On coverings of Banach spaces and their subsets by hyperplanes

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Terminology

We consider only real Banach spaces of dimension bigger than 1.

Definition

A **hyperplane** of X = a one-codimensional closed subspace of X = kernel of a non-zero bounded functional on X.

We denote the set of all hyperplanes of X by $\mathcal{H}(X)$.

We define the σ-ideal of subsets of X that can be covered by countably many hyperplanes:

$$\mathcal{H}_\sigma(X) = \{Y \subseteq X : \exists \mathcal{F} \subseteq (X) \quad Y \subseteq \bigcup \mathcal{F},\ \mathcal{F} \text{ countable}\}.$$

Terminology

$X \notin \mathcal{H}_\sigma(X)$, hyperplanes are nowhere dense subsets of X so the Baire cat. thm applies.

- $\mathfrak{add}(X)$ = the minimal cardinality of a family of sets from $\mathcal{H}_\sigma(X)$ whose union is not in $\mathcal{H}_\sigma(X)$
- $\mathfrak{cov}(X)$ = the minimal cardinality of a family of sets from $\mathcal{H}_\sigma(X)$ whose union is equal to X = minimal number of hyperplanes that we need to cover X.
- $\mathfrak{non}(X)$ = the minimal cardinality of a subset of X that is not in $\mathcal{H}_\sigma(X)$
- $\mathfrak{cof}(X)$ = the minimal cardinality of a family of sets from $\mathcal{H}_\sigma(X)$ such that each member of $\mathcal{H}_\sigma(X)$ is contained in some element of that family.

Obs: $cov(X) \geq \omega_1$. $A_\alpha \in \mathcal{H}_\sigma(X)$

If we have $\{A_\alpha\}_{\alpha<\kappa}$, $\bigcup A_\alpha = X$, then we extend $A_\alpha \subseteq \bigcup_n H_{\alpha,n}$

$$\bigcup_{\substack{\alpha<\kappa \\ n\in\mathbb{N}}} H_{\alpha,n} = X$$

$\mathfrak{add}$ and $\mathfrak{cof}$

Theorem

For any Banach space X we have: $\mathfrak{add}(X) = \omega_1, \mathfrak{cof}(X) = |\mathcal{H}| = |X^*|$.

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Lemma

If a hyperplane $H \in \mathcal{H}(X)$ is included in a countable union of hyperplanes $\bigcup H_i$, $H_i \in \mathcal{H}(X)$, then $H = H_i$ for some i.

Proof: Assume that $H \neq H_i$ for $i \in \mathbb{N}$. Consider $H \cap H_i \neq H$.

↑ hyperplanes in H

$H \cap H_i$ is a nowhere dense subset of H.

Baire category thm $\Rightarrow$ $\bigcup H \cap H_i \not\supseteq H$ – contradiction.

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Proof that $\mathrm{add}(X) = \omega_1$: Take ω_1 hyperplanes $\{H_\alpha : \alpha < \omega_1\}$ and assume that $\bigcup_{\alpha<\omega_1} H_\alpha \subseteq \bigcup_{n\in\mathbb{N}} G_n$. By Lemma we get that $H_\alpha = G_n$ for some n, so $\{H_\alpha : \alpha < \omega_1\}$ is in fact countable.

For $\mathrm{cof}(X)$ use the fact that $|X^*|$

Separable spaces

Proposition

If X is a separable Banach space, then $\mathfrak{cov}(X) = \mathfrak{c}$ and $\mathfrak{non}(X) = \omega_1$.

Both equalities follow from the following Klee's result:

Theorem

Let X be a separable Banach space. Then there is a set $Y \subseteq X$ of cardinality $\mathfrak{c}$ such that each infinite subset of Y is linearly dense in X. In particular $H \cap Y$ is finite for every $H \in \mathcal{H}(X)$

Nonseparable spaces: $\mathfrak{cov}$

Proposition

$\mathfrak{cov}(X) \leq \mathfrak{c}$ for every Banach space X. In particular CH implies that $\mathfrak{cov}(X) = \omega_1$ for each X.

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- spaces of density ω_1

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Equality $\mathfrak{cov}(X) = \omega_1$ holds in ZFC for the following classes of nonseparable Banach spaces:

- spaces of density ω_1
- spaces with fundamental biorthogonal system
- spaces $C(K)$ where K is compact and does not have small diagonal
- spaces X such that B_{X^*} (with the weak* topology) does not have small diagonal

Small diagonals

Definition

We say that a topological space K has small diagonal if for every uncountable $A \subseteq K^2 \backslash \Delta(K)$ there is uncountable $B \subseteq A$ such that $\overline{B} \subseteq K^2 \backslash \Delta(K)$.

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Consider the following sentence:

$$\text{For every compact Hausdorff space } K, K \text{ is metrizable} \iff K \text{ has small diagonal.} \qquad (*)$$

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Theorem (Dow-Juhasz-Szentmiklossy)

- PFA $\implies$ $(*)$
- $(*)$ is consistent with any possible size of $\mathfrak{c}$

Open question: Is $(*)$ provable in ZFC?

Nonseparable spaces: $\mathfrak{cov}$

Proposition

Assume that all spaces with small diagonals are metrizable. Then $\mathfrak{cov}(X) = \omega_1$ for all nonseparable Banach spaces.

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Corollary

- PFA implies that $\mathfrak{cov}(X) = \omega_1$ for all nonseparable Banach spaces.
- It is consistent with any possible size of $\mathfrak{c}$ that $\mathfrak{cov}(X) = \omega_1$ for all nonseparable Banach spaces.

Question

Is it provable in ZFC that $\mathfrak{cov}(X) = \omega_1$ for all nonseparable Banach spaces?

Nonseparable spaces: $\mathfrak{non}$

General inequality

Let X be a nonseparable Banach space of density κ. Then

$$\kappa \leq \mathfrak{non}(X) \leq \mathrm{cf}([\kappa]^\omega).$$

If κ has countable cofinality, then $\mathfrak{non}(X) > \kappa$.

- $[\kappa]^\omega =$ the family of all countable subsets of κ
- $\mathrm{cf}([\kappa]^\omega) =$ the minimal cardinality of a family $\mathcal{F} \subseteq [\kappa]^\omega$ such that each member of $[\kappa]^\omega$ is included in some member of $\mathcal{F}$.

Values of cf($[\kappa]^\omega$)

Proposition

$\mathrm{cf}([\omega_n]^\omega) = \omega_n$ for $n = 1, 2, 3, \ldots$

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If X is a Banach space of density ω_n, then $\mathfrak{non}(X) = \omega_n$.

Values of $\mathrm{cf}([\kappa]^\omega)$

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Corollary

If X is a Banach space of density ω_n, then $\mathfrak{non}(X) = \omega_n$.

Theorem

Assume GCH or MM. Then for $\kappa > \omega$

- $\mathrm{cf}([\kappa]^\omega) = \kappa$, if $\mathrm{cf}(\kappa) > \omega$
- $\mathrm{cf}([\kappa]^\omega) = \kappa^+$, if $\mathrm{cf}(\kappa) = \omega$

In particular, if X has density κ, then $\mathfrak{non}(X) = \mathrm{cf}([\kappa]^\omega)$.

Values of $\mathrm{cf}([\kappa]^\omega)$

Theorem (Magidor)

Assume that there is a supercompact cardinal. Then for each $n \in \mathbb{N}$ it is consistent that $\mathrm{cf}([\omega_\omega]^\omega) = \omega_{\omega+n}$.

Question

Is it possible that $\mathfrak{non}(X) < \mathrm{cf}([\kappa]^\omega)$ for some Banach space of density $\kappa > \omega$?

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Question

Is it possible that $\mathfrak{non}(X) < \mathrm{cf}([\kappa]^\omega)$ for some Banach space of density $\kappa > \omega$?

Proposition

If X has density $\kappa > \omega$ and admits a fundamental biorthogonal system, then $\mathfrak{non}(X) = \mathrm{cf}([\kappa]^\omega)$.

Overcomplete sets

Definition

Let X be a Banach space of density κ. We say that a subset $Y \subseteq X$ is overcomplete, if $|Y| = \kappa$ and each subset of Y of cardinality κ is linearly dense in X.

Examples of spaces that admit overcomplete sets:

- separable Banach spaces
- $\ell_p(\omega_1)$ for $p \in (1, \infty)$
- $c_0(\omega_1)$

Overcomplete sets

Proposition

If $\mathrm{cf}(\mathrm{dens}(X)) > \mathfrak{cov}(X)$, then X does not admit an overcomplete set.

Corollary

Assume that all compact Hausdorff spaces with small diagonals are metrizable. Then no Banach space of density ω_2 admits an overcomplete set.