On ω-Corson compact spaces and related classes of Eberlein compacta

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All topological spaces are Tikhonov.

Definition

A space K is an Eberlein compact space if K is homeomorphic to a weakly compact subset of a Banach space.

Equivalently, a compact space K is an Eberlein compactum if K can be embedded in the following subspace of the product $\mathbb{R}^\Gamma$:

$$c_0(\Gamma) = \{x \in \mathbb{R}^\Gamma : \text{ for every } \varepsilon > 0 \text{ the set } \{\gamma : |x(\gamma)| > \varepsilon\} \text{ is finite}\},$$

for some set Γ.

All metrizable compacta are Eberlein compact spaces.

Continuous images, closed subspaces, countable products of Eberlein compacta are Eberlein compact spaces.

A compact space K is Corson compact if, for some set Γ, K is homeomorphic to a subset of the Σ-product of real lines

$$\Sigma(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| \leq \omega\}.$$

Clearly, the class of Corson compact spaces contains all Eberlein compacta.

Let κ be an infinite cardinal number. A compact space K is κ-Corson compact if, for some set Γ, K is homeomorphic to a subset of the Σ_κ-product of real lines

$$\Sigma_\kappa(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| < \kappa\}.$$

Obviously, the class of Corson compact spaces coincides with the class of ω_1-Corson compact spaces.

Let $\{X_\gamma : \gamma \in \Gamma\}$ be the family of nonempty topological spaces, and let a_γ be a fixed point of X_γ.

The σ-product of the family $\{(X_\gamma, a_\gamma) : \gamma \in \Gamma\}$ is the following subspace of the product $\prod_{\gamma\in\Gamma} X_\gamma$

$$\sigma(X_\gamma, a_\gamma, \Gamma) = \{(x_\gamma)_{\gamma\in\Gamma} \in \prod_{\gamma\in\Gamma} X_\gamma : |\{\gamma \in \Gamma : x_\gamma \neq a_\gamma\}| < \omega\}.$$

If $X_\gamma = I = [0, 1]$ and $a_\gamma = 0$, for all $\gamma \in \Gamma$, then we denote the σ-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I, \Gamma)$.

If $X_\gamma = \mathbb{R}$ and $a_\gamma = 0$, for all $\gamma \in \Gamma$, then we denote the σ-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(\mathbb{R}, \Gamma)$.

If $X_\gamma = I^\omega$ and $a_\gamma = (0, 0, \dots)$, for all $\gamma \in \Gamma$, then we denote the σ-product $\sigma(X_\gamma, a_\gamma, \Gamma)$ by $\sigma(I^\omega, \Gamma)$.

For $\kappa = \omega$, $\Sigma_\kappa(\mathbb{R}^\Gamma) = \sigma(\mathbb{R}, \Gamma)$.

A compact space K is NY compact if K can be embedded into some σ-product of metrizable compacta.
We denote the class of NY compact spaces by $\mathcal{NY}$.

Proposition

For a compact space K we have

(a) *K is ω-Corson if and only if it can be embedded into some σ-product of metrizable finitely dimensional compacta if and only if it can be embedded into the σ-product $\sigma(I, \Gamma)$ for some set Γ.*

(b) *K is NY compact if and only if it can be embedded into the σ-product $\sigma(I^\omega, \Gamma)$ for some set Γ.*

A family $\mathcal{U}$ of subsets of a space X is T_0-separating if, for every pair of distinct points x, y of X, there is $U \in \mathcal{U}$ containing exactly one of the points x, y.

Given a family $\mathcal{U}$ of subsets of a space X, a point $x \in X$, and an infinite cardinal κ, we write $ord(x,\mathcal{U}) < \kappa$ if $|\{U \in \mathcal{U} : x \in U\}| < \kappa$.
We say that $\mathcal{U}$ is point-finite if $ord(x,\mathcal{U}) < \omega$ for all $x \in X$.

Proposition

Let κ be an uncountable cardinal number. For a compact space K, the following conditions are equivalent:

1. *K is κ-Corson;*
2. *There exists a family $\mathcal{U}$ consisting of cozero subsets of K which is T_0-separating, and $ord(x,\mathcal{U}) < \kappa$ for all $x \in K$.*

An analogous characterization for ω-Corson compacta does not work:

Proposition (M., Plebanek, Zakrzewski)

For a compact space K, the following conditions are equivalent:

- (a) *There exists a T_0-separating, point-finite family $\mathcal{U}$ consisting of cozero subsets of K;*
- (b) *K is a scattered Eberlein compact space.*

Recall that a space X is strongly countable-dimensional if X is a countable union of closed finite-dimensional subspaces.

Proposition (M., Plebanek, Zakrzewski)

Every ω-Corson compact space is Eberlein compact and strongly countably dimensional.

Proposition

A metrizable compact space K is ω-Corson if and only if it is strongly countably dimensional.

All scattered Eberlein compacta are ω-Corson.

A family $\mathcal{A}$ of subsets of a space X is closure preserving if, for any subfamily $\mathcal{A}' \subseteq \mathcal{A}$, we have

$$\overline{\bigcup \mathcal{A}'} = \bigcup \{\overline{A} : A \in \mathcal{A}'\}.$$

A space X is metacompact if every open cover of X has a point-finite open refinement.

Theorem (M., Plebanek, Zakrzewski)

For a compact space K, the following conditions are equivalent:

1. (a) *K is ω-Corson;*
2. (b) *K has a closure preserving cover consisting of finite dimensional metrizable compacta;*
3. (c) *K is hereditarily metacompact and each nonempty subspace A of K contains a nonempty relatively open separable, metrizable, finite dimensional subspace U.*

Theorem (M., Plebanek, Zakrzewski)

For a compact space K, the following conditions are equivalent:

(a) *K belongs to the class $\mathcal{NY}$;*

(b) *There exists a T_0-separating family $\mathcal{U} = \bigcup\{\mathcal{U}_\gamma : \gamma \in \Gamma\}$ consisting of cozero subsets of K, where each $\mathcal{U}_\gamma$ is a countable and the family $\{\bigcup \mathcal{U}_\gamma : \gamma \in \Gamma\}$ is point-finite;*

(c) *K has a closure preserving cover consisting of metrizable compacta;*

(d) *K is hereditarily metacompact and each nonempty subspace A of K contains a nonempty relatively open subspace U of countable weight.*

The equivalence of conditions (a-c) was proved by Nakhmanson and Yakovlev.

Corollary (Nakhmanson and Yakovlev)

The class $\mathcal{NY}$ is stable under continuous images

Corollary

A NY compact space K is ω-Corson if and only if it is strongly countably dimensional.

Corollary

For any sequence $(K_n)_{n\in\omega}$ of nonmetrizable Eberlein compacta, the product $\prod_{n\in\omega} K_n$ does not belong to $\mathcal{NY}$.

Theorem (Gruenhage)

For a compact space K, the following conditions are equivalent:

- (a) *K is Eberlein compact;*
- (b) *K^2 is hereditarily σ-metacompact;*
- (c) *$K^2 \setminus \Delta$ is σ-metacompact.*

Example (M., Plebanek, Zakrzewski)

There exist a zero-dimensional Eberlein compact space K such that K^2 is hereditarily metacompact, but K is not ω-Corson.

Theorem (M., Plebanek, Zakrzewski)

Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of nonmetrizable Eberlein compact spaces, then $\prod_{n\in\mathbb{N}} K_n$ is not hereditary metacompact.

The class of ω-Corson compact spaces is clearly stable under taking closed subspaces and finite products, but is not stable under taking continuous images, as the Hilbert cube is a continuous image of the Cantor set 2^ω.

Problem

Does every nonmetrizable compact space contain a closed nonmetrizable, zero-dimensional subspace?

Example (Koszmider 2016)

There exists (in ZFC) a nonmetrizable compact space without nonmetrizable zero-dimensional closed subspaces.

Example (M.)

Assuming the existence of a Luzin set, there exists a nonmetrizable Eberlein compact space K without closed nonmetrizable zero-dimensional subspaces.

Theorem (M.)

Assuming that $\mathfrak{b} > \omega_1$, each Eberlein (Corson) compact space K of weight $> \omega_1$ contains a closed nonmetrizable, zero-dimensional subspace L.

Theorem (M., Plebanek, Zakrzewski)

Assuming that $\mathfrak{b} > \omega_1$, each nonmetrizable compact space $K \in \mathcal{NY}$ contains a closed nonmetrizable zero-dimensional subspace L.

Problem

Is it consistent that every Eberlein compact space K of weight ω_1 contains a closed zero-dimensional subspace L of the same weight?

Problem

Does there exist in ZFC a compact space of weight ω_1 without nonmetrizable zero-dimensional closed subspaces?

Between ω-Corson and *NY* compacta

A compact space K belongs to the class $\mathcal{EC}_{\omega c}$ if, for some set Γ there is an embedding $\varphi : K \to \mathbb{R}^\Gamma$ and a countable subset Γ_0 of Γ such that, for each $x \in K$, the set $\mathrm{supp}(\varphi(x)) \setminus \Gamma_0$ is finite.

Proposition (M., Plebanek, Zakrzewski)

A compact space K belongs to $\mathcal{EC}_{\omega c}$ if and only if it can be embedded into the product $I^\omega \times L$ of the Hilbert cube I^ω and some ω-Corson compact space L.

Proposition (M., Plebanek, Zakrzewski)

Each ω-Corson compact space belongs to the class $\mathcal{EC}_{\omega c}$, and each member of $\mathcal{EC}_{\omega c}$ is NY compact.

For a locally compact space X, $\alpha(X)$ denotes the one point compactification of X. For an infinite cardinal number κ, $D(\kappa)$ denotes a discrete space of cardinality κ.

Example (M., Plebanek, Zakrzewski)

The space $\alpha(D(\omega_1) \times 2^\omega)$ is ω-Corson (hence belongs to $\mathcal{EC}_{\omega c}$), but its continuous image $\alpha(D(\omega_1) \times I^\omega)$ does not belong to $\mathcal{EC}_{\omega c}$.