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Weakly Radon-Nikodym Boolean algebras

Joint work with

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Based on

(i) *Weakly Radon-Nikodym Boolean algebras and independent sequences*, Fund. Math. 241 (2018).

(ii) *Abundance of independent sequences in compact spaces and Boolean algebras*, in preparation.

Such a name?

(S, Σ)

Banach spaces with the Radon-Nikodym property

$X,$ $\mu : \Sigma \to X$, $\lambda : \Sigma \to \mathbb{R}_+$, $\mu \ll \lambda$

$\exists f : S \to X$ $\mu(\cdot) = \int_{(\cdot)} f \, d\lambda$

Stegall: X^* has RNP if and only if X is Asplund.

X Asplund $\Leftrightarrow$ $\forall Y \underset{\text{sep}}{\subseteq} X$ Y^* sep.

Definition (Namioka). K is Radon-Nikodym compact if $K \hookrightarrow (X^*, weak^*)$ for some Asplund space X.

Definition (Glasner & Megrilishvili). K is weakly Radon-Nikodym compact if $K \hookrightarrow (X^*, weak^*)$ where X does not contain ℓ_1.

There is also WRNP of Banach spaces: X^* has WRNP iff X does not contain ℓ_1 (Janicka, Musiał).

WRN Boolean algebras

Definition'. $\mathcal{A}$ is **WRN** if its Stone space $ult(\mathcal{A})$ is weakly Radon-Nikodym compact.

Definition. $\mathcal{A} \in$ **WRN** if $\mathcal{A} = \langle \mathcal{G} \rangle$, where $\mathcal{G} = \bigcup_n \mathcal{G}_n$ and no $\mathcal{G}_n$ contains an infinite independent sequence.

$a_1, \dots a_n \in \mathcal{A}$ is independent if
$a_1^{\varepsilon_1} \cap \dots \cap a_n^{\varepsilon_n} \neq 0$ for $\varepsilon_i \in \{0,1\}$

Theorem. $\mathcal{A} \in$ **WRN** iff $\mathcal{A} = \bigcup_n \mathcal{F}_n$ and no $\mathcal{F}_n$ contains an infinite independent sequence.

Corollary.

(1) The class **WRN** is hereditary.

(2) No algebra from **WRN** contains an uncountable independent family.

Remark. It follows that 0-dim. image of **WRN** compact is again in **WRN**.

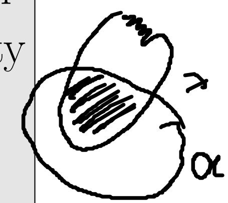
The class of Radon-Nikodym compacta is not closed under continuous images (Avilés & Koszmider) and nor is the class of weakly Radon-Nikodym compacta (Martínez-Cervantes).

Minimally generated Boolean algebras

Definition (Koppelberg). $\mathcal{A} \subseteq \mathcal{B}$ is a minimal extension if $\mathcal{B}$ is generated by $\mathcal{A}$ and some x having the property

$$(\forall a \in \mathcal{A})\, a \cap x \in \mathcal{A} \text{ or } a^c \cap x \in \mathcal{A}.$$

$\mathcal{A}$ is minimally generated if $\mathcal{A} = \bigcup_{\xi<\kappa} \mathcal{A}_\xi$ and every $\mathcal{A}_{\xi+1}$ is a minimal extension of $\mathcal{A}_\xi$.



$\mathcal{A}_0 = \{0,1\}$

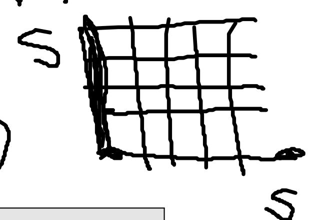
$K_0 \leftarrow K_1 \; \ldots \; K_\alpha \leftarrow K_{\alpha+1}$

K_α

The classes of min. generated algebras and **WRN** algebras share some properties:

- They contain interval algebras;
- contain no uncountable independent families.

interval algebra = an algebra generated by a chain!



$\mathcal{A} \otimes \mathcal{A} = clop(S \times S)$

Example. $\mathcal{A} = \langle\{(0,t) : t \in (0,1)\}\rangle$, $S = ult(\mathcal{A})$.
Then $\mathcal{A}$ is an interval algebra (so it is minimally generated) but $\cancel{\mathcal{A} \otimes \mathcal{A} = ult(S \times S)}$ is not minimally generated (Koppelberg). Clearly $\mathcal{A} \otimes \mathcal{A} \in$ **WRN**.

Theorem. There is a minimally generated algebra outside **WRN**.

Convering sequences

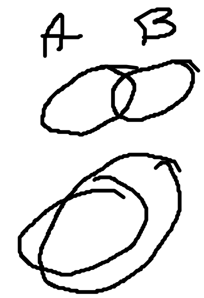
Problem (Haydon). Assume $\mathcal{A} \in \mathbf{WRN}$; does $ult(\mathcal{A})$ contain a converging sequence?

Remark. If $\mathcal{A}$ contains no uncountable independent family and $ult(\mathcal{A})$ has no converging sequences then $ult(\mathcal{A})$ is an Efimov space.

Theorem (Dow & Pichardo-Mendoza). Under CH there is a minimally generated $\mathcal{A}$ such that $ult(\mathcal{A})$ contains no converging sequences (so is an Efimov space).)

Towards positive answer to Haydon's problem

Theorem. Suppose that $\mathcal{A} = \langle \mathcal{G} \rangle$, where $\mathcal{G} = \bigcup_{n}^{\infty} \mathcal{G}_n$ and $\mathcal{G}_n$ contains no $n+1$ independent elements ($\mathcal{A} \in$ **UWRN**). Then $ult(\mathcal{A})$ is sequentially compact.



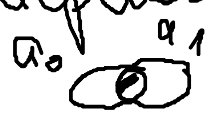
Example (Haydon). Take a family $\mathcal{G} \subseteq P(\omega)$ which is maximal with respect to the property: $(\forall A, B \in \mathcal{G})$

$$\text{if } A \neq B \text{ then one of } A \cap B, A \setminus B, B \setminus A \text{ is finite.}$$

Write $\mathcal{A} = \langle \mathcal{G} \rangle$; then $ult(\mathcal{A})$ is not sequentially compact. In particular, $\mathcal{A} \in$ **WRN** $\setminus$ **UWRN**.

$ult(\mathcal{A}) \supseteq \omega$, $0, 1, 2, \dots$ has no converging subsequence.

$\mathcal{G}$ contains no infinite independent sequence. $a_0, a_1, \dots \in \mathcal{A}$



$\mathcal{A} \in$ Eberlein $\Longleftrightarrow$ ult$(\mathcal{A})$ is Eberlein compact.

WRN and Eberlein

Definition. $\mathcal{A} \in$ **WRN** iff $\mathcal{A} = \langle \mathcal{G} \rangle$, where $\mathcal{G} = \bigcup_n \mathcal{G}_n$ and no $\mathcal{G}_n$ contains an infinite independent sequence.

Theorem. $\mathcal{A} \in$ **Eberlein** iff $\mathcal{A} = \langle \mathcal{G} \rangle$, where $\mathcal{G} = \bigcup_n \mathcal{G}_n$ and no $\mathcal{G}_n$ contains an infinite centered sequence.

WRN / independent $\simeq$ Eberlein / centered.

Definition.

- For a class $\mathfrak{C}$ of Boolean algebras, say that $\mathcal{A} \in \mathfrak{C}^{\perp}$ if $\mathcal{B} \subseteq \mathcal{A}, \mathcal{B} \in \mathfrak{C} \longrightarrow \mathcal{B}$ is countable
- For a class $\mathfrak{C}$ of compacta, say that $K \in \mathfrak{C}^{\perp}$ if every continuous image $L \in \mathfrak{C}$ of K is metrizable.

Fact. **Eberlein**$^{\perp}$ = **ccc**.

Remark. $\mathcal{A} \in$ **ccc** iff every uncountable subfamily of $\mathcal{A}$ contains an infinite centered sequence.

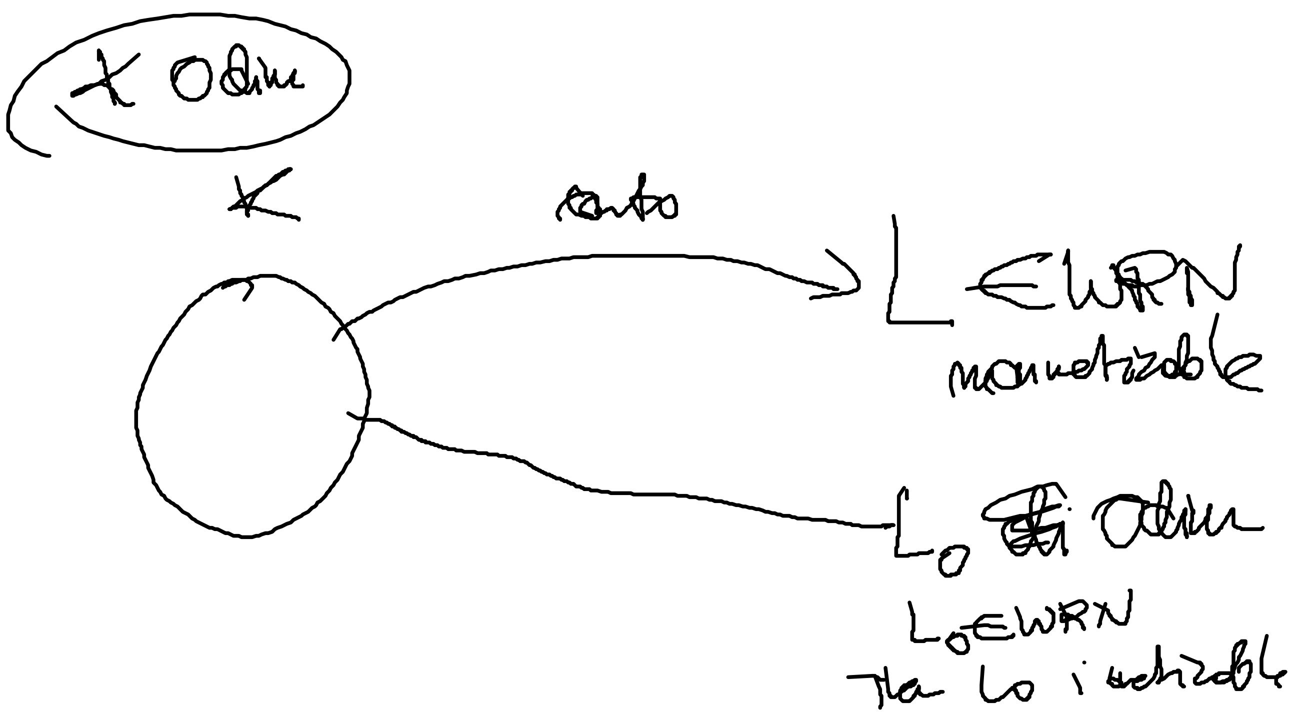
ccc $\Longleftrightarrow$ precaliber(ω_1, ω)

Theorem. $\mathcal{A} \in$ **WRN**$^{\perp}$ iff every uncountable subfamily of $\mathcal{A}$ contains an infinite ~~centered~~ sequence.

independent.

Boolean examples.
Every subalgebra of a free algebras is in $\mathbf{WRN}^\perp$.
If $\mathrm{MA}(\omega_1)$ does not hold then there is a nonmetrizable 0-dim. Corson compact space K such that $\mathrm{Clop}(K) \in \mathbf{WRN}^\perp$.

Compact examples.
Every dyadic space is in $\mathbf{WRN}^\perp$.



Theorem. There is a compact 0-dim. space K such that $\mathrm{Clop}(K) \in \mathbf{WRN}^\perp$ but $K \notin \mathbf{WRN}^\perp$.