of $X^{\mathbf{G}}$ and hence they contain information on the infinitesimal neighborhood of $X^{\mathbf{G}}$. If the map $\mathcal{D}_X(\operatorname{Spec} k) \to \widehat{\mathcal{D}}_X(\operatorname{Spec} k)$ is surjective, then the family $\{\mathbf{M}_n \to X\}_{n\in\mathbb{N}}$ can be lifted to a morphism $\mathbf{M} \to X$, which existence depends on the topology of X and this a priori is not encapsulated by the infinitesimal neighborhood of $X^{\mathbf{G}}$. We prove that $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is surjective by the two step argument. Let Z be a scheme representing $\widehat{\mathcal{D}}_X$. Then Z is a locally noetherian scheme with an action of $\mathbf{M}$ such that Z can be covered by open affine $\mathbf{M}$-stable subschemes. It follows that for such Z the category of coherent $\mathbf{G}$-sheaves on Z is canonically equivalent with appropriately defined category of coherent $\mathbf{G}$-sheaves on a formal $\mathbf{M}$-scheme $\widehat{Z}$ consisting of the sequence of formal neighborhoods of fixed points $Z^{\mathbf{M}}$ of Z. This type of phenomenon is called *coherent completeness* in [AHR20] and it resembles the celebrated Grothendieck's existence theorem ([FGI05, Theorem 8.4.2]). We derive from it that there exists a functor

$$\mathfrak{Coh}_{\mathbf{G}}(X) \to \mathfrak{Coh}_{\mathbf{G}}(Z)$$

Secondly, according to the result due to Hall and Rydh ([HR19, Theorem 1.1] or by preprint by Jelisiejew and the author [JS20, Theorem A.1]) there exists a canonical $\mathbf{G}$-equivariant morphism $Z \to X$ which induces the functor discussed above on categories of coherent $\mathbf{G}$-sheaves. Results of this type, which reconstruct a morphism of schemes $f : X \to Y$ (stacks, algebraic spaces) out of a certain monoidal functors $F : \mathfrak{Coh}(Y) \to \mathfrak{Coh}(X)$ in such a way that $f^* \cong F$ in the category of functors, are called *tannakian formalisms* in this work. This is justified by the fact that classical Tannaka duality ([Mil17, Note 9.4]) can be interpreted as the reconstruction of an algebraic group $\mathbf{G}$ from its category of linear representations considered as a monoidal category over vector spaces. From the existence of this $\mathbf{G}$-equivariant morphism $Z \to X$ (or in other words the morphism $\widehat{\mathcal{D}}_X \to X$) one can deduce that each family $\{\gamma_n : \mathbf{M}_n \times_k Y \to X\}_{n\in\mathbb{N}}$ of compatible $\mathbf{G}$-equivariant morphisms can be extended to a $\mathbf{G}$-equivariant morphism $\mathbf{M} \times_k Y \to X$. This is equivalent with the fact that the natural transformation $\mathcal{D}_X \to \widehat{\mathcal{D}}_X$ is surjective on every level and from this Theorem A is inferred.

3. Main ideas involved in the proof of Theorem B

Theorem B is less demanding and its proof can be explained by referring to the notion of tubular neighborhoods. A tubular neighborhood in differential topology ([BJ82, Definition 12.10]) is a certain differentiable map from the normal bundle of a submanifold to the ambient manifold, which induces a diffeomorphism of the normal bundle with the neighborhood of the submanifold. For differentiable manifolds tubular neighborhoods always exist ([BJ82, Theorem 12.11]). In the world of schemes they exists affine locally under some additional smoothness assumptions. Now if $r_X : X^+ \to X^{\mathbf{G}}$ is a smooth morphism at $s_X(x)$, then in some affine neighborhood of $s_X(x)$ there exists a morphism from the normal bundle of the closed subscheme $s_X : X^{\mathbf{G}} \hookrightarrow X^+$ to X^+. This morphism is étale (it is an analogon of a tubular neighborhood). Moreover, one can construct this morphism as equivariant with respect to some toric submonoid of $\mathbf{M}$. Then by some result from formal $\mathbf{M}$-geometry one can prove that this morphism is an isomorphism and hence r_X is locally isomorphic to vector bundle, which is what Theorem B asserts.

4. Relation of this thesis to joint works of Jelisiejew and the author

Theorems A and B are fruits of the collaboration of Jelisiejew and the author ([JS19], [JS20]). Let us now explain how approach presented in this thesis deviates from the content of these two papers.

In [JS19] there is some stress on the notion of the formal $\mathbf{M}$-scheme, but formal geometry is not studied (due to the usual brevity of research papers) in a systematic way. In particular, that work does not contain coherent completeness. This lack is filled in the second paper [JS20], but coherent completeness is studied there somewhat out of the context of formal geometry. Substantial part of the thesis is an exhaustive and unified exposition of the theory of the formal $\mathbf{M}$-schemes for a Kempf monoid $\mathbf{M}$.

There is also a minor technical difference between coherent completeness studied in [JS20] and in this thesis. Here we get rid of the notion and usage of *Serre subcategories*. The reader may judge, if this makes our presentation clearer than that of [JS20].
Moreover, there is a key difference between [JS19] and this work. The first relies on affine étale $\mathbf{G}$-equivariant neighborhoods obtained via the result of Alper, Hall and Rydh ([AHR20, Theorem 2.6]. This restricts the scope of generality of that paper to linearly reductive monoids. Here this was eliminated thanks to coherent completeness, tannakian formalism and properties of Kempf monoids. This makes Theorem A more general with respect to the class of algebraic monoids for which it holds than its counterpart [JS19, Theorem 6.17].
Thanks to an additional work we were able to obtain a slightly stronger result than [JS20, Theorem 1.1]. Namely Theorem A is derived for schemes locally of finite type over k and this can be further refined to locally noetherian case if one accepts unpublished result ([JS20, Theorem A.1]). In contrast [JS20, Theorem 1.1] is restricted to the quasi-compact case. Here representability is formulated as the isomorphism between $\widehat{\mathcal{D}}_X$ and $\mathcal{D}_X$, which is the original approach of [JS19] and seems natural, but is not expressed explicitly in [JS20] (again due to brevity).
The proof (in the present thesis) of Theorem B relies on formal geometry and the concept of the tubular neighborhood known from differential geometry. This is significantly different from the original approach of [JS19], which was based on affine étale $\mathbf{G}$-equivariant neighborhoods, and [JS20], which does not refer to any results in formal geometry.

5. Example

Actions of $\mathbb{G}_m$ are significantly less complex than actions of the higher dimensional tori. One instance of this phenomenom is [BBS85] which is entirely devoted to answering certain question concerning $\mathbb{C}^* \times \mathbb{C}^*$-actions. The answer is much simpler in the case of $\mathbb{C}^*$ and is provided by the earlier work [BBS83]. Here we give an example of an application of the generalized Białynicki-Birula decomposition for higher dimensional tori. Interestingly, the case of two dimensional tori is used in order to relate two commuting actions of $\mathbb{G}_m$.

Example 5.1. Let X be a smooth, quasi-projective scheme over k. Suppose that $\mathbb{G}_m$ acts on X with finitely many fixed points. Then Theorems A and B imply that X^+ (defined for $\mathbb{G}_m \subseteq \mathbb{A}^1_k$) is a disjoint sum of finitely many affine spaces (often called *cells*) corresponding to fixed points of the action. This also follows from classical Białynicki-Birula result ([BB73, Theorem 4.3]).
Assume now that X is equipped with two commuting actions of $\mathbb{G}_m$. Denote them by a_i for $i = 1,2$. Suppose that a_1 and a_2 admit the same fixed point locus $F \subseteq X$ which consists of finitely many points. Let x_0 be a point of F. Next suppose that W_i for $i = 1,2$ is a cell over x_0 of the Białynicki-Birula decomposition (defined for $\mathbb{G}_m \subseteq \mathbb{A}^1_k$) with respect to a_i. Then as we noted above W_i is isomorphic (as k-scheme) with an affine space. According to [JS19, Proposition 7.6] we may view W_i as a locally closed subscheme of X. We are going to prove that the intersection $W_1 \cap W_2$ is also isomorphic to an affine space. For this consider the action $a : \mathbb{G}_m \times_k \mathbb{G}_m \to X$ induced by a_1, a_2 and apply the Białynicki-Birula decomposition with respect to the monoid $\mathbb{A}^2_k$ which contains (in a canonical way) $\mathbb{G}_m \times_k \mathbb{G}_m$ as its group of units. Let W be a cell of this Białynicki-Birula decomposition corresponding to fixed point x_0 of $\mathbb{G}_m \times_k \mathbb{G}_m$. Similarly as above, from Theorems A and B we deduce that W is an affine space. Therefore, it suffices to prove that

$$W = W_1 \cap W_2$$

By definition W represents the functor

$$\mathcal{D}_1(Y) = \left\{\gamma : \mathbb{A}^2 \times_k Y \to X \,|\, \gamma \text{ is } \mathbb{G}_m \times_k \mathbb{G}_m\text{-equivariant and } \gamma\left(\{(0,0)\} \times_k Y\right) = \{x_0\}\right\}$$

Moreover note that $W_1 \cap W_2 = W_1 \times_X W_2$ and hence $W_1 \cap W_2$ represents the functor

$$\mathcal{D}_2(Y) = \left\{\gamma : \left(\mathbb{A}^2 \setminus \{(0,0)\}\right) \times_k Y \to X \;\middle|\; \begin{array}{l} \gamma \text{ is } \mathbb{G}_m \times_k \mathbb{G}_m\text{-equivariant and} \\ \gamma(\{(0,1)\} \times_k Y) = \{x_0\} = \gamma(\{(1,0)\} \times_k Y) \end{array} \right\}$$

Consider $\gamma \in \mathcal{D}_2(Y)$. Let U be an open affine and $\mathbb{G}_m \times_k \mathbb{G}_m$-stable neighborhood of x_0 (it exists according to the classical result of Sumihiro [CLS11, Theorem 3.1.7]). Since $\gamma(\{0\} \times_k \mathbb{A}^1_k \times_k Y) = \{x_0\} = \gamma(\mathbb{A}^1_k \times_k \{0\} \times_k Y)$, we deduce that γ factors through U. Next we have a cocartesian (pushout) square

$$\begin{array}{ccc} \mathbb{G}_m \times_k \mathbb{G}_m & \hookrightarrow & \mathbb{A}^1_k \times_k \mathbb{G}_m \\ \downarrow & & \downarrow \\ \mathbb{G}_m \times_k \mathbb{A}^1_k & \hookrightarrow & \mathbb{A}^2_k \end{array}$$

in the category of affine k-schemes with actions of $\mathbb{G}_m \times_k \mathbb{G}_m$. Hence we can extend γ uniquely to a morphism $\overline{\gamma} : \mathbb{A}^2_k \times_k Y \to U$. Thus there exists the unique morphism $\tilde{\gamma} : \mathbb{A}^2_k \times_k Y \to X$ which extends γ. This proves that functors $\mathcal{D}_1$ and $\mathcal{D}_2$ are isomorphic over X. Thus $W = W_1 \cap W_2$.

Note that the assumption that the cells W_1 and W_2 correspond to the same fixed point is essential. Indeed, consider the projective line $\mathbb{P}^1_k$ with two actions of $\mathbb{G}_m$ given by formulas $a_1\big(t, [x_0, x_1]\big) = [tx_0, x_1]$ and $a_2\big(t, [x_0, x_1]\big) = [t^{-1}x_0, x_1]$. These two actions commute and their schemes of fixed points coincide. Then the cells for a_1 are

$$\mathbb{A}^1_k \cong \big\{[x_0, x_1] \in \mathbb{P}^1_k \,|\, x_1 \neq 0\big\},\ \big\{[1, 0]\big\}$$

and the cells for a_2 are

$$\mathbb{A}^1_k \cong \big\{[x_0, x_1] \in \mathbb{P}^1_k \,|\, x_0 \neq 0\big\},\ \big\{[0, 1]\big\}$$

The intersection of an a_1-cell $\big\{[x_0, x_1] \in \mathbb{P}^1_k \,|\, x_1 \neq 0\big\}$ corresponding to $[0,1]$ and an a_2-cell $\big\{[x_0, x_1] \in \mathbb{P}^1_k \,|\, x_0 \neq 0\big\}$ corresponding to $[1, 0]$ is isomorphic to $\mathbb{G}_m$ as a k-scheme. Hence it is not an affine space.

This example is extracted from [JS19, Example 7.9], where we investigate structural properties of decompositions induced by monomial orderings on Hilbert schemes; the conclusion of the example was not known even in that special case.

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