Let $\mathbb{M}_{\omega_2}$ be an iterated Miller forcing of length $\omega_2$ with countable support and $G_{\omega_2}$ be an $\mathbb{M}_{\omega_2}$-generic over $V$ satisfying CH.

**Lemma 0.1.** For each set $X \subseteq 2^\omega \cap V[G_{\omega_2}]$ such that $X \in V[G_{\omega_2}]$, there is an ordinal number $\beta < \omega_2$ such that $X \cap V[G_{\beta}] \in V[G_{\beta}]$.

**Proof.** Let $X = \{ x_\alpha : \alpha < \omega_2 \}$. In $V$ we have a sequence $\langle \dot{x}_\alpha : \alpha < \omega_2 \rangle$, where each $\dot{x}_\alpha$ is a name for a real and $X = \{ \dot{x}_\alpha[G_{\omega_2}] : \alpha < \omega_2 \}$. Every antichain in $\mathbb{M}_{\omega_2}$ has size at most $\omega_1$. Fix $\alpha < \omega_2$. Without loss of generality we may assume that $\dot{x}_\alpha$ is the set of all pairs $\langle \langle i, k^\alpha_{i,\xi} \rangle, p^\alpha_{i,\xi} \rangle$, where $i < \omega$ and $\xi < \omega_1$ such that:

- $k^\alpha_{i,\xi} \in \{0, 1\}$,
- for each $i$, the set $A^\alpha_i := \{ p^\alpha_{i,\xi} : \xi < \omega_1 \}$ is a maximal antichain in $\mathbb{M}_{\omega_2}$,
- for each $\xi$, we have $p^\alpha_{i,\xi} \Vdash \dot{x}_\alpha(i) = k^\alpha_{i,\xi}$.

Since the set $\bigcup \{ \text{supp}(p) : p \in A^\alpha_i, i < \omega \}$ has size at most $\omega_1$, it is contained in some ordinal number $g(\alpha) > \alpha$. Let $C_0$ be the club of all fixed points of the map $g$, i.e., for all $\beta \in C_0$ and $\alpha < \beta$, we have $g(\alpha) < \beta$.

For each $\beta \in C_0$, we have $\{ x_\alpha : \alpha < \beta \} \in V[G_{\beta}]$: The sequence $\langle \dot{x}_\alpha : \alpha < \beta \rangle$ is in $V$. Since all antichains $A^\alpha_i$ for $\alpha < \beta$ and $i < \omega$ are subsets of $\mathbb{M}_{\beta}$, we have $G_{\omega_2} \cap A^\alpha_i = G_{\beta} \cap A^\alpha_i$ and thus,

$$\dot{x}_\alpha[G_{\omega_2}] = \{ \langle i, k^\alpha_{i,\xi} \rangle : i < \omega, \{ p^\alpha_{i,\xi} \} = G_{\omega_2} \cap A^\alpha_i = G_{\beta} \cap A^\alpha_i \} = \dot{x}_\alpha[G_{\beta}]$$

for all $\alpha < \beta$.

For each ordinal number $\alpha < \omega_2$, there is an ordinal number $h(\alpha) > \alpha$ such that $X \cap V[G_{\alpha}] \subseteq \{ x_\beta : \beta < h(\alpha) \}$. Let $C_1$ be a club in $\omega_2$ such that for all $\beta \in C_1$ and $\alpha < \beta$, we have $h(\alpha) < \beta$.

For each $\beta \in C_1$ with $\text{cf}(C_1 \cap \beta) = \omega_1$, we have $X \cap V[G_{\beta}] \subseteq \{ x_\alpha : \alpha < \beta \}$: Fix a real $x \in X \cap V[G_{\beta}]$. Then there is an ordinal number $\gamma < \beta$ such that $x \in X \cap V[G_{\gamma}]$, and thus $x \in \{ x_\alpha : \alpha < h(\gamma) \} \subseteq \{ x_\alpha : \alpha < \beta \}$.

Let $D$ be the set of all $\beta \in C_0 \cap C_1$ such that there is an increasing sequence in $C_0 \cap C_1$ of length $\omega_1$ whose supremum is $\beta$. Then the set $D$ is an $\omega_1$-club. Pick $\beta \in D$. Since $\beta \in C_0$, we have $\{ x_\alpha : \alpha < \beta \} \in V[G_{\beta}]$, and thus $\{ x_\alpha : \alpha < \beta \} \subseteq X \cap V[G_{\beta}]$. On the other hand, since $\beta \in C_1$ and $\text{cf}(C_1 \cap \beta) = \omega_1$, we have $X \cap V[G_{\beta}] \subseteq \{ x_\alpha : \alpha < \beta \}$. Finally, we have that $X \cap V[G_{\beta}] = \{ x_\alpha : \alpha < \beta \} \in V[G_{\beta}]$. \qed