

**REPORT ON  
“LIMITS OF SATURATED IDEALS OF POINTS WITH  
APPLICATIONS TO SECANT VARIETIES”**

**AUTHOR: TOMASZ MAŃDZIUK  
PHD DISSERTATION: UNIVERSITY OF WARSAW**

Broadly speaking, this thesis examines the geometry of zero-dimensional schemes. After the introduction and a chapter summarizing some of the relevant background material, the main body of the document has three chapters, each showcasing a different aspect of zero-dimensional schemes. As the author correctly indicates, the best results appear in Chapter 3 which concentrates on zero-dimensional schemes in projective space. Emphasizing multigraded Hilbert schemes (which parametrize all ideals with a fixed Hilbert function), the author provides three necessary [Proposition 3.1, Theorem 3.5, Theorem 3.40] and one sufficient condition [Theorem 3.12] for a point on a specific multigraded Hilbert scheme to lie in the unique irreducible component containing the locus of points corresponding to radical ideals. The focus on one particular Hilbert function is tolerable given its explicit connections to the classic Hilbert scheme (parametrizing subchemes with a fixed Hilbert polynomial). Chapter 4 strives to extend the analysis on projective space to all smooth projective toric varieties. The results in this context are somewhat limited. However, the insight [Theorem 4.15] into the behaviour of multigraded Hilbert schemes under morphisms is innovative and the application to products of projective space is worthwhile. Chapter 5 explores the well-known connection between zero-dimensional schemes and secant varieties (and their generalization called cactus varieties). This short chapter basically restates results (sometimes without proof) from a recent preprint.

In my judgement, the mathematical work presented in this document does meet or exceed the essential requirements for a PhD dissertation in the mathematical sciences. Indeed, I am reasonably convinced that the results are correct and new in a nontrivial sense. This work does solve some existing problems of reasonable importance and will appeal to an appreciable subset of the contemporary mathematical research community. For example, I believe that, after being subdivided into at least a couple independent articles, this research should be publishable in respected mathematical journals. Although I was able to understand the material without undue difficulty, the quality of the exposition is not nearly as strong as the underlying mathematics. Numerous stylistic, usage, and grammatical issues diminish the overall impact of the thesis. However, I expect that these details can be adequately addressed by making some relatively minor changes to the file; concrete possibilities are enumerated below. In summary, I am happy to recommend that this dissertation be accepted pending minor revisions.

---

*Date:* 31 January 2022.

## MINOR COMMENTS, QUESTIONS, AND SUGGESTIONS.

- Hyphens “-”, minus signs “−”, en-dashes “–”, and em-dashes “—” are slightly different punctuation. Throughout the document, these differences appear to be ignored; see any style guide (such as the Chicago Manual of Style) for the proper usage.
- There many phrases or expression in the document that fail to be sentences because they lack a verb. A mathematical expression cannot serve as the verb; typical examples have the form “then  $\$. . . \$$ ” or “In particular,  $\$. . . \$$ ”. Please correct these grammatical errors.
- Make quantifiers unambiguous: replace “for  $\$. . . \$$ ” with “for all  $\$. . . \$$ ” or “for some  $\$. . . \$$ ”.
- Avoid indicating future action when referencing this thesis. For instance, avoid “we shall prove” or “we will prove”. Since the document already exists, these phrases can suggest that the arguments in the document need to be completed in a future (i.e. they are not currently complete). I suggest searching for the words “will” and “shall” in the document and essentially eliminating all occurrences.
- Avoid starting the last sentence in every lemma, theorem, and proposition with “Then ...”. In most cases, the word “Then” can simply be omitted.
- Avoid using a comma as an abbreviation/replacement for the word “and”. Many example in the document involve a pair of integers: use “ $r$  and  $n$ ” rather than “ $r, n$ ”.
- Throughout the document, remove the phrase “In the above notation”. It serves no useful purpose.
- (page 1) The analogy with fruit seems overly simplistic and potential confusing. The anatomy of even simple fruits is more complicated: the outermost layer (epicarp) often contains the essential oils and flavours, the fleshy middle layer (mesocarp) is usually the part that is eaten, and inside layer (endocarp) surrounds the seeds.
- (page 2) Replace “positive integers  $r, n,$ ” with “positive integers  $r$  and  $n,$ ”.
- (page 3) I think that it would help the reader to explain the acronyms for “Slip” and “Slip”.
- (page 3) Why introduce the variables “ $x_0, x_1, \dots, x_n$ ” for the variables in the homogeneous coordinate ring for  $\mathbb{P}^n$  rather than using the “ $\alpha_0, \alpha_1, \dots, \alpha_n$ ” which seem to appear everywhere else.
- (page 4, Proposition 1.1) The phrase “If ... then  $[I] \notin \text{Slip}_{r,n}.$ ” is not a sentence. Try “If ..., then we have ...” or “If ..., then the point  $[I]$  does not belong to  $\text{Slip}_{r,n}$ ”.
- (page 5, Proposition 1.4) Replace “Assume that the following ... . Then there” with “When the following hold ... , there is not point  $[I] \in \dots$ ”.
- (page 11) Use `\hfill` before `\begin{enumerate}` if necessary to prevent the misalignment of the first item (as for example happens in the proof of Lemma 2.9).
- (page 12) Why not actually recall the definition of a generic initial ideal rather than just providing a reference?
- (page 12) Between binary operators like “ $>$ ” use `\dotsb` rather than `\dotsc`  
 $\dots$

- (page 12) Use `\left(` and `\right)` to automatically obtain the right size parentheses around the products with  $\prod$ .
- (page 13) Condition (\*) is often stated as the ideal  $J$  contains a power of the irrelevant ideal  $\mathfrak{m} = (\alpha_0, \alpha_1, \dots, \alpha_n)$ .
- (page 16) Why has the ambient polynomial become  $T$  rather than  $S$ ? The letter  $T$  seems an unfortunate choice when discussing tangent spaces.
- (page 17) Replace “Note that we” with “We”.
- (page 20) Having a separate (one-sentence) paragraph for each subsection seems unnecessary. Is more sentence variety possible?
- (page 21) For readability and completeness, one should introduce your notation for the category  $\mathbb{k}\text{-Alg}$  of  $\mathbb{k}$ -algebra (here one needs a hyphen not a minus sign), the category  $\mathbf{Set}$  of sets, and the category  $\mathbf{Sch}_{\mathbb{k}}$  of schemes over  $\mathbb{k}$ .
- (page 25–26) Since it is used so infrequently, I suggest that you avoid introducing the notation  $\mathbb{k}\text{-f.g. Alg}$  for the subcategory of finitely generated  $\mathbb{k}$ -algebra.
- (page 26) To explain the importance of  $n$ , I would remind the reader that the polynomial ring  $S$  has  $n + 1$  variables in Lemma 2.44.
- (page 27) Replace “ $r, n$ ” with “ $r$  and  $n$ ”. Better yet, rephrase Proposition 2.45 as “For any two positive integers  $r$  and  $n$ , the closure of  $\text{Sip}_{r,n}$  in  $\text{Hilb}_S^{h_{r,n}}$  is an irreducible component.” Why use  $S[\mathbb{P}^n]$  rather than  $S$ ?
- (page 27) Put the definition of  $\text{Slip}_{r,n}$  in a “Definition” environment.
- (page 27) Omit the “map considered above”. For example, say “Let  $\varphi_{r,n}: \text{Hilb}_S^{h_{r,n}} \rightarrow \text{Hilb}_r(\mathbb{P}^n)$  is defined on closed points by ...”.
- (page 36) Why not use  $\mathbf{A}$  to denote an abelian group throughout the document?
- (page 37) Replace “1-1” with “one-to-one”.
- (page 37) Eliminate the blank line(s) before “Moreover” in the statement of Theorem 2.77. The final sentence does not need to be a new paragraph.
- (page 39) Replace “have standard” with “have the standard”.
- (page 40) What does  $\mathbb{k}_{dp}$  denote or mean?
- (page 40) The clause that starts “That is, as ...” is not a sentence. Fix this grammatical error.
- (page 41) Why use `\righthalfcup` to indicate the natural action of  $S$  on  $S^*$ ? Are  $\bullet$  or  $\circ$  not much more common?
- (page 41–43) Replace “Let  $d, k$ ” with “Let  $d$  and  $k$ ”.
- (page 45) Replace “zero dimensional” with “zero-dimensional”.
- (page 45) Be more explicit about the semicontinuity result you are using and, ideally, include reference.
- (page 47) Tables should have use the `\begin{table}`–`\end{table}` environment and have a caption.
- (page 47) Replace “check mark” with “check mark ( $\checkmark$ )”.
- (page 47) Replace “question mark” with “question mark (?)”.
- (page 49) Why use  $(\star)$  rather than an equation number like (3.10)?
- (page 50–53) Omit the clause “In the above notation” in all of the lemmas. If you are relying on a global notation, then clause is not helpful.
- (page 54) Why not include the Hilbert function of  $S/J$  in Example 3.32?
- (page 54) Can you give more precise references in [21] and [58]?
- (page 55) Replace “pages 111,112” with “pp. 111–112”.

- (page 55) Combine Propositions 3.35 and 3.36 into one statement?
- (page 56) State Proposition 3.39 before proving it.
- (page 56) Replace “conditions 1.-3.” with “conditions 1–3” (conditions~1--3). In particular, omit the period(s) “.” from citations of the conditions.
- (page 58) Replace “non-zero” with “nonzero”.
- (page 58) Why is “Notation 3.43” rather than “Remark 3.43”? What ‘notation’ is being introduced? In the subsequent Lemma and Proposition, this paragraph is treated as part of the hypotheses.
- (page 72) Why do you stress the existence of a monomial ideal?
- (page 75) Why not merge Lemma 3.90 and 3.91 into one statement?
- (page 82) I am not sure what the sentence “we present our results in their natural generality” means. When would one choose an “unnatural” level of generality?
- (page 83) Replace “referring” with “referring”.
- (page 84) Replace “Note that the ... . Some” with “Although the ... , some”.
- (page 85) Omit “We recall this here.”.
- (page 85) Omit “ $= X_{\Sigma_X}$ ”. Doesn’t the notation  $\Sigma_X$  already implies that the fan comes from  $X$ ?
- (page 86) Replace “ $X, Y$ ” with “ $X$  and  $Y$ ”.
- (page 88) I would also indicate that the isomorphism  $f_*(\mathcal{O}_X) = \mathcal{O}_Y$  implies that  $f$  is surjective and has connected fibers (the converse also holds in characteristic zero).
- (page 89) It is unusual to rely (i.e. Theorem 4.12) so heavily on unpublished working notes, but to not include the proof.
- (page 90) Why start new paragraphs with “The first claim” and “The second claim”?
- (page 92) The assertion that “it is hard or impossible to sensibly define” seems overly vague. What should the word ‘sensible’ mean in this context? Is it actually impossible in some case?
- (page 93) What does the  $*$  in  $\Sigma_Y^*(\tau)$  indicate?
- (page 94) Why not put both  $S[Y]$  and  $S[X]$  inline? Too many mathematical expressions on this page are displayed.
- (page 101) Why introduce the symbol  $\theta_2$ ? It seems to be used only 3 times—would it not be clearer to simply repeat the 3 term polynomial on these occasions.
- (page 103) Omit the sentence “Recall the definitions of border rank ... Subsection 2.4.2”.
- (page 107) Why not include the *Macaulay2* calculation?
- (page 110) Replace “Note that the” with “The”.
- (page 118) The reference [11] appears to be incomplete.
- (page 119) The reference [24] appears to be incomplete.
- (page 121) The reference [58] appears to be incomplete.
- (page 122) The reference [65] appears to be incomplete.
- (page 122) The reference [71] appears to be incomplete.
- (page 122) The reference [72] appears to be incomplete.