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Report on the PhD thesis of Tomasz Galazka

To whom it may concern:

The thesis is written under supervision of dr. hab. Adam Osekowski at the university of Warsaw. The main topic concerns martingale inequalities, and applications to harmonic analysis. The overall goal of the thesis is to prove sharp estimates in martingale theory and harmonic analysis. This field of mathematics is highly involved and technical. The leading method in the field is the Bellman function method, and this method is the main method in the thesis.

Chapter 2: A dual approach to Burkholder's L^p estimates

Chapter 2 is based on the preprint "A dual approach to Burkholder's L^p estimates" written jointly with Banuelos and Osekowski. In a celebrated result by Burkholder of 1984 it was proved that for martingales $f = (f_n)$ and (g_n) with values in a Hilbert space H one has

$$\|g\|_p \leq (p^* - 1)\|f\|_p, \quad p \in (1, \infty). \tag{1}$$

where g is differentially subordinated to f, which means $\|dg_n\| \leq \|df_n\|$, where $df_n = f_n - f_{n-1}$. The number p^* arises in many estimates in martingale theory and harmonic analysis and is defined by $p^* = \max\{p, p'\}$, where p' is the Hölder conjugate of p. Afterwards, the estimate (1) was extended to the continuous time setting by Wang in 1995. A Bellman approach to prove such estimates was invented by Burkholder and explained well in his lecture notes from 1991. In the thesis a new method has been discovered which can be seen as a dual method. The main result of chapter 2 is the following:

Suppose that X, Y, Z are H-valued local martingales such that Y is differentially subordinate to X, then

$$\|[Y, Z]_\infty\|_1 \leq (p^* - 1)\|X\|_p\|Z\|_{p'}, \quad p \in (1, \infty). \tag{2}$$

Here $[Y, Z]$ denotes the cross-variation between Y and Z. By optimizing over Z, one can obtain the estimate $\|Y\|_p \leq (p^* - 1)\|X\|_p$ as an immediate corollary of (2). This shows that (2) is a very powerful duality type estimate.

Earlier duality type estimates occur in famous works of Nazarov, Treil, Volberg, but did not give sharp constants. Thus the chapter can be seen as a major improvement of these deep results. There were earlier attempts to use other more complicated Bellman functions (depending on 4 variables) to get sharp estimates by Banuelos and Osekowski. However, it seems to me that the formulation (2) is new, and the proof uses a more natural Bellman function depending on only two variables. There are some known results in harmonic analysis (in particular Littlewood-Paley theory and functional calculus), where variants of $\|[Y,Z]_\infty\|_1$ appear. These turn out to be extremely powerful. The probabilistic version (2) probably implies several of these, and I expect it to be of great use in the area. Some applications to Littlewood-Paley theory are already given as an illustration in the chapter.

Finally let me remark that it could be interesting to know whether there are probabilistic proofs/anaolgues of the more recent extensions of the Littlewood-Paley estimates in for instance

- Convexity of power functions and bilinear embedding for divergence-form operators with complex coefficients by Carbonaro and Dragičević 2020.

In the latter paper a related Bellman function is used.

Chapter 3. Sharp estimates for martingale transforms with unbounded transforming sequences

Chapter 3 can be viewed as an extension of some of the results in Chapter 2. As a consequence of Chapter 2 one has

$$\|Y\|_p \le (p^* - 1)\|X\|_p, \tag{3}$$

if $|H| \le 1$ and $Y = H \cdot X$ (the stochastic integral). The surprising thing in Theorem 3.3 is that one actually has

$$\|Y\|_p \le C_{p,q,r}\|X\|_q\|H^*\|_r, \tag{4}$$

where one needs that H is left-continuous for technical reasons. Here $1/p = 1/q + 1/r$ and the constant $C_{p,q,r}$ is explicit and optimal. By letting $r \to \infty$ one actually recovers (3) from (4) (for the case of left-continuous H). Moreover, one actually only needs that Y is differentially subordinate to $H \cdot X$ in (4).

In my opinion the estimate (4) with optimal constants is really surprising. It seems to be something completely new in the literature, and I expect that it will have a lot of impact. The strange thing is that usually in harmonic analysis, off-diagonal estimates (i.e. different integrability order of the functions in considerations) are somewhat simpler and not really singular. Now with (4) the situation is different since the optimal constants allow to let $r \to \infty$. It would be interesting to see which implications (4) has in harmonic analysis.

The proof of (4) uses two Bellman functions again (one of Burkholder, and one of Baernstein II and Montgomery-Smith). The proof of the estimate (4) is very clever and uses Ito's formula for the Bellman functions in consideration. The proof of the sharpness of the constants is quite deep and technical, and sharpness is already reached in the discrete setting.

Versions of (4) with weak type norms for Y are also obtained (again with sharp constants). It could be interesting to see if one can also take other Banach function space norms of Y, H^* and X (such as weak type norms of H or X). In principle such results seem likely to hold given the standard estimates one could get from real interpolation. Of course optimal constants will be much harder to obtain in this framework, and probably are less important.

It would also be interesting to know whether the assumption that H is left-continuous can be weakened. Finally, it could be interesting to see if there are UMD-valued generalizations of (4).

Chapter 4. Sharp $L^p \to L^{q,\infty}$ estimates for Hilbert and Riesz transforms on compact Lie groups

Chapter 4 discusses estimates of the form $\|g\|_{q,\infty} \leq C\|f\|_p$ where g is a certain transform of f. Here f and g are defined on a finite measure space and $p < q$. Of main interest in this chapter are the Hilbert and Riesz transforms on the (d-dimensional) torus. Again in many cases sharp constants are obtained, but this time the descriptions of the constants are rather technical. Martingale versions of the estimates are considered as well, and play a crucial role in the proofs. Extensions to compact manifolds are also considered, but optimality of the constants remains open here.

Of main interest in the chapter is the following estimate for the Hilbert transform

$$\|Hf\|_{q,\infty} \leq C_{p,q}\|f\|_p, \tag{5}$$

for $q < p$. Here a standard renorming for the weak quasi-norm is used. The precise constant $C_{p,q}$ in the limiting case $q = p$ was obtained by Osekowski for al $p \in (1, \infty)$ in his 2014 paper titled "Sharp inequalities for Riesz transforms".

A surprising feature of the estimate (5) is that the constant does not depend on q if $q \leq p \leq 2$, and thus $C_{p,q} = C_{p,p}$. This is no longer true if $p > 2$. It is surprising that such precise information on the Hilbert transform can be deduced. It could be interesting to obtain similar results in the case $\|Hf\|_{q,\infty}$ is replaced by $\|Hf\|_q$. This would shine new light on the operator norm of H. Maybe it could even open new doors to open problems related to other operators such as the Beurling-Ahlfords transform, which are stlil not well-understood.

The proofs of the results of this chapter are extensions of the argument in Osekowski's paper mentioned before. But still the results are surprising and nontrivial to obtain.

Chapter 5. Sharp analytic version of Fefferman's inequality

The $H^1 - BMO$ duality due to Fefferman is very important in harmonic analysis and probability theory. One of the main results due to Feffereman is that for $f \in H^1(T)$ and $g \in BMO$ with mean zero one has

$$\left| \int_T \bar{f} g dx \right| \leq C\|f\|_{H^1(T)}\|g\|_{BMO}. \tag{6}$$

The best constant C in the real case is known since works of Getoor and Sharpe, and also Osekowski. Its precise value is very relevant as it enters many other results which are deduced from (6). In the thesis a different norm on BMO is used, which is more natural in the conformal setting. It is shown that one can take $C = \sqrt{e^2 + 1}$ and that this constant is optimal. It is even optimal if one considers the weaker estimate

$$Re \int_T \bar{f} g dx \leq C\|f\|_{H^1(T)}\|g\|_{BMO}. \tag{7}$$

The proof of the lower estimate $C \geq \sqrt{e^2 + 1}$ is not based on an example of functions f and g as one would expect. Instead the theory of analytic envelopes and plurisuperharmonic functions is used.

For the proof of $C \leq \sqrt{e^2+1}$ some of the insights in the proof of the lower estimate are actually relevant. The key in the proof is eventually another Bellman function argument, but this time a non-standard. Together with Ito's formula for complex martingales, and a representation theorem using Poisson transforms and Brownian motion, the proof is then completed.

It would be interesting to see what can be said if *mathbbT* is replaced by the n-dimensional torus T^n. But it seems that the techniques of chapter 5 are not immediately extendable to this setting.

Final assessment

It was a great please to read the thesis of Galazka. It is clearly sufficient for obtaining a doctoral degree. In my opinion the thesis is actually exceptionally good and certainly belongs to the top 15% of theses I have seen in mathematics. The variety of mathematical tools is enormous and ranges from probability theory to harmonic analysis to complex analysis, and geometry. Obtaining new results in such a wide context is quite exceptional at the level of a PhD. Working with Bellman function is not easy to learn, and it is clear that Galazka has mastered this technique very well. He even found several new Bellman functions and applied them to prove estimates in probability theory and harmonic analysis.

I have almost no criticism on the presentation. The thesis is written in an excellent way. One minor point which is a personal taste: I would have included a longer introduction (instead of just a summary), to explain some of the main results. On the other hand, I should say that the individual chapters contain excellent introductions. These are very helpful for the reader.

What I cannot judge as external reviewer is what the contribution of the PhD student is, and what the contribution of the supervisor is. All papers are jointly with Osekowski, so this is impossible for me to say. However, let me add that even if the contribution of Galazka to the joint papers is minimally 50%, I would still propose to award the thesis a distinction.

Besides the thesis, Galazka recently completed a highly impressive work on noncommutative martingales, jointly with Jiao, Osekowski and Wu. Based on the thesis and the mentioned recent preprint of Galazka, I expect that Galazka could have a far-reaching mathematical career. Having a distinction on this piece of work would set the right stage in this respect.

Sincerely,

Prof. dr. ir. M.C. Veraar

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