$\boxed{\text{Exm}}$ $A \subseteq \mathbb{N}$

① $\overline{d}(A) = \limsup_{n\to\infty} \frac{|A\cap n|}{n}$

asymptotic density

② $\overline{\delta}(A) = \limsup \frac{\sum_{i\in A\cap n} \frac{1}{i}}{\sum_{i\in n} \frac{1}{i}} =$

$= \limsup_{n\to\infty} \frac{\qquad}{\ln n}$

logarithmic density

③ uniform density (Banach density)

$\overline{u}(A) = \limsup_{n\to\infty} \left(\max \frac{|A\cap [k+1, k+n]|}{n} \right.$

<u>Def.</u> Abstract upper density

$\delta : \mathcal{P}(\omega) \to [0,1]$

with

(1) $\delta(\omega) = 1$

(2) F - finite $\to \delta(F) = 0$

(3) $A \subseteq B \to \delta(A) \leq \delta(B)$

(4) $\delta(A \cup B) \leq \delta(A) + \delta(B)$

Remark

$\mathcal{I}_\delta = \{A : \delta(A) = 0\}$

it is an ideal

on ω i.e.

(1) $\mathrm{Fin} \subseteq \mathcal{I}_\delta$

(2) $\mathbb{N} \notin \mathcal{I}_\delta$

(3) $A, B \in \mathcal{I}_\delta \rightarrow A \cup B \in \mathcal{I}_\delta$

(4) $A \subseteq B \in \mathcal{I}_\delta \rightarrow A \in \mathcal{I}_\delta$.

Exam

I - ideal on $\mathbb{N}$

then

$$\delta = \mathbb{1}_{P(\omega) \setminus I}$$

abstract upper density

$$\mathcal{Z}_\delta = I.$$

Question (2013)

(G. Grekos)

Is it true that for every ideal $\mathcal{I}$ there is a „nice" density δ s.t. $\mathcal{Z}_\delta = \mathcal{I}$?

Nice = translation invariant

If I is translation invariant, the

$$\delta(A) = \begin{cases} 0 & A \in I \\ 1 & A \notin I \end{cases}$$

is translation invariant density s.t. $Z_\delta = I$.

Def. A a.u.d δ is rich if

$\forall r \in [0,1] \ \exists A \ \delta(A) = r$

(δ = rich = onto)

Thm (Di Nasso – Jin, 2018) (Acta Arith).

If I is a summable ideal then there is a.u.d δ which is rich and $\mathcal{Z}_\delta = \mathcal{I}$.

$$\boxed{\begin{gathered}\text{summable ideals}\\ f : \mathbb{N} \to [0, \infty)\\ \mathcal{I}_f = \{A : \sum_{n \in A} f(n) < \infty\}\end{gathered}}$$

$$\left.\begin{aligned}\mathcal{I}_d &= \mathcal{Z}_{\bar{d}}\\ \mathcal{I}_\delta &= \mathcal{Z}_{\bar{\delta}}\\ \mathcal{I}_u &= \mathcal{Z}_{\bar{u}}\end{aligned}\right\}\ \text{are not summable.}$$

$$\boxed{f(n) = \frac{1}{n}}$$

Jacek Tryba

Thm

If I has the Baire property, the there is rich a.u.d δ w/ $\mathcal{I}_\delta = I$.

Sketch of proof of Di Nasso-Jin

- $I = I_f$ (f)
- They show that there is an infinite

I-almost disjoint family in I^+

- Using the function f and this family they

  I-AD

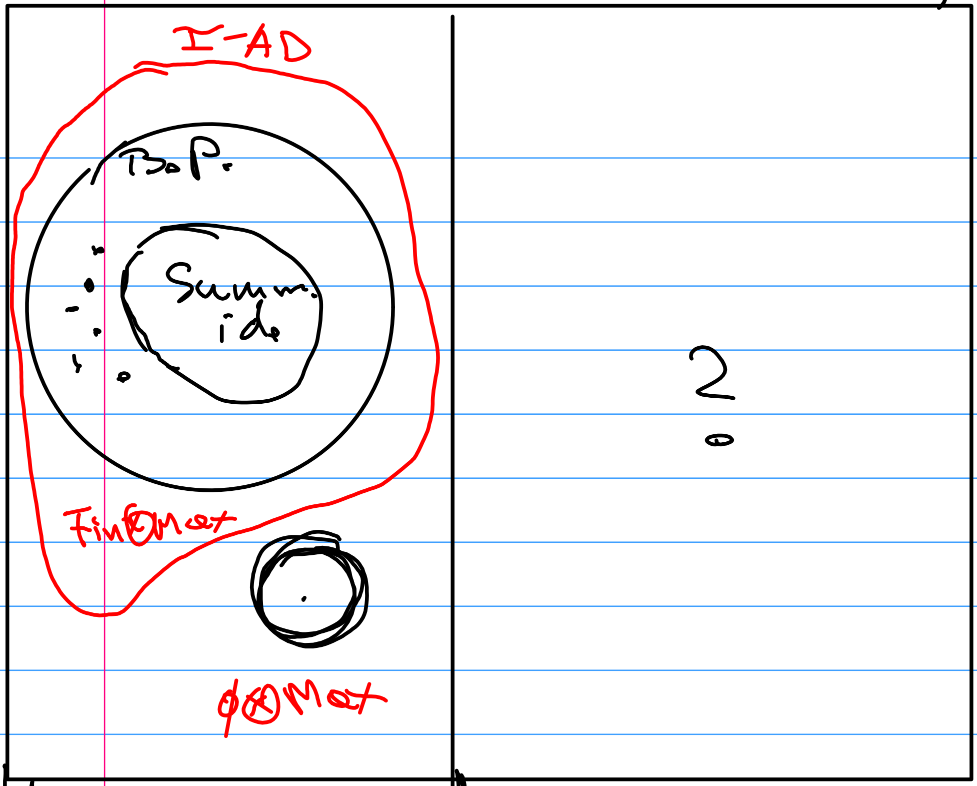
  construct a rich a.u.d δ w/ $Z_\delta = I$.

- Proof is 2 pages long.

Sketch of our proof

- $\mathcal{I}$ w/ B.P.
- There is $\mathcal{I}$-almost disjoint family in $\mathcal{I}^+$ of cardinality $\mathfrak{c}$
- Using this $\mathcal{I}$-AD family we construct a rich a.u.d δ w/ $\mathcal{Z}_\delta = \mathcal{I}$.
- Proof is $\frac{1}{2}$ page long.

All ideals

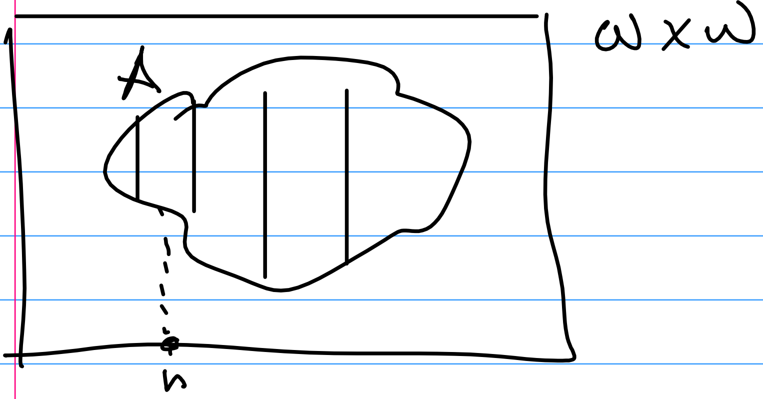


There is a rich a.u. of J $\Sigma J = I$

There is no.

Exm 1

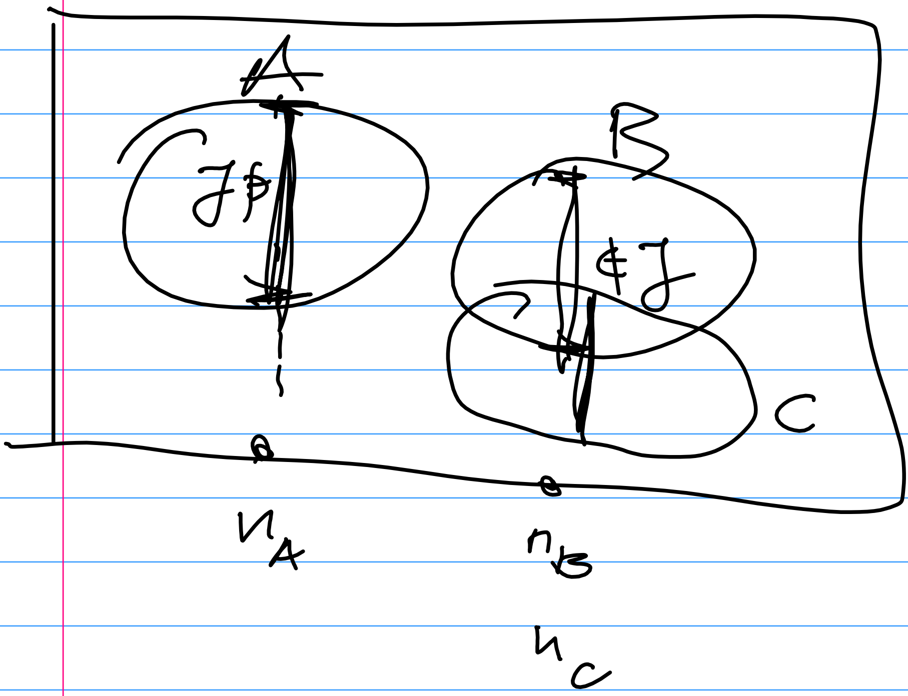
- $\mathcal{J}$ - max - ideal
- $I = \{\emptyset\} \otimes \mathcal{J}$



$$A \in I \iff \forall_n \; A_{(n)} \in \mathcal{J}.$$

- $$\delta(A) = \sum_{A_{(n)} \notin \mathcal{J}} \frac{1}{2^n}$$
- $\mathrm{ran}(\delta) = [0,1]$

- $\mathcal{J}$ - a.u.d
- $\mathcal{Z}_{\mathcal{J}} = \overline{\mathcal{I}}$
- There is no uncountable $\mathcal{I}$-AD family in $\mathcal{I}^+$



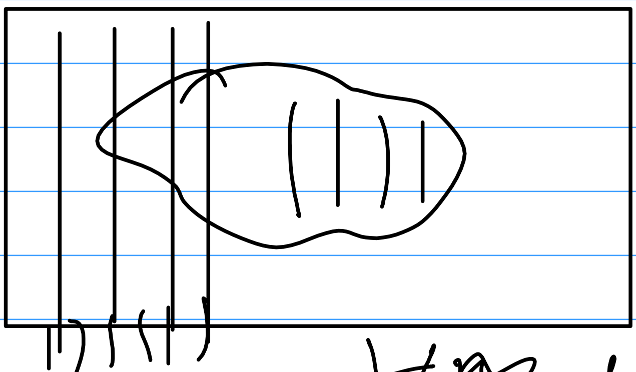
$B_{(n)} \cap C_{(n)} \notin \mathcal{J}$

B and C are not

I-a.d.

Exm 2

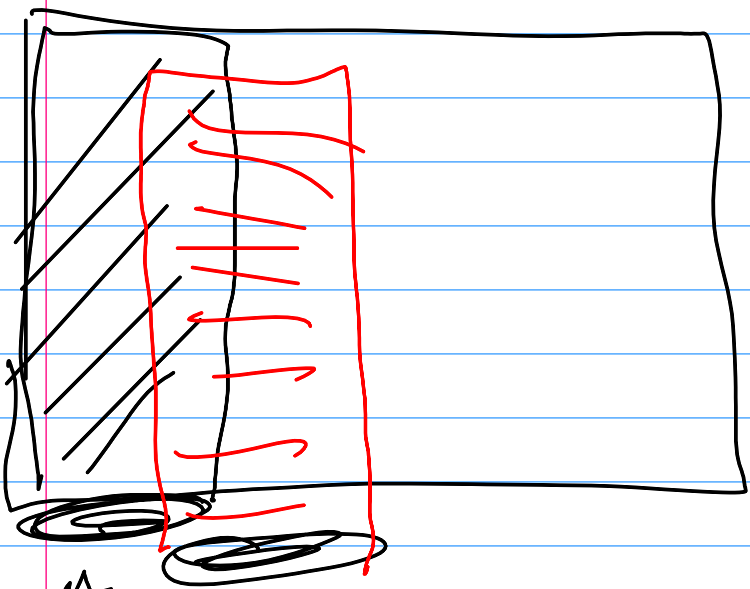
- $\mathcal{J}$ max. ideal
- $I = Fin \otimes \mathcal{J}$



$A \in I \iff \forall^{\infty}_{n}\ A_{(n)} \in \mathcal{J}$

- There is I-AD family of card-$\mathfrak{c}$.

Take Fin-A.D. family of cardinality $\mathfrak{c}$ on ω.



A

• $\emptyset \otimes \mathrm{Max}$ & B.P.

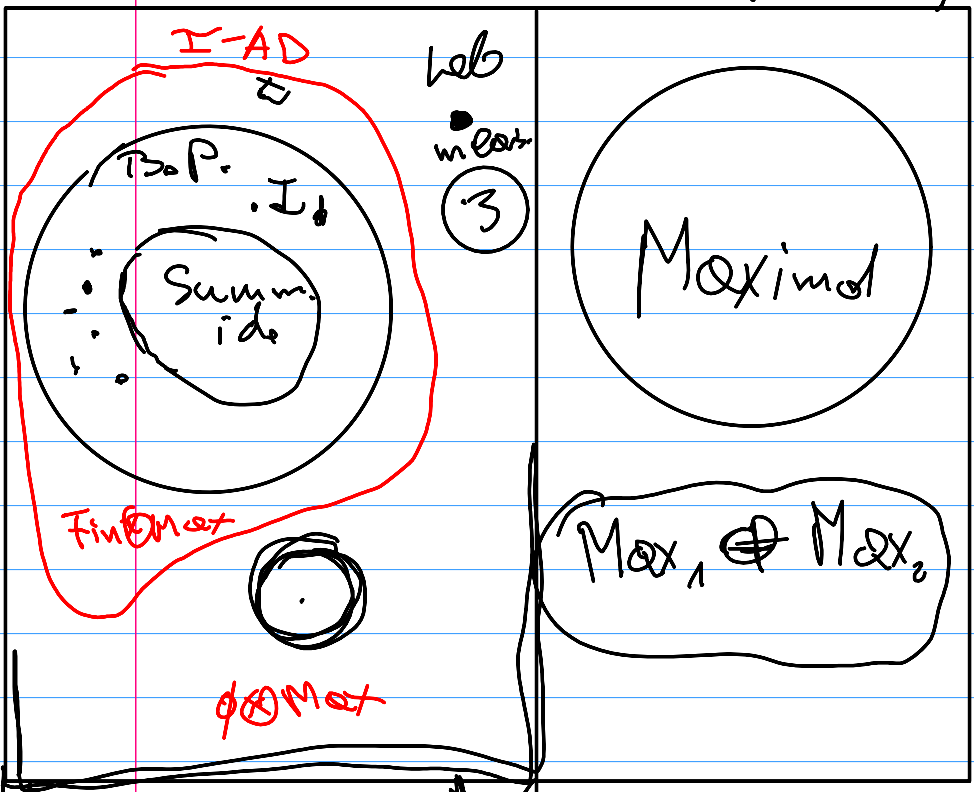
non-meager

$\underline{\emptyset \otimes \mathrm{Max}} \subseteq \underbrace{\mathrm{Fin} \otimes \mathrm{Max}}_{\text{non-meager}}$

$\Downarrow$

not B.P.

All i [illegible]



There is a rich a.u. of J

$Z_J = I$

There is no rich a.u.

$Z_J = I$.

Exam 2

$\mathcal{I}$ - maximal

Let δ s.t. $\mathcal{Z}_\delta = \mathcal{I}$.

$A \in \mathcal{I} \Leftrightarrow \delta(A) = 0$

$A \notin \mathcal{I} \Leftrightarrow \omega \setminus A \in \mathcal{I}$

$1 = \delta(\omega) \leq \delta(A) + \delta(\omega \setminus A)$

$= \delta(A) + 0 = \delta(A) \leq 1$

$A \notin \mathcal{I} \Leftrightarrow \delta(A) = 1$

$\operatorname{ran}(\delta) = \{0, 1\}$.

not rich.

$\boxed{Q}$ How about Lebesgue measurability?

Exm 3

- J - max. ideal

Def

$$\delta(A) = \lim_{J} \frac{|A \cap n|}{n}$$

- δ - rich finitely additive **measure**

$$\bar{I} := \mathcal{Z}_\delta$$

- There is no uncountably I-AD family in I^+

b/ δ - is a finite measure, so c.c.c.

- Law of Large Numbers:

$\hookrightarrow$

$$\left\{ A \subseteq \mathbb{N} : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1_A(k) = \frac{1}{2} \right\} =$$

---

of measure 1

$$= \left\{ A \subset \mathbb{N} : \lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n} = \frac{1}{2} \right\}$$

Then

$$I = \mathcal{I}_\delta \subseteq P(\omega) \setminus L$$

I is of measure zero.

TaA + Rich

Thm (Di Nasso – Jin, 2018) (Acta Arith).

If I is a summable ideal then there is a.u.d S which is rich and translation invariant.

Thm The same as above for translation invariant ideals w/ the Baire property.

Proof

- For any T.A. ideal w/ B.P there is $\mathcal{I}$ translation almost disjoint family in $\mathcal{I}^+$ of card $\mathfrak{c}$.

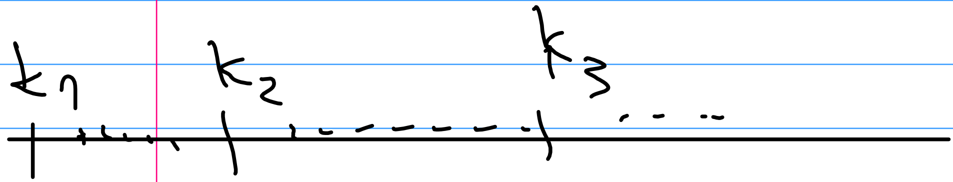
$\mathcal{I} = $ T.A.I
$\forall \cup$ $A \cap (B+k) \in \overline{\mathcal{I}}$
k $A \neq B$

- Talagrand characterization of id. w/ B.P.

$\exists_{k_1 < k_2 < \dots}$ $\underbrace{\left(\exists_n^\infty [k_n, k_{n+1}) \subseteq A \Rightarrow A \notin \mathcal{I} \right)}$

• WLOG

$$\lim_{n \to \infty} (k_{n+1} - k_n) = \infty$$



$$I_n = [k_n, k_{n+1})$$

• Take Fin-AD family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ of card $\mathfrak{c}$.

• For $A \in \mathcal{A}$

$$C_A = \bigcup_{n \in A} I_{2n}$$

$$\boxed{C_A = \bigcup_{n \in A} I_n}$$

$\mathcal{A} = \{1, 4, \dots\}$

- $|\mathcal{C}| = ?$

$\mathcal{C} = \{C_A : A \in \mathcal{A}\}$

- $C_A \in \mathcal{I}^+$ (by Talagrand)

- $\mathcal{C}$ is $\mathcal{I}$-T.A.D.

  (Fin-T.A.D.)

$A \neq B$, $A, B \in \mathcal{A}$

$k \in \mathbb{Z}$, $\boxed{C_A \cap (C_B + k) \in \mathrm{Fin}}$

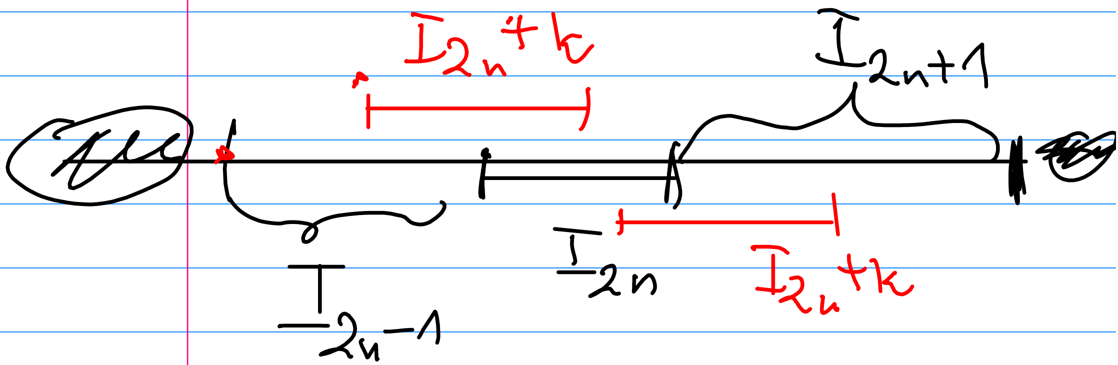
$\mathcal{I}$

Since $|I_n| \to \infty$,

$$\exists_N \ \forall_{n>N} \ |I_n| > k$$

Now

$$\forall_{n>N} \left(I_{2n} + k \subseteq I_{2n-1} \cup I_{2n} \cup I_{2n+1} \right)$$



$$\forall_{n>N} \ \forall_{m \neq n} \ I_{2m} \cap (I_{2n} + k) = \phi$$

$$\boxed{C_A \cap (C_B + k)} =$$

$$= \bigcup_{m\in A} I_{2m} \cap \left(\bigcup_{n\in B} I_{2n} + k\right) =$$

$$= \bigcup_{m\in A} \bigcup_{n\in B} \left(\underbrace{I_{2m} \cap (I_{2n} + k)}_{m=n \ \neq \emptyset}\right)$$

$$\subseteq \underbrace{\underbrace{\bigcup_{n\le N} I_n}_{\text{Fin}} \cup \underbrace{\bigcup_{n\in A\cap B} \left(I_{2n} \cap (I_{2n} + k)\right)}_{\text{Finite}}}_{\text{Finite}}$$

Finite

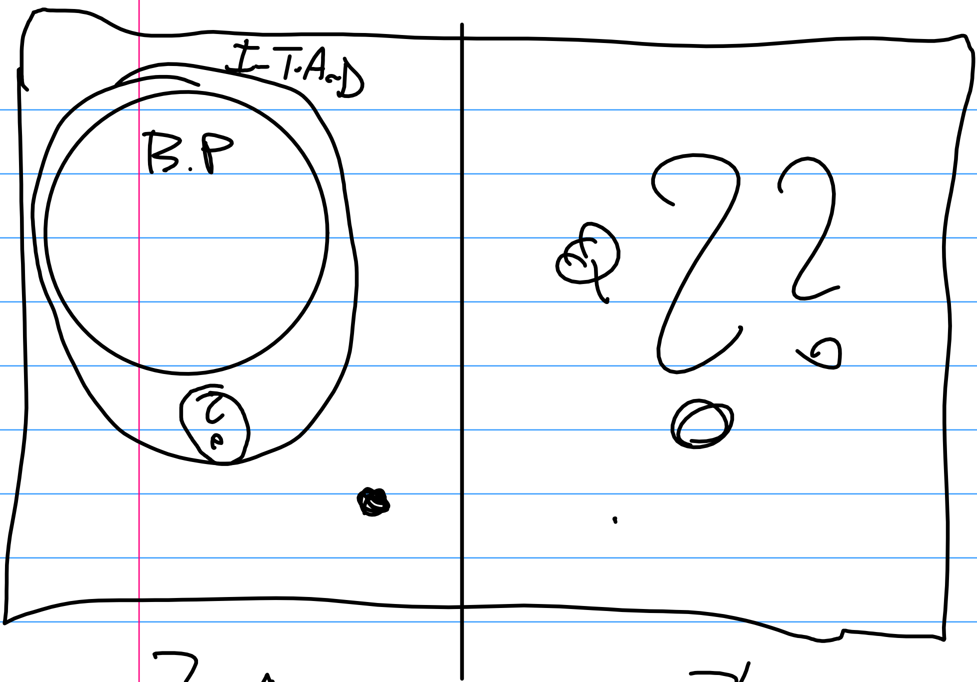
- Let $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ be I-T.A.D family.

- Extend $\mathcal{A}$ to maximal I-T.A.D.

- $(0,1) = \{r_\alpha : \alpha < \mathfrak{c}\}$

Define:

$$\boxed{\delta(A) = \sup\{r_\alpha : \exists k \in \mathbb{Z} \\ CA_\alpha \cap (A+k) \notin I^+\}}$$



Rich + T.A.