A universal coregular countable second-countable space

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Lviv & Kielce

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Grassmannians and Projective Spaces

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The space $Gr_k(X)$ is called the k-th Grasmannian of X.

We shall be interested in the simplest case of 1-Grassmannians.

In this case $Gr_1(X)$ is the space of lines in X,

or else the projective space of X.

It is well-known and well-studied space.

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Topologically, $Gr_1(X)$ is the quotient space X^*/F^* of the space $X \setminus \{0\}$ by the action of the multiplicative group $F^* = F \setminus \{0\}$.

So, $Gr_1(X)$ carries the quotient topology with respect to the orbit map $X^* \to Gr_1(X)$, which is open (but not necessarily closed).

General topologists know that quotient topologies are dangerous and can provide many surprises.

Let us consider the simplest surprising case.

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Rational Projective space

In the countable power $\mathbb{Q}^\omega$ of the fields of rationals $\mathbb{Q}$, consider the countable linear subspace

$$\mathbb{Q}^{<\omega} = \{(x_i)_{i\in\omega} \in \mathbb{Q}^\omega : |\{i \in \omega : x_i \neq 0\}| < \omega\}$$

consisting of all eventually zero sequences of rational numbers.

The space $\mathbb{Q}^{<\omega}$ carries the Tychonoff product topology inherited from $\mathbb{Q}^\omega$. This is the topology of simple convergence.

It is clear that $X = \mathbb{Q}^{<\omega}$ is a countable metrizable space without isolated points, so is homeomorphic to $\mathbb{Q}$ according to the classical

Theorem (Sierpiński)

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Topological characterizations of $\mathbb{Q}$

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The metrizability in this theorem can be weakened to the second countability (= existence of a countable base of the topology) according to another classical

Theorem (Urysohn)

A topological space X is metrizable and separable if and only if X is regular and second-countable.

Those two theorems imply

Corollary (Sierpiński–Urysohn)

A topological space X is homeomorphic to $\mathbb{Q}$ if and only if X is countable, regular, second-countable and has no isolated points.

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Sierpiński–Urysohn Theorem

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A topological space X is homeomorphic to $\mathbb{Q}$ if and only if X is countable, regular, second-countable and has no isolated points.

Let us recall that a topological space X is regular if for any open set $U \subset X$ and point $x \in U$ there exists an open set V such that

$$x \in V \subseteq \overline{V} \subseteq U.$$

The projective space $\mathbb{Q}\mathrm{P}^\infty$

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Let us return to our linear topological space $X = \mathbb{Q}^{<\omega}$ and its projective space

$$\mathbb{Q}\mathrm{P}^\infty = X^*/\mathbb{Q}^*.$$

It is clear that the space $\mathbb{Q}\mathrm{P}^\infty$ is countable, second-countable, and has no isolated points.
What about the regularity of $\mathbb{Q}\mathrm{P}^\infty$?

Surprise (first noticed by Gelfand and Fuks in 1967)

The space $\mathbb{Q}\mathrm{P}^\infty$ is not regular.
Moreover, it is countable and connected!

How this is possible?

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The connectedness of $\mathbb{Q}\mathrm{P}^\infty$

Take any non-empty open set $U \subseteq \mathbb{Q}\mathrm{P}^\infty$ and let $q^{-1}[U]$ be its preimage under the quotient map $q : \mathbb{Q}^{<\omega} \setminus \{0\} \to \mathbb{Q}\mathrm{P}^\infty$.

The set $q^{-1}[U]$ is open and $\mathbb{Q}^*$-conical, i.e., $\mathbb{Q}^* \cdot q^{-1}[U] = q^{-1}[U]$.

Since $q^{-1}[U]$ is open in the Tychonoff product topology, it contains an open set of form $V \times \mathbb{Q}^{\omega\setminus n}$ for some $n = \{0, \dots, n-1\} \in \omega$ and some open set $V \subseteq \mathbb{Q}^n \setminus \{0\}$.

Being $\mathbb{Q}^*$-conical, the set $q^{-1}[U]$ contains the $\mathbb{Q}^*$-cone

$$\mathbb{Q}^* \cdot (V \times \mathbb{Q}^{\omega\setminus n}) = (\mathbb{Q}^* \cdot V) \times \mathbb{Q}^{\omega\setminus n}$$

and then its closure

$$\overline{q^{-1}[U]} \supset \overline{\mathbb{Q}^* \cdot V} \times \mathbb{Q}^{\omega\setminus n} = \{0\}^n \times \mathbb{Q}^{\omega\setminus n}$$

contains the linear subspace of finite codimension.

Since the quotienient map q is open, the closure $\overline{U}$ contains the image $q[\{0\}^n \times \mathbb{Q}^{\omega\setminus n}]$ for some $n \in \omega$.

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The superconnectedness of $\mathbb{Q}\mathrm{P}^\infty$

Therefore, for any nonempty open set $U \subseteq \mathbb{Q}\mathrm{P}^\infty$ the closure $\overline{U}$ contains the image $q[\{0\}^n \times \mathbb{Q}^{\omega\setminus n}]$ for some $n \in \omega$.

Consequently, for any nonempty open sets $U_1, \cdots U_k \subseteq \mathbb{Q}\mathrm{P}^\infty$ there exists $n \in \omega$ such that

$$\overline{U}_1 \cap \cdots \cap \overline{U}_k \supset q[\{0\}^n \times \mathbb{Q}^{\omega\setminus n}] \neq \emptyset.$$

So, $\mathbb{Q}\mathrm{P}^\infty$ is connected and moreover, $\mathbb{Q}\mathrm{P}^\infty$ is superconnected!

Definition

A topological space X is called *superconnected* if for any nonempty open sets $U_1, \dots, U_k$ the intersection $\overline{U}_1 \cap \cdots \cap \overline{U}_k$ is not empty.

Remark

Each superconnected space X is connected: assuming that X is disconnected, we could write X as the union $X = U_1 \cup U_2$ of two non-empty disjoint open sets and then $\overline{U}_1 \cap \overline{U}_2 = U_1 \cap U_2 = \emptyset$.

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The coregularity of $\mathbb{Q}\mathrm{P}^\infty$

Therefore the countable second-countable space $\mathbb{Q}\mathrm{P}^\infty$ is superconnected and not regular (otherwise it would be metrizable and disconnected).

But it is not regular to a very small extent.

Observation

For any for any nonempty open sets $U_1, \dots, U_k \subseteq \mathbb{Q}\mathrm{P}^\infty$ the completement $\mathbb{Q}\mathrm{P}^\infty \setminus (\overline{U}_1 \cap \dots \cap \overline{U}_k)$ is a regular space! Because $\mathbb{Q}\mathrm{P}^\infty \setminus (\overline{U}_1 \cap \dots \cap \overline{U}_k) \supseteq q[(\mathbb{Q}^n \setminus \{0\}) \times \mathbb{Q}^{\omega \setminus n}]$.

Definition

A topological space X is coregular if X is Hausdorff and for any nonempty open sets $U_1, \dots, U_k \subseteq X$ the complement $X \setminus (\overline{U}_1 \cap \dots \cap \overline{U}_k)$ is a regular space.

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Superconnected coregular spaces

Unified Definition

A Hausdorff topological space X is superconnected and coregular if for any nonempty open sets $U_1, \dots, U_k \subseteq X$ the intersection $\overline{U}_1 \cap \dots \cap \overline{U}_k$ is not empty and its complement $X \setminus (\overline{U}_1 \cap \dots \cap \overline{U}_k)$ is a regular space.

If $\{U_n\}_{n\in\omega}$ is a countable base of the topology in a superconnected coregular Hausdorff space, then for every $n \in \omega$ the set

$$X_n = \overline{U}_1 \cap \dots \cap \overline{U}_n$$

is non-empty and its complement $X \setminus X_n$ is a regular topological space. Moreover the sequence $(X_n)_{n\in\omega}$ is decreasing and has empty intersection $\bigcap_{n\in\omega} X_n = \emptyset$.

Superconnected coregular spaces

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Is there any topological characterization of the space $\mathbb{Q}P^\infty$, analogical to the topological characterization of the space $\mathbb{Q}$?

Well, let us list what we know about the space $\mathbb{Q}P^\infty$:

- countable,
- second-countable,
- Hausdorff;
- superconnected;
- coregular;
- locally metrizable.

Do these properties uniquely identify the topology of $\mathbb{Q}P^\infty$?

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Main Theorem

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A topological space X is homeomorphic to the space $\mathbb{Q}P^\infty$ if and only if X is countable, second-countable and possesses a decreasing sequence of non-empty closed sets $(X_n)_{n\in\omega}$ such that

- $X_0 = X$, $\bigcap_{n\in\omega} X_n = \emptyset$, *and* $X_{n+1} \subseteq X_n$ *for all* n;
- *for every* $n \in \omega$ *the complement* $X \setminus X_n$ *is a regular topological space;*
- *for every* $n \in \omega$ *and a nonempty relatively open set* $U \subseteq X_n$ *the closure* $\overline{U}$ *contains some* X_m.

The sequence $(X_n)_{n\in\omega}$ with the above properties is called a superskeleton of X. If every set X_{n+1} is nowhere dense in X_n, then the superskeleton is called canonical.

A canonical superskeleton in $\mathbb{Q}P^\infty$ is the sequence $(X_n)_{n\in\omega}$ of closed subsets $X_n = q[\{0\}^n \times \mathbb{Q}^{\omega\setminus n}]$.

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Proof of the Main Theorem

The proof is technically very difficult and exploits the classical back-and-forth method of Cantor.

Given a canonical supersekeleton $(X_n)_{n\in\omega}$ in a coregular superconnected space X, we construct inductively two sequences $(x_i)_{i\in\omega}$ in X and $(y_i)_{i\in\omega}$ in $\mathbb{Q}\mathrm{P}^\infty$ so that the correspondence $h : x_n \to y_n$ determines a homeomorphism between X and $\mathbb{Q}\mathrm{P}^\infty$ mapping the sets X_n of the supersekeleton in X to the corresponding sets in the canonical superskeleton in the space $\mathbb{Q}\mathrm{P}^\infty$.

The construction of the sequences $(x_i)_{i\in\omega}$ and $(y_i)_{i\in\omega}$ is inductive with many conditions. Besides the points x_i and y_i we also construct their basic neighborhoods $U_{i,j}$ and $V_{i,j}$ in order to guarantee that the bijection $h : x_n \to y_n$ will be a homeomorphism.

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A screenshot of a piece of the proof

Inductively we shall construct sequences of points $\{x_n\}_{n\in\omega} \subseteq X$, $\{y_n\}_{n\in\omega} \subseteq Y$, a double sequences of open sets $\{U_{n,k}\}_{n,k\in\omega} \subseteq \tau_X$, $\{V_{n,k}\}_{k,n\in\omega} \subseteq \tau_Y$, and a function $\ell : \Gamma \to \omega$ such that for any $\gamma \in \Gamma$ the following conditions are satisfied:

(1) If $\gamma = n$ for some number $n \in \omega$, then
- (1a) $\ell(\gamma) = \ell_X(x_n) = \ell_Y(y_n)$;
- (1b) $x_n \notin \{x_k\}_{k\in\downarrow\gamma}$ and $y_n \notin \{y_k\}_{k\in\downarrow\gamma}$;
- (1c) $\{(i,j) \in \downarrow\gamma : x_n \in U_{i,j}\} = \{(i,j) \in \downarrow\gamma : y_n \in V_{i,j}\}$;
- (1d) $\{(i,j) \in \downarrow\gamma : x_n \in \overline{U}_{i,j}\} = \{(i,j) \in \downarrow\gamma : y_n \in \overline{V}_{i,j}\}$;
- (1e) If $n \in \Omega$, then $x_n = \xi(n)$ and $y_n = f(x_n)$;
- (1f) If $n \in \overrightarrow{\Omega}$, then $x_n = \min(X' \setminus \{x_k\}_{k\in\downarrow\gamma})$ and $y_n \notin \overline{B}$;
- (1g) If $n \in \overleftarrow{\Omega}$, then $y_n = \min(Y' \setminus \{y_k\}_{k\in\downarrow\gamma})$ and $x_n \notin A$.

(2) If $\gamma = (n,k)$ for some $n,k \in \omega$, then
- 2a) $\ell(\gamma) \geq 2 + \max\{\ell(\alpha) : \alpha \in \downarrow\gamma\}$;
- 2b) for any $m \in \omega \cap \downarrow\gamma$ with $m \neq n$, we have $x_m \notin \overline{U}_{n,k}$ and $y_m \notin \overline{V}_{n,k}$;
- 2c) $x_n \in U_{n,k} \subseteq O_k^X(x_n) \subseteq X \setminus X_{1+\ell(n)}$ and $y_n \in V_{n,k} \subseteq O_k^Y(x_n) \subseteq Y \setminus Y_{1+\ell(n)}$;
- 2d) $\{(i,j) \in \downarrow\gamma : U_{n,k} \subseteq U_{i,j}\} = \{(i,j) \in \downarrow\gamma : x_n \in U_{i,j}\}$ and $\{(i,j) \in \downarrow\gamma : V_{n,k} \subseteq V_{i,j}\} = \{(i,j) \in \downarrow\gamma : y_n \in V_{i,j}\}$;
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- 2f) $X_{\ell(\gamma)} = \partial U_{n,k}$ and $Y_{\ell(\gamma)} = \partial V_{n,k} \subseteq \overline{V_{n,k} \cap Y_{\ell(n)}}$;
- 2g) if $n \in \Omega$, then $f(U_{n,k} \cap A) = V_{n,k} \cap B$;
- 2h) If $n \notin \Omega$, then $U_{n,k} \cap A = \emptyset = V_{n,k} \cap \bar{B}$;
- 2i) If $\overleftarrow{\Omega} \neq \emptyset$, then $X_{\ell(\gamma)} = \partial U_{n,k} \subseteq \overline{U}_{n,k} \cap X_{\ell(n)}$.

We recall that a topological space X is *coregular* if it is Hausdorff and for any nonempty open sets $U_1, \dots, U_n$ the complement $X \setminus (\overline{U}_1 \cap \dots \cap \overline{U}_n)$ is a regular topological space.

So, every regular topological space X is coregular.

The coregular space $\mathbb{Q}\mathrm{P}^\infty$ has the following universal property.

Theorem

Every countable second-countable coregular topological space is homeomorphic to a subspace of $\mathbb{Q}\mathrm{P}^\infty$.

Coregular Universality of $\mathbb{Q}P^\infty$

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A subset of a topological space is called *regular open* if it is equal to the interior of its closure.

A topological space is called *semiregular* if it has a base of the topology consisting of regular open sets.

Proposition

Every coregular space is semiregular.

In particular

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The superconnected countable space $\mathbb{QP}^\infty$ is semiregular.

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Homogeneity of the space $\mathbb{Q}\mathrm{P}^\infty$

It is easy to see that for any lines ℓ, ℓ' in the ltp $\mathbb{Q}^{<\omega}$ there exists a linear homeomorphism H of $\mathbb{Q}^{<\omega}$ such that $H(\ell) = \ell'$.

This implies that the projective space $\mathbb{Q}\mathrm{P}^\infty$ is topologically homogeneous: for any points $x, y \in \mathbb{Q}\mathrm{P}^\infty$ there exists a homeomorphism h of $\mathbb{Q}\mathrm{P}^\infty$ such that $h(x) = y$.

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Deep and shallow subsets

A subset A of a topological space X is called

- *deep* if for any non-empty open sets $U_1, \dots, U_n \subseteq X$ the set $A \setminus (\overline{U_1} \cap \dots \cap \overline{U_n})$ is finite.
- *shallow* if there exist non-empty open sets $U_1, \dots, U_n \subseteq X$ such that $A \cap (\overline{U_1} \cap \dots \cap \overline{U_n}) = \emptyset$.

Fact 1: For any deep (shallow) set A in a topological space X and any homeomorphism $h : X \to X$ the set $h(A)$ is deep (shallow).
Fact 2: Any infinite set in a second-countable space contains an infinite subset which is either deep or shallow.
Fact 3: Any finite set in a Hausdorff space is shallow.

Theorem (Dychotomic Homogeneity of $\mathbb{Q}P^\infty$)

Let A, B be two closed discrete subsets of $\mathbb{Q}P^\infty$. If the sets A, B are either both deep or both shallow, then any bijection $f : A \to B$ extends to a homeomorphism h of $\mathbb{Q}P^\infty$ such that $h(A) = B$.

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- *shallow* if there exist non-empty open sets $U_1, \dots, U_n \subseteq X$ such that $A \cap (\overline{U_1} \cap \dots \cap \overline{U_n}) = \emptyset$.

Fact 1: For any deep (shallow) set A in a topological space X and any homeomorphism $h : X \to X$ the set $h(A)$ is deep (shallow).

Fact 2: Any infinite set in a second-countable space contains an infinite subset which is either deep or shallow.

Fact 3: Any finite set in a Hausdorff space is shallow.

Theorem (Dychotomic Homogeneity of $\mathbb{Q}\mathrm{P}^\infty$)

Let A, B be two closed discrete subsets of $\mathbb{Q}\mathrm{P}^\infty$. If the sets A, B are either both deep or both shallow, then any bijection $f : A \to B$ extends to a homeomorphism h of $\mathbb{Q}\mathrm{P}^\infty$ such that $h(A) = B$.

Since finite subsets are shallow, we have

Corollary (Finite homogeneity of $\mathbb{Q}P^\infty$)

Any bijection $h : A \to B$ between finite subsets of $\mathbb{Q}P^\infty$ extends to a homeomorphism of $\mathbb{Q}P^\infty$.

Theorem (Discrete homogeneity of $\mathbb{Q}$)

Any bijection $h : A \to B$ between closed discrete subspaces $A, B \subset \mathbb{Q}$ extends to a homeomorphism of $\mathbb{Q}$.

How about $\mathbb{Q}P^\infty$?

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$\mathbb{Q}P^\infty$ contains two closed discrete subsets A, B (one shallow and other deep) such that no homeomorphism of $\mathbb{Q}P^\infty$ sends A onto B.

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Topological copies of $\mathbb{Q}P^\infty$ is "nature"

The space $\mathbb{Q}P^\infty$ is an orbit space of the action of the multiplicative group $\mathbb{Q}^*$ on $\mathbb{Q}^{<\omega} \setminus \{0\}$, so it is natural to look for topological copies of the space $\mathbb{Q}P^\infty$ among orbit spaces of group actions.

By a *group act* we understand a topological space X endowed with an action $\alpha : G \times X \to X$ a group G. The action α satisfies the following axioms:

- for every $g \in G$ the map $\alpha(g, \cdot) : X \to X$, $\alpha(g, \cdot) : x \mapsto gx := \alpha(g, x)$, is a homeomorphism of X;
- for the identity 1_G of the group G and every $x \in X$ we have $1_G x = x$;
- $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$.

In this case we also say that X is a *G-space*.

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Some properties of G-spaces

We say that a G-space X has *closed orbits* if for any point $x \in X$ its *orbit* $Gx = \{gx : g \in G\}$ is a closed subset of X.

A subset $A \subseteq X$ is called *G-invariant* if it coincides with its G-saturation $GA = \bigcup_{x \in A} Gx$.

The action of G on X induces the equivalence relation

$$E = \{(x, gx) : x \in X, \ g \in G\}.$$

The quotient space X/E by this equivalence relation is called the *orbit space* of the G-space and is denoted by X/G.

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Copies of $\mathbb{Q}\mathrm{P}^\infty$ among orbit spaces

Theorem

Let X be a G-space with closed G-orbits, possessing a vanishing sequence $(X_n)_{n\in\omega}$ of nonempty G-invariant closed subsets such that

1. *for any $n \in \omega$ and nonempty open G-invariant set $U \subseteq X_n$, the closure $\overline{U}$ contains some set X_m;*
2. *for any $n \in \omega$, point $x \in X \setminus X_n$, and open G-invariant neighborhood $U \subseteq X$ of $x \in U$, there exists an open G-invariant neighborhood $V \subseteq X$ of x such that $\overline{V} \subseteq U \cup X_n$.*

Then the orbit space X/G has a superskeleton.
If X is first-countable and X/G is countable, then the space X/G is homeomorphic to $\mathbb{Q}\mathrm{P}^\infty$.

Singular G-spaces

Definition

A topological space X endowed with a continuous action $\alpha : G \times X \to X$ of a Hausdorff topological group G is called *singular* if it has the following properties:

(i) the topological space X is regular and infinite;

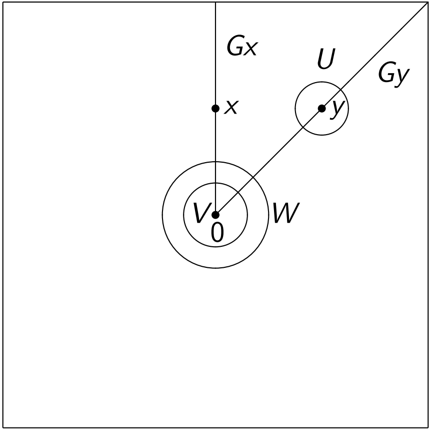
(ii) the set $\mathrm{Fix}_G(X) = \{x \in X : Gx = \{x\}\}$ is a singleton;

(iii) for every $x \in X \setminus \mathrm{Fix}_G(X)$ the map $\alpha_x : G \to X$, $\alpha_x : g \mapsto gx = \alpha(g, x)$, is injective and open;

(iv) the orbit Gx of every point $x \in X \setminus \mathrm{Fix}_G(X)$ contains the singleton $\mathrm{Fix}_G(X)$ in its closure $\overline{Gx}$;

(v) for any points $x \in X \setminus \mathrm{Fix}_G(X)$ and $y \in X$, there exists a neighborhood $U \subseteq X$ of y such that for any neighborhood $W \subseteq X$ of the singleton $\mathrm{Fix}_G(X)$, there exists a neighborhood $V \subseteq X$ of $\mathrm{Fix}_G(X)$ such that $\alpha_u(\alpha_x^{-1}(V)) \subseteq W$ for every $u \in U$.

(v) for any points $x \in X \setminus \mathrm{Fix}_G(X)$ and $y \in X$, there exists a neighborhood $U \subseteq X$ of y such that for any neighborhood $W \subseteq X$ of the singleton $0 = \mathrm{Fix}_G(X)$, there exists a neighborhood $V \subseteq X$ of $\mathrm{Fix}_G(X)$ such that $\alpha_u(\alpha_x^{-1}(V)) \subseteq W$ for every $u \in U$.



1. The complex plane $\mathbb{C}$ endowed with the action of the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers.
2. Any subfield $\mathbb{F} \subseteq \mathbb{C}$ endowed with the action of the multiplicative group $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$.
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4. The closed half-line $\overline{\mathbb{R}}_+ = [0, \infty)$ endowed with the action of the multiplicative group $\mathbb{R}_+$.
5. The space $\mathbb{Q}$ of rationals, endowed with the action of the multiplicative group $\mathbb{Q}_+$ of positive rational numbers.
6. The one-point compactification $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$ of the discrete space $\mathbb{Z}$ endowed with the natural action of the additive group $\mathbb{Z}$ of integer numbers.
7. The one-point compactification of any non-compact locally compact topological group G, endowed with the natural action of the topological group G.

Examples of singular G-spaces:

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Projective spaces of singular G-spaces

Given a singular G-space X, consider the G-space X^ω endowed with the Tychonoff product topology and the coordinatewise action of the group G.

Let s be the unique point of the singleton $\mathrm{Fix}(X;G)$.

Consider the subspaces of X^ω:

$$X^{<\omega} := \{x \in X^\omega : |\{n \in \omega : x(n) \neq s\}| < \omega\} \text{ and } X_\circ^{<\omega} := X^{<\omega} \setminus \{s\}^\omega.$$

The orbit space $X_\circ^{<\omega}/G$ is called the *infinite projective space* of the singular G-space X and is denoted by $X\mathrm{P}^\infty$.

If $X = \mathbb{F}$ is a non-discrete topological field endowed with the action of its multiplicative group $\mathbb{F}^*$, then $\mathbb{F}^{<\omega}$ is a topological vector space over the field $\mathbb{F}$ and $\mathbb{F}\mathrm{P}^\infty$ is the projective space of $\mathbb{F}^{<\omega}$ in the standard sense. In particular, $\mathbb{Q}\mathrm{P}^\infty$ is the projective space of the tvp $\mathbb{Q}^{<\omega}$ over the topological field $\mathbb{Q}$ of rational numbers.

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Copies of $\mathbb{Q}\mathrm{P}^\infty$ among projective spaces

Theorem

The infinite projective space $X\mathrm{P}^\infty$ of any singular G-space X possesses a canonical superskeleton.
If the singular G space X is countable and metrizable, then its infinite projective space $X\mathrm{P}^\infty$ is homeomorphic to the space $\mathbb{Q}\mathrm{P}^\infty$.

Some projective geometry

Let $\mathbb{F}$ be a topological field. Three elements $\mathbb{F}^*x, \mathbb{F}^*y, \mathbb{F}^*z$ of the projective space $\mathbb{F}\mathrm{P}^\infty$ are called *collinear* if the union $\mathbb{F}^*x \cup \mathbb{F}^*y \cup \mathbb{F}^*z$ is contained in some 2-dimensional vector subspace of $\mathbb{F}^{<\omega}$.

For two topological fileds $\mathbb{F}_1, \mathbb{F}_2$ a map $f : \mathbb{F}_1\mathrm{P}^\infty \to \mathbb{F}_2\mathrm{P}^\infty$ is called *affine* if for any collinear elements $\mathbb{F}_1^*x, \mathbb{F}_1^*y, \mathbb{F}_1^*z \in \mathbb{F}_1\mathrm{P}^\infty$, the elements $f(\mathbb{F}_1^*x), f(\mathbb{F}_1^*y), f(\mathbb{F}_1^*z)$ are collinear in the projective space $\mathbb{F}_2^*\mathrm{P}^\infty$.

A bijective map $f : \mathbb{F}_1\mathrm{P}^\infty \to \mathbb{F}_2\mathrm{P}^\infty$ is called an *affine isomorphism* if both maps f and f^{-1} are affine.

If an affine isomorphism $f : \mathbb{F}_1\mathrm{P}^\infty \to \mathbb{F}_2\mathrm{P}^\infty$ is also a homeomorphism, then f is called an *affine topological isomorphism*.

The projective spaces $\mathbb{F}_1\mathrm{P}^\infty, \mathbb{F}_2\mathrm{P}^\infty$ are called *affinely isomorphic* (resp. *affinely homeomorphic*) if there exists an affine topological ismorphism $f : \mathbb{F}_1\mathrm{P}^\infty \to \mathbb{F}_2\mathrm{P}^\infty$.

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Projective rigidity theorem

In spite of the fact that for any countable subfields $\mathbb{F}_1, \mathbb{F}_2 \subseteq \mathbb{C}$, the infinite projective spaces $\mathbb{F}_1\mathrm{P}^\infty$ and $\mathbb{F}_2\mathrm{P}^\infty$ are homeomorphic (to $\mathbb{Q}\mathrm{P}^\infty$), we have the following rigidity result for affine isomorphisms between infinite projective spaces.

Theorem

Two (topological) fields $\mathbb{F}_1, \mathbb{F}_2$ are (topologically) isomorphic iff their infinite projective spaces $\mathbb{F}_1\mathrm{P}^\infty$, $\mathbb{F}_2\mathrm{P}^\infty$ are affinely isomorphic (affinely homeomorphic).

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The spaces $\mathbb{C}$, $\mathbb{R}$, $\bar{\mathbb{R}}_+$ endowed with suitable group actions are singular G-spaces.

By a preceding theorem, the infinite projective spaces $\mathbb{C}P^\infty$, $\mathbb{R}P^\infty$, $\overline{\mathbb{R}}_+P^\infty$ possess (canonical) superskeleta.

Each of these spaces has a countable base of the topology consisting of sets, homeomorphic to the space $\mathbb{R}^{<\omega}$, so is a (non-metrizable) $\mathbb{R}^{<\omega}$-manifold.

It can be shown that the $\mathbb{R}^{<\omega}$-manifolds $\mathbb{C}P^\infty$, $\mathbb{R}P^\infty$, $\overline{\mathbb{R}}_+P^\infty$ are pairwise non-homeomorphic (because of different homotopical properties of complements $Y_0 \setminus Y_n$ of their canonical skeleta).

The distinguishing topological property of the space $\overline{\mathbb{R}}_+P^\infty$ is possessing a superskeleton $(Y_n)_{n\in\omega}$ such that for every $n < m$ in ω the complement $Y_n \setminus Y_m$ is contractible.

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By a preceding theorem, the infinite projective spaces $\mathbb{C}\mathrm{P}^\infty$, $\mathbb{R}\mathrm{P}^\infty$, $\overline{\mathbb{R}}_+\mathrm{P}^\infty$ possess (canonical) superskeleta.

Each of these spaces has a countable base of the topology consisting of sets, homeomorphic to the space $\mathbb{R}^{<\omega}$, so is a (non-metrizable) $\mathbb{R}^{<\omega}$-manifold.

It can be shown that the $\mathbb{R}^{<\omega}$-manifolds $\mathbb{C}\mathrm{P}^\infty$, $\mathbb{R}\mathrm{P}^\infty$, $\overline{\mathbb{R}}_+\mathrm{P}^\infty$ are pairwise non-homeomorphic (because of different homotopical properties of complements $Y_0 \setminus Y_n$ of their canonical skeleta).

The distinguishing topological property of the space $\overline{\mathbb{R}}_+\mathrm{P}^\infty$ is possessing a superskeleton $(Y_n)_{n\in\omega}$ such that for every $n < m$ in ω the complement $Y_n \setminus Y_m$ is contractible.

Characterization of $\overline{\mathbb{R}}_+\mathrm{P}^\infty$?

Fact: The space $\overline{\mathbb{R}}_+\mathrm{P}^\infty$ has a superskeleton $(Y_n)_{n\in\omega}$ such that for every $n < m$ in ω the complement $Y_n \setminus Y_m$ is contractible.

This fact and the topological characterization of $\mathbb{Q}\mathrm{P}^\infty$ suggests the following topological characterization of the space $\overline{\mathbb{R}}_+\mathrm{P}^\infty$.

Conjecture

A Hausdorff topological space X is homeomorphic to $\overline{\mathbb{R}}_+\mathrm{P}^\infty$ iff X has a superskeleton $(X_n)_{n\in\omega}$ such that for every n the set X_{n+1} is a Z-set in X_n and the space $X_n \setminus X_m$ is homeomorphic to $\mathbb{R}^{<\omega}$.

A closed subset A of a topological space X is called a *Z-set* in X if the set $C([0,1]^\omega, X \setminus A)$ is dense in the function space $C([0,1]^\omega, X)$, endowed with the compact-open topology.

Remark: It can be shown that the spaces $\mathbb{R}\mathrm{P}^\infty$, $\mathbb{C}\mathrm{P}^\infty$, $\overline{\mathbb{R}}_+\mathrm{P}^\infty$ contain dense subspaces, homeomorphic to $\mathbb{Q}\mathrm{P}^\infty$.

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References

T. Banakh, Ya. Stelmakh,
A universal coregular countable second-countable space,
preprint (arxiv.org/abs/2003.06293).

Thank you!