Abstract colorings, games and ultrafilters

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Colorings of $\mathbb{N}$

A coloring of $\mathbb{N}$ is a function $\chi\colon \mathbb{N} \to \{1, 2, \ldots, k\}$

$$\mathbb{N} = \underbrace{C_1 \cup C_2 \cup \cdots \cup C_k}_{\text{pairwise disjoint}}$$

Theorem (Theorems)

For each coloring of $\mathbb{N}$

- Schur 1916:
  there is a monochromatic set $\{a, b, a+b\}$
- van der Waerden 1927:
  for each n, there is a monochromatic arithmetic progression of length n
- Deuber 1973:
  for all $m, p, c \in \mathbb{N}$, there is a monochromatic (m, p, c)-set

  $m, p, c \in \mathbb{N}$, $x \in \mathbb{N}^m$

  $$S(m,p,c,x) := \left\{ cx_t + \sum_{i=t+1}^{m+1} \lambda_i x_i : t \in \{1, \ldots, m\}, (\forall\, i \in \{t+1, \ldots, m+1\})(|\lambda_i| \le p) \right\}$$

$$S(2, \mathbf{2}, \mathbf{1}, x) = \{\mathbf{1}x_1 + \mathbf{0}x_2,\ \mathbf{1}x_1 + \mathbf{1}x_2,\ \mathbf{1}x_1 + \mathbf{2}x_2,\ \mathbf{1}x_1 - \mathbf{2}x_2,\ \mathbf{1}x_1 - \mathbf{1}x_2,\ \mathbf{1}x_2\} =$$
$$= \{x_1,\ x_1 + x_2,\ x_1 + 2x_2, x_1 - 2x_2, x_1 - x_2, x_2\}$$

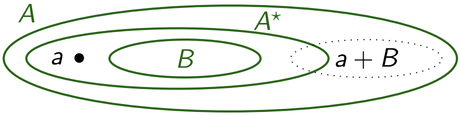
Semigroup $\beta\mathbb{N}$

- $\beta\mathbb{N}$: all ultrafilters on $\mathbb{N}$
- Basic open sets $[A] := \{ p \in \beta\mathbb{N} : A \in p \}$, $A \subseteq \mathbb{N}$
- $\beta\mathbb{N} \supseteq \mathbb{N}$: identify x with $\{ A \subseteq \mathbb{N} : x \in A \}$
- Extend $+ \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, to $+ \colon \beta\mathbb{N} \times \beta\mathbb{N} \to \beta\mathbb{N}$ such that:
  - for each $x \in \mathbb{N}$ the function $q \mapsto x + q$ is continuous
  - for each $q \in \beta\mathbb{N}$ the function $p \mapsto p + q$ is continuous
  - $+$ is associative on $\beta\mathbb{N}$
- $(\beta\mathbb{N}, +)$ is a compact right-topological semigroup

$$A \in p + q \longleftrightarrow \{ x \in \mathbb{N} : (\exists B \in q)(x + B \subseteq A) \} \in p$$

- $\beta\mathbb{N} \ni e$ is idempotent: $e + e = e$

$$(\forall A \in e)(\exists A^\star \in e)(\forall a \in A^\star)(\exists B \in e)(a + B \subseteq A)$$



Hindman's Theorem

Lemma (Numakura 1952)

Every nonempty compact right-topological semigroup has an idempotent.

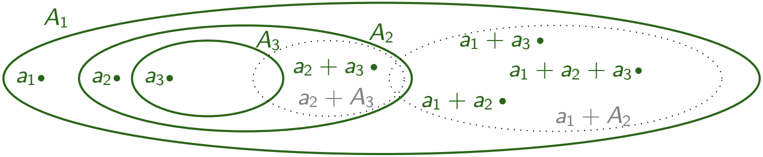
$a_1, a_2, \ldots \in \mathbb{N}$, $F = \{i_1, \ldots, i_n\}$ increasing enumeration

$$a_F := a_{i_1} + \cdots + a_{i_n} \qquad \mathrm{FinSum}(a_1, a_2, \ldots) := \{\, a_F : F \in \mathrm{Fin}(\mathbb{N}) \,\}$$

Theorem (Hindman 1974)

For each coloring of $\mathbb{N}$, there is a sequence $a_1, a_2, \ldots \in \mathbb{N}$ such that $\mathrm{FinSum}(a_1, a_2, \ldots)$ is monochromatic.

- Pick an idempotent $e \in \beta\mathbb{N}$ and a monochromatic $A_1 \in e$



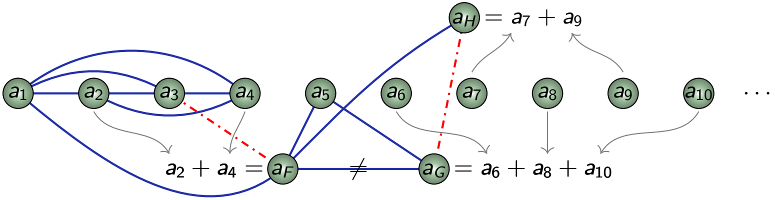
- $a_{i_1} + a_{i_2} + \cdots + a_{i_m} \in A_{i_1}$ for $i_1 < i_2 < \cdots < i_m$

Colorings of graphs

Theorem (Ramsey 1930)

For each coloring of $[\mathbb{N}]^2$, there is an infinite set $A \subseteq \mathbb{N}$ such that $[A]^2$ is monochromatic.

- $a_1, a_2, \ldots \in \mathbb{N}$ is proper: $a_F \neq a_G$ for all $F, G \in \mathrm{Fin}(\mathbb{N})$ with $F < G$
- sumgraph of $\underbrace{a_1, a_2, \ldots}_{\text{proper}}$: $\{\, \{a_F, a_G\} : F, G \in \mathrm{Fin}(\mathbb{N}) \text{ with } F < G \,\}$



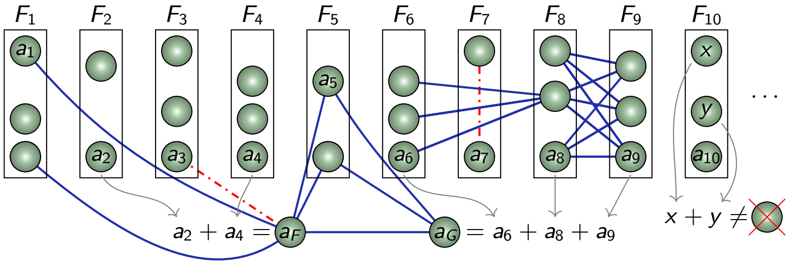
Theorem (Milliken 1975, Taylor 1976) $(\mathbb{N}, +)$

For each coloring of $[\mathbb{N}]^2$, there is a proper sequence $a_1, a_2, \ldots \in \mathbb{N}$ whose sumgraph is monochromatic.

Colorings of graphs

- partite graph of $\underbrace{F_1, F_2, \ldots}_{\text{pairwise disjoint}} \in \mathrm{Fin}(\mathbb{N})$: $\{ \{a_i, a_j\} : a_i \in F_i, a_j \in F_j, i \neq j \}$
- partite sumgraph of $F_1, F_2, \ldots \in \mathrm{Fin}(\mathbb{N})$, all sequences in $F_1 \times F_2 \times \cdots$ are proper

$$\{ \{a_F, a_G\} : (a_1, a_2, \ldots) \in F_1 \times F_2 \times \cdots \text{ and } F, G \in \mathrm{Fin}(\mathbb{N}) \text{ with } F < G \}$$



Colorings of graphs

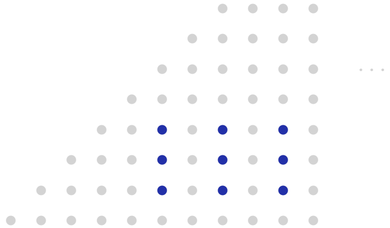
$$S(m,p,c,x) := \left\{ cx_t + \sum_{i=t+1}^{m+1} \lambda_i x_i : t \in \{1,\ldots,m\}, (\forall\, i \in \{t+1,\ldots,m+1\})(|\lambda_i| \le p) \right\}$$

$$S(2,2,1,x) = \{x_1, x_1+x_2, x_1+2x_2, x_2, x_1-x_2, x_1-2x_2\}$$

Theorem (Bergelson–Hindman 1988) $(\mathbb{N},+)$

Let $\mathcal{R}_1, \mathcal{R}_2, \ldots$ be an enumeration of all families of (m,p,c)-sets. For each coloring of $[\mathbb{N}]^2$, there are sets $R_1 \in \mathcal{R}_1, R_2 \in \mathcal{R}_2, \ldots$ such that the partite sumgraph of $R_1, R_2, \ldots$ is monochromatic.

- $[\mathbb{N}]^2 = \{ (a,b) \in \mathbb{N}^2 : a > b \}$
- there is a monchromatic set M such that for each n, there are arithmetic progressions $A_1, A_2 \subseteq \mathbb{N}$ of length n with $A_1 \times A_2 \subseteq M$

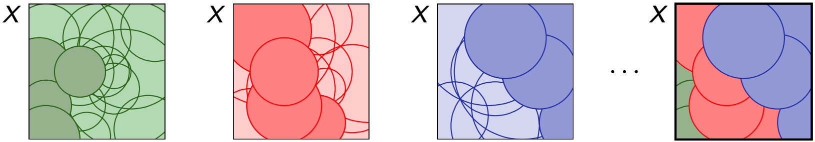


Colorings of covers

- $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$:

$$(\forall\, A_1, A_2, \ldots \in \mathcal{A})(\,\exists\, \text{finite } F_1 \subseteq A_1, F_2 \subseteq A_2, \ldots)(\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{B})$$

- O : all countable open covers of X
- $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$:



$\mathcal{F}_1 \subseteq \mathcal{U}_1$ $\mathcal{F}_2 \subseteq \mathcal{U}_2$ $\mathcal{F}_3 \subseteq \mathcal{U}_3$

- ω-cover: $(\forall\, \text{finite } F \subseteq X\,)(\,\exists\, U \in \mathcal{U} \setminus \{X\}\,)(F \subseteq U)$
- Ω: all countable ω-covers of X
- λ-cover: $(\forall\, x \in X\,)(\{\, U \in \mathcal{U} : x \in U\,\}$ is infinite$)$
- Λ: all countable λ-covers of X

Colorings of covers

Theorem (Scheepers 1999)

If X is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, then for every $\mathcal{U}\in\Omega$ and a coloring of $[\tau]^2$, there are finite sets $\mathcal{F}_1,\mathcal{F}_2,\dots\subseteq\mathcal{U}$ such that

- *$\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\in\Lambda$ and the partite graph of $\mathcal{F}_1,\mathcal{F}_2,\dots$ is monochromatic.*

Theorem (Scheepers 1996)

If X is $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$, then for every $\mathcal{U}\in\Omega$ and a coloring of $[\tau]^2$, there are finite sets $\mathcal{F}_1,\mathcal{F}_2,\dots\subseteq\mathcal{U}$ such that

- *$\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\in\Omega$ and the partite graph of $\mathcal{F}_1,\mathcal{F}_2,\dots$ is monochromatic.*

- Y has countable fan tightness:

$$(\forall\, A_1,A_2,\dots\subseteq Y, y\in\bigcap_{n\in\mathbb{N}}\overline{A_n})(\exists\ \text{fin}\ F_1\subseteq A_1,F_2\subseteq A_2,\dots)(y\in\overline{\bigcup_{n\in\mathbb{N}}F_n})$$

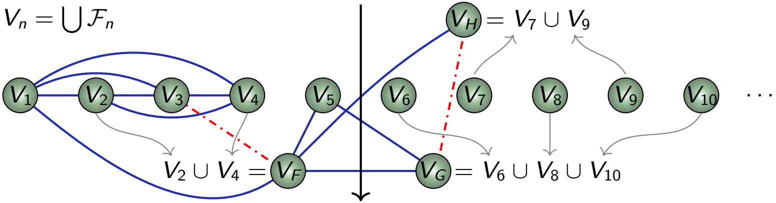
- Just, Miller, Scheepers, Szeptycki 1996: X is $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)\leftrightarrow X$ is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ in all finite powers $\leftrightarrow \mathsf{C_p}(X)$ has countable fan tightness

Colorings of covers

Theorem (Tsaban 2018) $(\tau, \cup)$

If X is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, then for each decreasing sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Lambda$ such that $\mathcal{U}_1$ has no finite subcover and a coloring of $[\tau]^2$, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that

- $\bigcup \mathcal{F}_n \in \Lambda$ and the sumgraph of $\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \ldots$ is monochromatic.



Theorem (Milliken 1975, Taylor 1976) $(\mathbb{N}, +)$

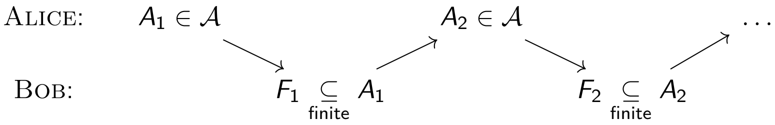
For each coloring of $[\mathbb{N}]^2$, there is a proper sequence $a_1, a_2, \ldots \in \mathbb{N}$ whose sumgraph is monochromatic.

Topological games

- $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$:

$$(\forall A_1, A_2, \ldots \in \mathcal{A})(\exists \text{finite } F_1 \subseteq A_1, F_2 \subseteq A_2, \ldots)(\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{B})$$

- $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$



If $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{B}$, then BOB wins. Otherwise, ALICE wins.

- BOB has a winning strategy in $\mathsf{G}_{\mathrm{fin}}([\mathbb{N}]^\infty, [\mathbb{N}]^\infty)$
- If X is σ-compact, then BOB has a winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$

Theorem (Hurewicz 1925)

X is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ iff ALICE has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.

Semigroup βS

- βS: all ultrafilters on S
- Basic open sets $[A] := \{ p \in \beta S : A \in p \}$, $A \subseteq S$
- $\beta S \supseteq S$: identify x with $\{ A \subseteq S : x \in A \}$
- Extend $+ \colon S \times S \to S$, to $+ \colon \beta S \times \beta S \to \beta S$ such that:
  - for each $x \in S$ the function $q \mapsto x + q$ is continuous
  - for each $q \in \beta S$ the function $p \mapsto p + q$ is continuous
  - $+$ is associative on βS
- $(\beta S, +)$ is a compact right-topological semigroup

$$A \in p + q \longleftrightarrow \{ x \in S : (\exists B \in q)(x + B \subseteq A) \} \in p$$

- $\beta S \ni e$ is idempotent: $e + e = e$

$$(\forall A \in e)(\exists A^\star \in e)(\forall a \in A^\star)(\exists B \in e)(a + B \subseteq A)$$

Lemma (Numakura 1952)

Every nonempty compact right-topological semigroup has an idempotent.

Superfilters and idempotents

- $[S]^\infty \supseteq \mathcal{A}$ is a superfilter on S:
  - $\mathcal{A} \ni A \subseteq B \longrightarrow B \in \mathcal{A}$
  - $A \cup B \in \mathcal{A} \longrightarrow A \in \mathcal{A}$ or $B \in \mathcal{A}$
- $[S]^\infty$
- every ultrafilter
- Ω
- $\{ A \in [X]^\infty : \mathbf{x} \in \overline{A} \}$

Lemma (Tsaban 2018)

Let $a_1, a_2, \dots \in S$ be proper and $\mathcal{A}$ be a translation invariant superfilter on S.

$$\{ p \in \beta S : \{ \operatorname{FinSum}(a_n, a_{n+1}, \dots) : n \in \mathbb{N} \} \subseteq p \subseteq \mathcal{A} \}$$

is a closed and nonempty subsemigroup of $(\beta S, +)$.

- $S = (\mathbb{N}, +)$, $\mathcal{A} = [\mathbb{N}]^\infty$
- $\Omega \ni \mathcal{U} = \{ U_1, U_2, \dots \}$, $S = (\mathcal{U}, \max)$, $\mathcal{A} = \{ \mathcal{V} \in \Omega : \mathcal{V} \subseteq \mathcal{U} \}$,

$$\mathcal{A} \ni \mathcal{V} \to \{ \max\{B, V\} : V \in \mathcal{V} \} \in \mathcal{A}$$

- $\Omega \ni \mathcal{U}$ with no finite subcover and closed under $\cup$, $S = (\mathcal{U}, \cup)$, $\mathcal{A} = \{ \mathcal{V} \in \Omega : \mathcal{V} \subseteq \mathcal{U} \}$

$$\mathcal{A} \ni \mathcal{V} \to \{ B \cup V : V \in \mathcal{V} \} \in \mathcal{A}$$

Superfilters and idempotents

- $\beta S \ni p$ is large for $\emptyset \neq \mathcal{R} \subseteq \mathrm{Fin}(S)$: $(\forall A \in p)(\exists R \in \mathcal{R})(R \subseteq A)$
- There is a large $p \in \beta S$ for $\emptyset \neq \mathcal{R} \subseteq \mathrm{Fin}(S)$ iff for each coloring of S, there is a monochromatic set in $\mathcal{R}$

Lemma (Deuber–Hindman 1987)

The set

$$\{\, p \in \beta N : [\mathbb{N}]^\infty \supseteq p \text{ is large for each family of } (m,p,c)\text{-sets} \,\}$$

is a closed and nonempty subsemigroup of $(\beta\mathbb{N}, +)$

Lemma (Tsaban 2018)

Let $a_1, a_2, \ldots \in S$ *be proper and* $\mathcal{A}$ *be a translation invariant superfilter on* S. *The set*

$$\left\{ p \in \beta S : \mathcal{A} \supseteq p \text{ is large for } \{\, \{x\} : x \in \mathrm{FinSum}(a_n, a_{n+1}, \ldots) \,\}, n \in \mathbb{N} \right\}$$

is a closed and nonempty subsemigroup of $(\beta S, +)$.

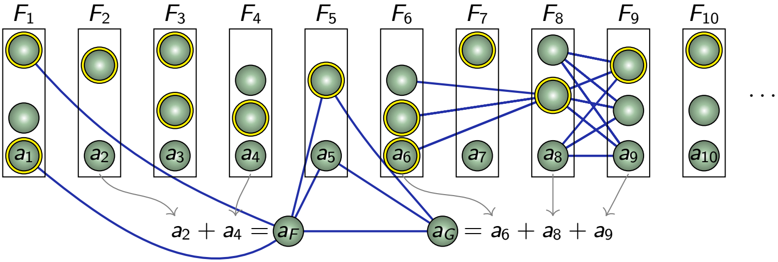
The main result

Theorem (Sz 2020)

- *Assume that* $\mathcal{A}, \mathcal{B} \subseteq [S]^\infty$, $\mathcal{B}$ *is closed under supersets in* $[S]^\infty$,
- *there is an idempotent* $e \subseteq \mathcal{A}$, *large for* $\mathcal{R}_1, \mathcal{R}_2, \ldots \subseteq \mathrm{Fin}(S)$,
- ALICE *has no winning strategy in* $\mathrm{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$.

For each coloring of $[S]^2$, *there are sets* $F_1, F_2, \ldots \in \mathrm{Fin}(S)$ *such that*

- $\mathcal{R}_n \ni R_n \subseteq F_n \subseteq \bigcup \mathcal{R}_n$ *and* $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{B}$,
- *the partite sumgraph of* $F_1, F_2, \ldots$ *is monochromatic.*



Applications

Theorem (Bergelson–Hindman 1988) $(\mathbb{N},+)$

Let $\mathcal{R}_1, \mathcal{R}_2, \dots$ be an enumeration of all families of (m,p,c)-sets. For each coloring of $[\mathbb{N}]^2$, there are sets $R_1 \in \mathcal{R}_1, R_2 \in \mathcal{R}_2, \dots$ such that

- the partite sumgraph of $R_1, R_2, \dots$ is monochromatic.

Theorem (Sz 2020)

- *Assume that $\mathcal{A}, \mathcal{B} \subseteq [S]^\infty$, $\mathcal{B}$ is closed under supersets in $[S]^\infty$,*

  $S = (\mathbb{N},+)$, $\mathcal{A} = \mathcal{B} = [\mathbb{N}]^\infty$
- *there is an idempotent $e \subseteq \mathcal{A}$, large for $\mathcal{R}_1, \mathcal{R}_2, \dots \subseteq \mathrm{Fin}(S)$*

  Deuber–Hindman + Numakura
- *Alice has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$.*

  Bob has a winning strategy in $\mathsf{G}_{\mathrm{fin}}([\mathbb{N}]^\infty, [\mathbb{N}]^\infty)$

For each coloring of $[S]^2$, there are sets $F_1, F_2, \dots \in \mathrm{Fin}(S)$ such that

- $\mathcal{R}_n \ni R_n \subseteq F_n \subseteq \bigcup \mathcal{R}_n$ *and* $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{B}$,
- *the partite sumgraph of $F_1, F_2, \dots$ is monochromatic.*

Applications

Theorem (Scheepers 1996)

If X is $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$, then for every $\mathcal{U}\in\Omega$ and a coloring of $[\tau]^2$, there are sets $\mathcal{F}_1,\mathcal{F}_2,\dots\in\mathrm{Fin}(\mathcal{U})$ such that

- $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\in\Omega$ *and the partite graph of $\mathcal{F}_1,\mathcal{F}_2,\dots$ is monochromatic.*

Theorem (Sz 2020)

- *Assume that $\mathcal{A},\mathcal{B}\subseteq[S]^\infty$, $\mathcal{B}$ is closed under supersets in $[S]^\infty$,*
  $\Omega\ni\mathcal{U}=\{U_1,U_2,\dots\}$, $S=(\mathcal{U},\max)$, $\mathcal{A}=\{\mathcal{V}\in\Omega:\mathcal{V}\subseteq\mathcal{U}\}$, $\mathcal{B}=\Omega$
- *there is an idempotent $e\subseteq\mathcal{A}$, large for $\mathcal{R}_1,\mathcal{R}_2,\dots\subseteq\mathrm{Fin}(S)$,*
  $\mathcal{R}_n=\{\{U_n\},\{U_{n+1}\},\dots\}$, $\bigcup\mathcal{R}_n=\{U_n,U_{n+1},\dots\}=\mathsf{FinSum}(U_n,U_{n+1},\dots)$
- Alice *has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$.*
  X *is* $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)\to$ Alice *has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Omega,\Omega)$

For each coloring of $[S]^2$, there are sets $F_1,F_2,\dots\in\mathrm{Fin}(S)$ such that

- $\mathcal{R}_n\ni R_n\subseteq F_n\subseteq\bigcup\mathcal{R}_n$ *and* $\bigcup_{n\in\mathbb{N}}F_n\in\mathcal{B}$, $\quad\mathcal{F}_n\subseteq\bigcup\mathcal{R}_n=\{U_n,U_{n+1},\dots\}\subseteq\mathcal{U}$
- *the partite sumgraph of $F_1,F_2,\dots$ is monochromatic.*

Applications

Theorem (Scheepers 1999)

If X is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, then for every $\mathcal{U}\in\Omega$ and a coloring of $[\tau]^2$, there are sets $\mathcal{F}_1,\mathcal{F}_2,\ldots\in\mathrm{Fin}(\mathcal{U})$ such that

- $\bigcup_{n\in\mathbb{N}}\mathcal{F}_n\in\Lambda$ *and the partite graph of* $\mathcal{F}_1,\mathcal{F}_2,\ldots$ *is monochromatic.*

Theorem (Sz 2020)

- *Assume that* $\mathcal{A},\mathcal{B}\subseteq[S]^\infty$, $\mathcal{B}$ *is closed under supersets in* $[S]^\infty$,
  $\Omega\ni\mathcal{U}=\{U_1,U_2,\ldots\}$, $S=(\mathcal{U},\max)$, $\mathcal{A}=\{\mathcal{V}\in\Omega:\mathcal{V}\subseteq\mathcal{U}\}$, $\mathcal{B}=\Lambda$
- *there is an idempotent* $e\subseteq\mathcal{A}$, *large for* $\mathcal{R}_1,\mathcal{R}_2,\ldots\subseteq\mathrm{Fin}(S)$
  $\mathcal{R}_n=\{\{U_n\},\{U_{n+1}\},\ldots\}$, $\bigcup\mathcal{R}_n=\{U_n,U_{n+1},\ldots\}=\mathsf{FinSum}(U_n,U_{n+1},\ldots)$
- $\textsc{Alice}$ *has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$.
  X *is* $\mathsf{S}_{\mathrm{fin}}(\Omega,\Lambda)\to\textsc{Alice}$ *has no win strategy in* $\mathsf{G}_{\mathrm{fin}}(\Omega,\Lambda)$, *also in* $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\Lambda)$

For each coloring of $[S]^2$, *there are sets* $F_1,F_2,\ldots\in\mathrm{Fin}(S)$ *such that*

- $\mathcal{R}_n\ni R_n\subseteq F_n\subseteq\bigcup\mathcal{R}_n$ *and* $\bigcup_{n\in\mathbb{N}}F_n\in\mathcal{B}$ $\quad\mathcal{F}_n\subseteq\bigcup\mathcal{R}_n=\{U_n,U_{n+1},\ldots\}\subseteq\mathcal{U}$
- *the partite sumgraph of* $F_1,F_2,\ldots$ *is monochromatic*

Applications

Theorem (Scheepers 1999)

If $Y = \mathrm{C_p}(X)$ has countable fan tightness, then for every $A \subseteq Y$ with $\mathbf{0} \in \overline{A}$ and a coloring of $[\mathrm{P}(Y)]^2$, there are finite sets $F_1, F_2, \ldots \subseteq A$ such that

- $\mathbf{0} \in \overline{\bigcup\{ F_n : n \in \mathbb{N} \}}$ *and the partite graph of $F_1, F_2, \ldots$ is monochromatic.*

Theorem (Sz 2020)

- *Assume that $\mathcal{A}, \mathcal{B} \subseteq [S]^\infty$, $\mathcal{B}$ is closed under supersets in $[S]^\infty$,*
  $A = \{a_1, a_2, \ldots\}$, $S = ([A]^\infty, \max)$, $\mathcal{A} = \mathcal{B} = \{ B \in [A]^\infty : \mathbf{0} \in \overline{B} \}$
- *there is an idempotent $e \subseteq \mathcal{A}$, large for $\mathcal{R}_1, \mathcal{R}_2, \ldots \subseteq \mathrm{Fin}(S)$*
  $\mathcal{R}_n = \{\{a_n\}, \{a_{n+1}\}, \ldots\}$, $\bigcup \mathcal{R}_n = \{a_n, a_{n+1}, \ldots\} = \mathrm{FinSum}(a_n, a_{n+1}, \ldots)$
- ALICE *has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$.*
  Y has countable fan tightness $\to$ ALICE has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{A})$

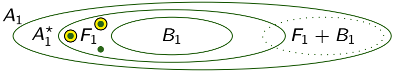
For each coloring of $[S]^2$, there are $F_1, F_2, \ldots \in \mathrm{Fin}(S)$ such that

- $\mathcal{R}_n \ni R_n \subseteq F_n \subseteq \bigcup \mathcal{R}_n$ *and* $\bigcup_{n \in \mathbb{N}} F_n \in \mathcal{B}$, $\quad F_n \subseteq \bigcup \mathcal{R}_n = \{a_n, a_{n+1}, \ldots\} \subseteq A$
- *the partite sumgraph of $F_1, F_2, \ldots$ is monochromatic*

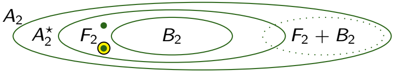
Theorem (Sz 2020)

Assume that $\mathcal{A}, \mathcal{B} \subseteq [S]^{\infty}$, $\mathcal{B}$ is closed under supersets in $[S]^{\infty}$, there is an idempotent $e \subseteq \mathcal{A}$, large for $\mathcal{R}_1, \mathcal{R}_2, \ldots \subseteq \mathrm{Fin}(S)$, and Alice has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$. For each coloring of $[S]^2$, there are sets $F_1, F_2, \ldots \in \mathrm{Fin}(S)$ such that $\mathcal{R}_n \ni R_n \subseteq F_n \subseteq \bigcup \mathcal{R}_n$ and $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{B}$, the partite sumgraph of $F_1, F_2, \ldots$ is monochromatic.

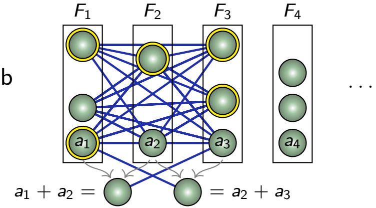
- $\{\, t \in S : \{s,t\}$ is red $\} \cup \{\, t \in S : \{s,t\}$ is blue $\} = S \setminus \{s\} \in e$
- $\mathcal{R}_1 \ni R_1 \subseteq F_1 \subseteq A_1^{\star} \subseteq A_1 = M \cap \bigcup \mathcal{R}_1$ — $e \ni M$ is monochromatic
- There is $e \ni B_1 \subseteq A_1^{\star}$ with $F_1 + B_1 \subseteq A_1$



- $\mathcal{R}_2 \ni R_2 \subseteq F_2 \subseteq A_2^{\star} \subseteq A_2 = \bigcap_{s\in F_1} \{\, t \in S \setminus \{s\} : \{s,t\}$ is blue $\} \cap B_1 \cap \bigcup \mathcal{R}_2$
- There is $e \ni B_2 \subseteq A_2^{\star}$ with $F_2 + B_2 \subseteq A_2$



- $\mathcal{R}_3 \ni R_3 \subseteq F_3 \subseteq A_3^{\star} \subseteq A_3 = \bigcap_{s\in F_1\cup F_2\cup F_1+F_2} \{\, t \in S \setminus \{s\} : \{s,t\}$ is blue $\} \cap B_2 \cap \bigcup \mathcal{R}_3$
- There is $e \ni B_3 \subseteq A_3^{\star}$ with $F_3 + B_3 \subseteq A_3$, etc.
- There is a play $(A_1^{\star}, F_1, A_2^{\star}, F_2, \ldots)$ won by Bob
- $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{B}$ □



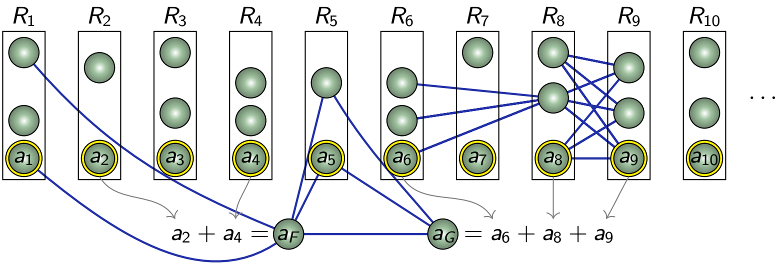
A modification...

Theorem (Sz 2020)

- *Assume that* $\mathcal{A}, \mathcal{B} \subseteq [S]^\infty$,
- *there is an idempotent* $e \subseteq \mathcal{A}$*, large for* $\mathcal{R}_1, \mathcal{R}_2, \ldots \subseteq \mathrm{Fin}(S)$,
- ALICE *has no winning strategy in* $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$.

For each coloring of $[S]^2$*, there are elements* $a_1, a_2, \ldots \in S$ *and sets* $R_1 \in \mathcal{R}_1, R_2 \in \mathcal{R}_2, \ldots$ *such that*

- $\mathcal{R}_n \ni R_n \ni a_n$ *and* $\{ a_n : n \in \mathbb{N} \} \in \mathcal{B}$,
- *the partite sumgraph of* $R_1, R_2, \ldots$ *is monochromatic.*



...and its consequences

Theorem (Scheepers 1999)

If X is $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, then for every $\mathcal{U}\in\Omega$ and a coloring of $[\tau]^2$, there is $\mathcal{V}\subseteq\mathcal{U}$ such that $\mathcal{V}\in\Lambda$ and the graph $[\mathcal{V}]^2$ is monochromatic.

Theorem (Scheepers 1996)

If X is $\mathsf{S}_1(\Omega,\Omega)$, then for every $\mathcal{U}\in\Omega$ and a coloring of $[\tau]^2$, there is $\mathcal{V}\subseteq\mathcal{U}$ such that $\mathcal{V}\in\Omega$ and the graph $[\mathcal{V}]^2$ is monochromatic.

- Y has countable strong fan tightness:

  $$(\forall A_1,A_2,\ldots\subseteq Y, y\in\bigcap_{n\in\mathbb{N}}\overline{A_n})(\exists a_1\in A_1, a_2\in A_2,\ldots)(y\in\overline{\{a_n : n\in\mathbb{N}\}})$$

- Sakai 1988: X is $\mathsf{S}_1(\Omega,\Omega)\leftrightarrow X$ is $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ in all finite powers $\leftrightarrow \mathsf{C_p}(X)$ has countable strong fan tightness

Theorem (Pawlikowski 1994)

X is $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ iff Alice *has no winning strategy in $\mathsf{G}_1(\mathrm{O},\mathrm{O})$.*

Richer structures

Theorem (Scheepers 1999)

If X is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, then for every $\mathcal{U} \in \Omega$ and a coloring of $[\tau]^2$, there are finite sets $\mathcal{F}_1, \mathcal{F}_2, \ldots \subseteq \mathcal{U}$ such that

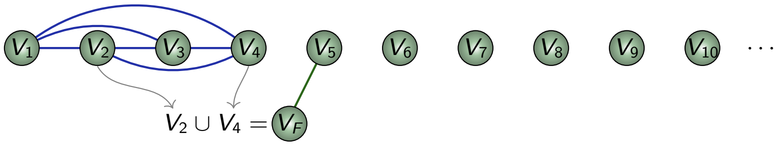
- *$\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \Lambda$ and the partite graph of $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is monochromatic.*

Theorem (Tsaban 2018) $(\tau, \cup)$

If X is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, then for each decreasing sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Lambda$ such that $\mathcal{U}_1$ has no finite subcover and a coloring of $[\tau]^2$, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that

- $\bigcup \mathcal{F}_n \in \Lambda$ and the sumgraph of $\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \ldots$ is monochromatic.

No: $X = \mathrm{Fin}(\mathbb{N})$, $U_n = \{\, F \in X : n \notin F \,\}$, $\mathcal{U} = \{U_1, U_2, \ldots\} \in \Omega$

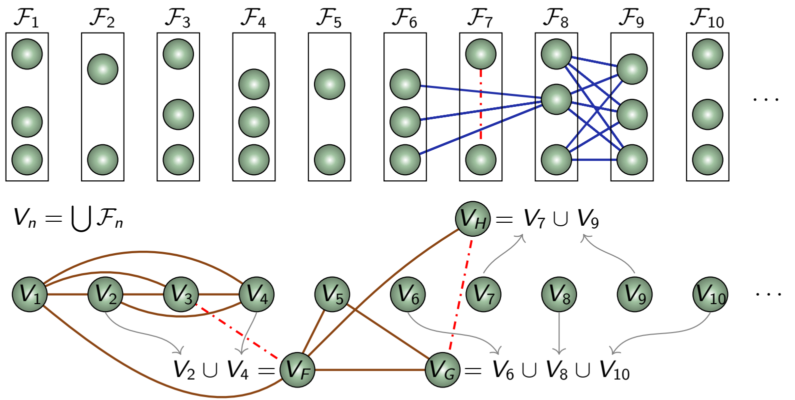


Richer structures

Theorem (Sz 2020) $(\tau, \cup)$

If X is $\mathrm{S_{fin}}(\mathrm{O},\mathrm{O})$, then for every $\mathcal{U} \in \Omega$ with no finite subcover and a coloring of $[\tau]^2$ there are finite sets $\mathcal{F}_1, \mathcal{F}_2, \ldots \subseteq \mathcal{U}$ such that

- $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \Lambda$ and the partite graph of $\mathcal{F}_1, \mathcal{F}_2, \ldots$ is monochromatic,
- the sumgraph of $\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \ldots$ is monochromatic.



Higher dimensions

- A partite k-sumgraph of $F_1, F_2, \ldots \in \mathrm{Fin}(S)$ is the set of all k-edges

$$\{a_{G_1}, \ldots, a_{G_k}\},$$

where $(a_1, a_2, \ldots) \in F_1 \times F_2 \times \cdots$ and $G_1 < \cdots < G_k \in \mathrm{Fin}(\mathbb{N})$.

Theorem (Sz 2020)

- *Assume that $\mathcal{A}, \mathcal{B} \subseteq [S]^\infty$, $\mathcal{B}$ is closed under supersets in $[S]^\infty$, $k \geq 2$*
- *there is an idempotent $e \subseteq \mathcal{A}$, large for $\mathcal{R}_1, \mathcal{R}_2, \ldots \subseteq \mathrm{Fin}(S)$,*
- ALICE *has no winning strategy in $\mathrm{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$.*

For each coloring of $[S]^k$, there are sets $F_1, F_2, \ldots \in \mathrm{Fin}(S)$ such that

- *$\mathcal{R}_n \ni R_n \subseteq F_n \subseteq \bigcup \mathcal{R}_n$ and $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{B}$,*
- *the partite k-sumgraph of $F_1, F_2, \ldots$ is monochromatic.*

Comments about covering properties

$$
\begin{array}{ccccccc}
 & & \mathsf{S}_{\mathrm{fin}}(\Omega,\Omega) & \longrightarrow & \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}) & & \\
 & & \uparrow \ \mathfrak{d} & & \uparrow \ \mathfrak{d} & & \\
\mathsf{S}_1(\Omega,\Gamma) & \longrightarrow & \mathsf{S}_1(\Omega,\Omega) & \longrightarrow & \mathsf{S}_1(\mathrm{O},\mathrm{O}) & \longrightarrow & \text{strong measure zero} \\
\mathfrak{p} & & \mathsf{cov}(\mathcal{M}) & & \mathsf{cov}(\mathcal{M}) & &
\end{array}
$$