Banach spaces and set theory

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Undecidable problems in Banach space theory

Questions concerning Banach spaces that cannot be resolved on the basis of standard set theory, i.e. that may have a positive or negative answer depending on the adoption of different additional set theoretic assumptions.

Examples of additional set theoretic assumptions:

CH - the Continuum Hypothesis: $\aleph_1 = \mathfrak{c}$ (continuum)

MA - Martin's Axiom

Universal Banach spaces

For a compact space K, $C(K)$ is the Banach space of real-valued continuous functions on K (with the sup norm).

For a Banach space X, X^* denotes the dual space, and B_{X^*} is the closed unit ball in X^*.

Theorem (Banach-Mazur, 1932)

The Banach space $C([0,1])$ contains an isometric copy of any separable Banach space.

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A sketch of a proof:

1. For a Banach space X, the map $e : X \to C((B_{X^*}, w^*))$ given by
$$e(x)(x^*) = x^*(x) \quad \text{for } x \in X, x^* \in B_{X^*}$$
is an isometric embedding;
2. If a Banach space X is separable, then the compact space (B_{X^*}, w^*) is metrizable;
3. Every nonempty metrizable compact space is a continuous image of the Cantor set $\mathbf{C}$;
4. If $\varphi : K \to L$ is a continuous surjection of compact spaces, then the map $f \mapsto f \circ \varphi$ is an isometric embedding of $C(L)$ into $C(K)$;
5. $C([0,1])$ contains an isometric copy of $C(\mathbf{C})$.

For a topological space X, $d(X)$ - the density of X is the minimal cardinality of a dense subspace of X, and $w(X)$ - the weight of X is the minimal cardinality of a (topological) base of X.
For metrizable spaces X, we have $d(X) = w(X)$.

Let $\mathcal{B}$ be a class of Banach spaces. We say that $X \in \mathcal{B}$ is injectively isomorphically (isometrically) universal for $\mathcal{B}$ if for every $Y \in \mathcal{B}$ there is an isomorphic (isometric) embedding of Y into X.

Problem 1

Let κ be an infinite cardinal number, and $\mathcal{B}_\kappa$ be the class of Banach spaces X of density $d(X) \leq \kappa$. Does there exist a Banach space which is injectively isomorphically (isometrically) universal for $\mathcal{B}_\kappa$?

$C([0,1])$ is injectively isometrically universal for $\mathcal{B}_{\aleph_0}$ - the class of separable Banach spaces.

Let $\mathcal{K}$ be a class of compact spaces. We say that $K \in \mathcal{K}$ is surjectively universal for $\mathcal{K}$ if for every nonempty $L \in \mathcal{K}$ there is a continuous surjection of K onto L.

Problem 2

Let κ be an infinite cardinal number, and $\mathcal{K}_\kappa$ be the class of compact spaces K of weight $w(K) \leq \kappa$. Does there exist a compact space which is surjectively universal for $\mathcal{K}_\kappa$?

YES to Problem 2 $\Longrightarrow$ **YES** to Problem 1 $(w(K) = d(C(K)))$.

A compact space K is totally disconnected if it has a base consisting of closed and open subsets.

Problem 3

Let κ be an infinite cardinal number, and $\mathcal{TDK}_\kappa$ be the class of totally disconnected compact spaces K of weight $w(k) \leq \kappa$. Does there exist a compact space which is surjectively universal for $\mathcal{TDK}_\kappa$?

Problems 2 and 3 are equivalent (every compact space is a continuous image of a totally disconnected compact space of the same weight).

Let $\mathcal{BA}$ be a class of Boolean algebras. We say that $A \in \mathcal{BA}$ is injectively universal for $\mathcal{BA}$ if for every $B \in \mathcal{BA}$ there is an isomorphic embedding of B into A.

Problem 4

Let κ be an infinite cardinal number, and $\mathcal{BA}_\kappa$ be the class of Boolean algebras A of cardinality $|A| \leq \kappa$. Does there exist a Boolean algebra which is injectively universal for $\mathcal{BA}_\kappa$?

Problems 3 and 4 are equivalent (the Stone duality).

The cases of $\kappa = \aleph_1$ or $\kappa = \mathfrak{c}$ (continuum)

Theorem (Esenin-Volpin, 1949)

Assuming the Continuum Hypothesis there exists a compact space surjectively universal for $\mathcal{K}_\mathfrak{c}$

$\beta\mathbb{N}$ is the Čech-Stone compactification of the space of natural numbers $\mathbb{N}$, and $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$.

The algebra of all closed and open subsets of $\mathbb{N}^*$ is isomorphic to the Boolean algebra $\mathcal{P}(\mathbb{N})/fin$, and the Banach space $C(\mathbb{N}^*)$ is isometric to the quotient space ℓ_∞/c_0.

$w(\mathbb{N}^*) = d(\ell_\infty/c_0) = \mathfrak{c}$.

Theorem (Parovičenko, 1963)

Every Boolean algebra of size $\leq \aleph_1$ embeds isomorphically into algebra $\mathcal{P}(\mathbb{N})/fin$.
Hence, assuming the Continuum Hypothesis the space $\mathbb{N}^*$ is surjectively universal for $\mathcal{K}_\mathfrak{c}$, and the space ℓ_∞/c_0 is injectively isometrically universal for $\mathcal{B}_\mathfrak{c}$.

Theorem (Dow-Hart, 2001)

It is consistent that there is no surjectively universal compact space for $\mathcal{K}_{\mathfrak{c}}$.

Theorem (Shelah-Usvyatsov, 2006)

It is consistent that there is no injectively isometrically universal Banach space for $\mathcal{B}_{\mathfrak{c}}$.

Theorem (Brech-Koszmider, 2012)

It is consistent that there are no injectively isomorphically universal Banach spaces for $\mathcal{B}_{\aleph_1}$ and $\mathcal{B}_{\mathfrak{c}}$.

Biorthogonal systems

X - a Banach space

A family of pairs $\{(x_\gamma, x^*_\gamma) : \gamma \in \Gamma\}$ in $X \times X^*$ is called a biorthogonal system in $X \times X^*$ if

$$x^*_\alpha(x_\beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

If $\{e_\gamma : \gamma \in \Gamma\}$ is an orthonormal basis in a Hilbert space $(H, < \cdot, \cdot >)$, and $e^*_\gamma \in H^*$ is defined by $e^*_\gamma(x) =< x, e_\gamma >$, then $\{(e_\gamma, e^*_\gamma) : \gamma \in \Gamma\}$ is a biorthogonal system in $H \times H^*$.

Theorem (Markushevich, 1943)

Every infinite dimensional separable Banach space has a biorthogonal system $\{(x_n, x^*_n) : n \in \mathbb{N}\}$ such that $\overline{\text{span}\{x_n : n \in \mathbb{N}\}} = X$ and $\overline{\text{span}\{x^*_n : n \in \mathbb{N}\}}^{w^*} = X^*$.

Problem 5

Let X be a nonseparable Banach space X. Does there exist an uncountable biorthogonal system in $X \times X^*$?

Example (Kunen, 1980)

Assuming the Continuum Hypothesis there exists a nonseparable space $C(K)$ without any uncountable biorthogonal system.

Theorem (Todorčević, 2006)

Assuming Martin's Maximum axiom every Banach space X of density $\aleph_1$ has a biorthogonal system $\{(x_\gamma, x^*_\gamma) : \gamma \in \Gamma\}$ of size $\aleph_1$, and such that $\overline{\mathrm{span}\{x_\gamma : \gamma \in \Gamma\}} = X$.

In particular, every nonseparable Banach space contains an uncountable biorthogonal system.

Twisted sums

A twisted sum of Banach spaces Y and Z is a short exact sequence

$$0 \to Y \to X \to Z \to 0$$

where X is a Banach space and the maps are bounded linear operators.

Such twisted sum is called trivial if the exact sequence splits, i.e., if the map $Y \to X$ admits a left inverse (equivalently, if the map $X \to Z$ admits a right inverse).

The twisted sum is trivial iff the range of the map $Y \to X$ is complemented in X; in this case, $X \cong Y \oplus Z$.

We write $\mathrm{Ext}(Z, Y) = 0$ if every twisted sum of Y and Z is trivial.

Example

$$0 \to c_0 \to \ell_\infty \to \ell_\infty / c_0 \to 0$$

Phillips (1940): c_0 is not complemented in l_∞

For a compact space K, $C(K)$ is the Banach space of real-valued continuous functions on K (with the sup norm).

For a closed $A \subset K$, $C(K|A) = \{f \in C(K) : f|A \equiv 0\}$,

In the sequence $0 \to c_0 \to \ell_\infty \to \ell_\infty / c_0 \to 0$

we can replace all spaces by isometric function spaces obtaining

$$0 \to C(\beta\mathbb{N}|\mathbb{N}^*) \to C(\beta\mathbb{N}) \to C(\mathbb{N}^*) \to 0$$

This twisted sum is nontrivial because there is no isomorphic embedding of $C(\mathbb{N}^*)$ into $C(\beta\mathbb{N})$ ($C(\beta\mathbb{N})$ has a separating sequence of functionals and $C(\mathbb{N}^*)$ does not have).

By the classical Sobczyk theorem any isomorphic copy of the space c_0 is complemented in any separable superspace. This implies $\mathrm{Ext}(Y, c_0) = 0$ for every separable Banach space Y. In particular

Remark

If K is a metrizable compact space, then every twisted sum of c_0 and $C(K)$ is trivial.

Problem 6 (Cabello, Castillo, Kalton, Yost, 2000)

Let K be a nonmetrizable compact space. Does there exist a nontrivial twisted sum of c_0 and $C(K)$?

Some classes of compacta K with $\mathrm{Ext}(C(K), c_0) \neq 0$

(Castillo, Correa-Tausk, 2016) For a non-metrizable K, there exists a nontrivial twisted sum of c_0 and $C(K)$ in any of the following cases:

- K is a weakly compact subspace of a Banach space;
- the weight $w(K)$ of K is equal to $\aleph_1$ and $((C(K))^*, w^*)$ is not separable;
- $C(K)$ contains an isomorphic copy of ℓ_∞;
- K contains a copy of $2^{\mathfrak{c}}$;
- K is an ordinal space, i.e., $K = [0, \kappa]$ for some ordinal number κ.

Theorem (Plebanek-M., 2018)

(MA + ¬CH) *The spaces c_0 and $C(2^{\aleph_1})$ do not have a nontrivial twisted sum.*

A topological space X is scattered if no nonempty subset $A \subseteq X$ is dense-in-itself.
For an ordinal α, $X^{(\alpha)}$ is the αth Cantor-Bendixson derivative of the space X. For a scattered space X, the scattered height

$$ht(X) = \min\{\alpha : X^{(\alpha)} = \emptyset\}.$$

Theorem (Plebanek-M., 2018)

(MA + ¬CH) *let K be a separable scattered compact space of height 3 and weight $\aleph_1$. Then every twisted sum of c_0 and $C(K)$ is trivial.*

In the above theorems $\aleph_1$ can be replaced by any infinite cardinal number $\lambda < \mathfrak{c}$.

Theorem (Avilés-Plebanek-M., 2020)

(CH) *If K is a compact nonmetrizable space then $\mathrm{Ext}(C(K), c_0) \neq 0$.*

Some consequences of $\mathrm{Ext}(C(K), c_0) = 0$

Theorem (Plebanek-M., 2018)

(MA + ¬CH) *let K be a separable scattered compact space of height 3 and weight $\lambda < \mathfrak{c}$. Then $\mathrm{Ext}(C(K), c_0) = 0$.*

Theorem (Cabello Sánchez-Castillo-Plebanek-Salguero-Alarcón-M., 2020)

(MA + ¬CH) *let K and L be separable scattered compact space of height 3 and weight $\lambda < \mathfrak{c}$. Then the Banach spaces $C(K)$ and $C(L)$ are isomorphic, and $C(K)$ is isomorphic to its square $C(K) \oplus C(K)$.*

Theorem (Pol-M., 2009)

There exist $2^{\mathfrak{c}}$ pairwise nonisomorphic Banach spaces $C(K)$ for separable scattered compact spaces K of height 3 and weight $\mathfrak{c}$.

Theorem (M., 1988)

There exists a separable scattered compact space K of height 3 and weight $\mathfrak{c}$ such that $C(K)$ is not isomorphic (not weakly homeomorphic) to its square $C(K) \oplus C(K)$.

A bounded operator $T : Y \to \ell_\infty / c_0$ can be lifted to ℓ_∞ if there is a bounded operator $\widetilde{T} : Y \to \ell_\infty$ such that $T = Q \circ \widetilde{T}$, where $Q : \ell_\infty \to \ell_\infty / c_0$ is the quotient operator.

Theorem (Avilés-Plebanek-M., 2020)

For an infinite dimensional Banach space Y the following are equivalent:

1. (i) $\mathrm{Ext}(Y, c_0) = 0$;
2. (ii) *every continuous function* $\mathbb{N}^* \longrightarrow (Y^*, weak^*)$ *extends to a continuous function* $\beta\mathbb{N} \longrightarrow (Y^*, weak^*)$;
3. (iii) *every bounded operator* $T : Y \to \ell_\infty / c_0$ *can be lifted to* ℓ_∞.

easy application: $\mathrm{Ext}(\ell_1(\kappa), c_0) = 0$, for any cardinal number κ.

Theorem

For an infinite dimensional Banach space Y the following are equivalent:

1. $\mathrm{Ext}(Y, c_0) = 0$;
2. *every continuous function* $\mathbb{N}^* \longrightarrow (Y^*, \text{weak}^*)$ *extends to a continuous function* $\beta\mathbb{N} \longrightarrow (Y^*, \text{weak}^*)$;

Corollary

If Y is a Banach space satisfying $\mathrm{Ext}(Y, c_0) = 0$, *then* $|C(\mathbb{N}^*, B_{Y^*})| \leq |Y^*|$.

Lemma

If K is a compact space of weight $\aleph_1$, then $|C(\mathbb{N}^*, K)| \geq 2^{\aleph_1}$.

Corollary

If Y is a Banach space of density $\aleph_1$ and $|Y^*| < 2^{\aleph_1}$, then $\mathrm{Ext}(Y, c_0) \neq 0$.

Some open questions

Problem

Let K be a compact space of weight $\geq \mathfrak{c}$. Is $\mathrm{Ext}(C(K), c_0) \neq 0$?

Problem

Let K be a scattered compact space of weight $\geq \mathfrak{c}$. Is $\mathrm{Ext}(C(K), c_0) \neq 0$?

Problem

Let K be a scattered compact space of countable height and weight $\geq \mathfrak{c}$. Is $\mathrm{Ext}(C(K), c_0) \neq 0$?

A. Avilés, W. Marciszewski, G. Plebanek, *Twisted sums of c_0 and $C(K)$-spaces: a solution to the CCKY problem*, Adv. Math. 369 (2020), 107168, 31 pp.

F. Cabello Sánchez, J.M.F. Castillo, *Homological methods in Banach space theory*, Cambridge Studies in Advanced Mathematics, 203. Cambridge University Press, Cambridge; Cambridge University Press, Cambridge, 2023, xi+547 pp.

F. Cabello Sánchez, J.M.F. Castillo, W. Marciszewski, G. Plebanek, A. Salguero-Alarcón, *Sailing over three problems of Koszmider*, J. Funct. Anal. 279 (2020), no. 4, 108571, 22 pp.

P. Hájek, V. Montesinos Santalucía, J. Vanderwerff, V. Zizler, *Biorthogonal systems in Banach spaces*, CMS Books in Mathematics 26. Springer, New York, 2008, xviii+339 pp.

P. Koszmider, *Universal objects and associations between classes of Banach spaces and classes of compact spaces*, Zb. Rad. (Beogr.) 17(25) (2015), Selected topics in combinatorial analysis, 93–115.

W. Marciszewski, G. Plebanek, *Extension operators and twisted sums of c_0 and $C(K)$-spaces*, J. Funct. Anal. 274 (2018), no. 5, 1491–1529.