Forcing-theoretic framework for the Fraïssé theory

A doctoral dissertation

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Abstract

In this dissertation, we study possible generalizations of the Fraïssé theory, with the use of the set-theoretic forcing. Mainly, we investigate structures added generically by partial orders of the form

$$\mathrm{Fn}(S, \mathcal{K}, \lambda) = \{A \in \mathcal{K} |\ F(A) \in [S]^{<\lambda}\},$$

where $F(A)$ denotes the underlying set of a structure A, S is any uncountable set, and λ is an infinite cardinal. Forcings

$$\mathrm{Fn}(\omega_1, \mathcal{K}, \omega)$$

are of our special interest. If $\mathcal{K}$ above is the class of all linear orders, the corresponding generic filters produce instances of so-called *increasing sets*. Introduced by Avraham and Shelah in the 1980s, they were used for example to prove the following result:

Theorem (Avraham-Shelah, [6]). *It is consistent with $ZFC + MA +$ "$2^\omega = \omega_2$" that there exists an uncountable set $A \subseteq \mathbb{R}$ with the property that each uncountable function $f \subseteq A \times A$ is non-decreasing on an uncountable set.*

The main innovation we introduce, is an extension of the notion of increasing set to structures different from linear orders. This results in what we call *rectangular models*. They allow to prove other similar results, for instance

Theorem (Kostana). *It is consistent with $ZFC + MA +$ "$2^\omega = \omega_2$" that there exists an uncountable, separable rational metric space (X, d) such that each uncountable 1-1 partial function $f \subseteq X \times X$ is an isometry on an uncountable set.*

It should be emphasized, that despite the obvious model-theoretic aspect, this is a dissertation about the set theory. The apparatus of model theory is very basic. We only assume, that the reader knows the definitions of objects like: a first order theory, a model, a homomorphism, the automorphism group of a model etc. We presuppose the knowledge of general topology at the level of the Baire Category Theorem and the Stone duality. In a few places, mainly in Section 2.3, we refer to elementary probability and measure theory. On the other hand, the set-theoretic machinery is rather sophisticated, and this refers particularly to forcing-theoretic arguments. Although almost all forcing notions appearing in the dissertation are c.c.c. (and indeed most of them resemble the Cohen forcing) arguments sometimes get quite technical and involved.

Keywords: Fraïssé limit, homogeneous structure, generic structure, random graph, saturated models, Baumgartner's Theorem
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Streszczenie

Przedmiotem pracy jest zbadanie możliwych uogólnień teorii Fraïsségo przy użyciu metody forcingu. Głównym obiektem naszych zainteresowań są struktury dodawane przez filtry generyczne, dla forcingów postaci

$$\mathrm{Fn}(S, \mathcal{K}, \lambda) = \{A \in \mathcal{K} | \ F(A) \in [S]^{<\lambda}\},$$

gdzie $F(A)$ oznacza uniwersum struktury A, S jest zbiorem nieprzeliczalnym, zaś λ jest nieskończoną liczbą kardynalną. Szczególną uwagę poświęcamy forcingom

$$\mathrm{Fn}(\omega_1, \mathcal{K}, \omega).$$

Jeśli $\mathcal{K}$ powyżej jest klasą porządków liniowych, to odpowiednie filtry generyczne dają przykłady tzw. *zbiorów rosnących* (ang. *increasing sets*). Te zbiory, wprowadzone przez Avrahama i Shelaha w latach 80-tych, pozwalają udowodnić na przykład

Twierdzenie (Avraham-Shelah, [6])**.** *Niesprzecznie z $ZFC + MA +$ "$2^\omega = \omega_2$" istnieje nieprzeliczalny zbiór $A \subseteq \mathbb{R}$ o tej własności, że każda nieprzeliczalna częściowa $f \subseteq A \times A$ jest niemalejąca na zbiorze nieprzeliczalnym.*

Główną nowością w pracy jest rozszerzenie pojęcia zbioru rosnącego na struktury inne niż porządki liniowe. Wprowadzamy tzw. *modele prostokątne*, które pozwalają uzyskać podobne wyniki dla innych struktur, na przykład:

Twierdzenie (Kostana)**.** *Niesprzecznie z $ZFC + MA +$ "$2^\omega = \omega_2$" istnieje nieprzeliczalna, ośrodkowa, wymierna przestrzeń metryczna (X, d) o tej własności, że każda nieprzeliczalna częściowa funkcja różnowartościowa $f \subseteq X \times X$ jest izometrią na pewnym zbiorze nieprzeliczalnym.*

Należy wyraźnie zaznaczyć, że mimo oczywistych aspektów teoriomodelowych jest to zdecydowanie praca o teorii mnogości. Aparat teoriomodelowy z którego korzystamy jest bardzo podstawowy. Zakładamy znajomość jedynie takich pojęć jak: teoria pierwszego rzędu, model, homomorfizm, czy grupa automorfizmów. Zakładamy również znajomość topologii ogólnej na poziomie Twierdzenia Baire'a czy też dualności Stone'a. W kilku miejscach, głównie w podrozdziale 2.3, pojawiaja się prosty rachunek prawdopodobieństwa i teoria miary. Z drugiej strony, maszyneria teoriomnogościowa, a w szczególności forcingowa, jest dość zaawansowana. Choć niemal wszystkie forcingi występujące w pracy są c.c.c. (a istotnie większość z nich przypomina forcing Cohena) argumenty są czasem techniczne i złożone.

Declarations

I hereby declare that this dissertation is my own work.

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Contents

Chapter 1

Introduction

1.1 History

The story begins with the discovery by Georg Cantor, that the order of rational numbers is the unique, up to isomorphism, countable, dense linear order without endpoints [8]. It is hard to say, whether his proof, from the 1895 article *Beiträge zur Begründung der transfiniten Mengenlehre*, can be called the *back-and-forth argument*. What is important, is that this theorem is now a textbook example of a situation, where one can show that two structures are isomorphic by means of successively extending finite partial isomorphism between them. This idea, today known as the *back-and-forth* method, has found numerous applications since then, and – what is perhaps even more important – served as the starting ground for many other mathematical theories.

Probably the most fruitful of them was formulated in 1954 by Roland Fraïssé [15]. By this time, model theory has become a well-established branch of mathematical logic. The generality it offered, allowed Fraïssé to formulate conditions for a class of finite models in a first order language that allow to prove a version of Cantor's Theorem for this class. Specifically, these conditions imply the existence of a unique up to isomorphism countable model canonically ascribed to this class. One of the features of the Fraïssé Theorem is that when we have this countable model, say M, in hand, we can recover the original class of finite models – these are just those, which can be isomorphically embedded in M. The model M is *homogeneous*, i.e. each finite partial automorphism extends to a full automorphism of M. If the language does not contain function symbols, M is also ω-categorical, which means that it is determined among all countable models by its first order theory.

The notion of *elementary equivalence* is an important weakening of isomorphism. Roughly speaking, it says that two models are indistinguishable from the viewpoint of the first order logic, although they still may be very different (for example have different cardinalities). A classical result from model theory says that all dense linear orders without endpoints are elementary equivalent in the language of linear orders. This is proved using the concept of the *Ehrenfeucht-Fraïssé game*, which is another emanation of the "back-and-forth" method.

In 1963 Ivan Parovičenko introduced what is now known as the Parovičenko space [40]. It is a compact, zero-dimensional topological space, satisfying some further conditions which ensure its uniqueness if the Continuum Hypothesis holds. In this case, it is homeomorphic to the Čech-Stone remainder of the countable discrete space, namely $\beta\mathbb{N} \setminus \mathbb{N}$. By the Stone duality, describing the Parovičenko space is equivalent to describing the Parovičenko Boolean algebra, and the proof of uniqueness of this space is reduced to the proof of uniqueness of the Boolean algebra. The only reason why Parovičenko's reasoning doesn't fit into the framework developed by Fraïssé, is that we deal with countable models instead of finite, and the resulting "big" model has size ω_1. The principle is however the same, and it is not surprising that a suitable generalization of Fraïssé theory was developed in order to take uncountable models into account. This line of research was pursued, among others, by Jónsson [25] and Morley & Vaught [36] in the 1960's.

In the meantime, category theory was gaining more and more recognition as a clear and expressive language for a significant part of mathematics. It could not have been missed, that Fraïssé theory can be re-introduced in this language, with the benefit of some more generality. The category-theoretic framework for Fraïssé theory was introduced by Droste and Göbel in [13], [12], and was further generalized by Kubiś [32]. The theory originally developed in the language of models and homomorphisms has declared independence from model theory, and was ready to become an influential technology in general topology [24], topological dynamics [26], theory of Polish groups [43], [27], or functional analysis [16], [4]. The article [35] is an excellent survey of what is known about homogeneous structures and their automorphism groups, and [27] contains an extensive introduction to the study of topological properties of these groups.

However, we do not present the theory in the most general possible setting. It was already done by Kubiś in [32], together with a brief historical introduction and numerous examples. Somewhat in the other direction, we would like to stay in the realm of the model theory, and study connections of Fraïssé theory with set-theoretic forcing. Specifically, building on the intuition that the Baire Category Theorem is the heart of the theory, we investigate how models studied by Fraïssé can be obtained as generic filters for partial orders, and what uncountable models we can build in a similar way.

1.2 Preliminaries

Chapters 2 and 3 are mainly of introductory nature, with a bit of a survey of more recent results. We present in them the classical Fraïssé theory, and its uncountable variant – the Fraïssé-Jónsson theory – respectively. They rely only on basic notions from model theory, with occasional exceptions (like the measure space from 2.3). In Chapter 4 we introduce and study uncountable generic structures, added by specific forcing notions, that resemble those arising from Fraïssé theory. We show that they are almost always rigid, at least in the corresponding generic extensions, which is a significant difference from the countable case. In Chapter 5 we study these structures under the assumption of Martin's Axiom. In this case, their automorphism groups are much richer, and in some cases we can provide conditions characterizing these models uniquely. Exposition of of Chapters 4 and 5 follow our preprints [29] and [31]. All theorems appearing in these chapters are author's original, unless stated otherwise. Apart from them, author's original contribution are a few results at the end of subsection 3.2.4.

We use terms *model* and *structure* exchangeably. By *embedding*, we always mean an isomorphism onto its image. A subset of a structure is a *substructure*, if the identity inclusion is an embedding. This convention for example ensures that a *subgraph* is always understood to be an *induced subgraph*. For any function f, we denote by $\operatorname{dom} f$ and $\operatorname{rg} f$ its domain and image respectively. For a structure A, its automorphism group is denoted by $\operatorname{Aut} A$. When considering functions between linear orders, we use terminology *increasing–strictly increasing*, so a constant function is increasing, but not strictly increasing (unless the domain has only one element). Concerning forcing-theoretic terminology, we write $p \leq q$ when p *is stronger than* q, and $p \perp q$ if p and q are *incomparable*. We say that a language is *purely relational*, if it doesn't contain function or constant symbols. For a model A we denote by $F(A)$ the underlying set (universe) of A.

Usually, a mathematical structure X is *homogeneous* if for each pair of points $x, y \in X$ there exists an automorphism of X which maps one of these points to the other. One can replace points with sets from a given class. We can therefore speak about structures *homogeneous with respect to* finite/countable/compact etc. sets. Our default convention is that

$$\text{homogeneous} = \text{homogeneous with respect to finite substructures.}$$

We will depart from this convention in Chapter 2.

Chapter 2

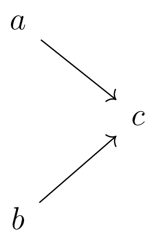
Classical Fraïssé Theory

2.1 Fraïssé Classes

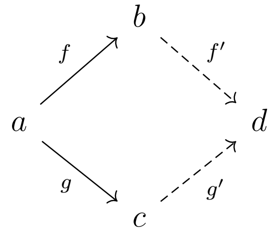
In this chapter we are looking at some classes $\mathcal{K}$ consisting of finite structures in some countable first order language L. Let us make a standing convention that whenever we say *the class of structures*, we strictly speaking mean *the class of isomorphism types*. Unless it is specifically stated otherwise, we only care about isomorphism types of models, not really specific sets of which they consist. We will be looking also at embeddings between structures from $\mathcal{K}$.

Definition 2.1.1. For a class $\mathcal{K}$ we will say that

- $\mathcal{K}$ has the *Joint Embedding Property* (JEP), if for each $a, b \in \mathcal{K}$ there exists $c \in \mathcal{K}$ such that there exist embeddings $a \hookrightarrow c$, and $b \hookrightarrow c$.



- $\mathcal{K}$ has the *Amalgamation Property* (AP), if for each pair of embeddings $f : a \hookrightarrow b$, $g : a \hookrightarrow c$, there exists $d \in \mathcal{K}$, together with a pair of embeddings $f' : b \hookrightarrow d$, $g' : c \hookrightarrow d$, such that $f' \circ f = g' \circ g$.



- $\mathcal{K}$ is *hereditary* if for any $b \in \mathcal{K}$ and any embedding $a \hookrightarrow b$, $a \in \mathcal{K}$.

Notice, that if $\mathcal{K}$ has a weakly initial object, namely a structure which embeds into any element of $\mathcal{K}$, then the JEP follows from the AP. This assumption is typically satisfied, however there are classes with the AP but not the JEP – for instance the class of all finite fields.

Definition 2.1.2. A class $\mathcal{K}$ is a *Fraïssé class* if is satisfies all properties listed above, and has at most countably many models, up to isomorphism.

For checking the Amalgamation Property, we can assume that both initial arrows are identity inclusions. The latter ones however, not always are inclusions, since structures may be "glued together". From time to time we are going to use variants of the AP, which ensures that they aren't.

Definition 2.1.3. A class $\mathcal{K}$ has the *Strong Amalgamation Property* (SAP) if for any structures $a, b, c \in \mathcal{K}$ and embeddings $f : a \hookrightarrow b$, $g : a \hookrightarrow c$, there exists $d \in \mathcal{K}$, together with embeddings $f' : b \hookrightarrow d$, $g' : c \hookrightarrow d$, satisfying $f' \circ f = g' \circ g$, and moreover $\operatorname{rg} f' \cap \operatorname{rg} g' = \operatorname{rg} (f' \circ f)$.

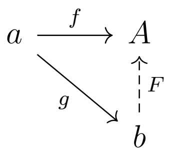
The Strong Amalgamation Property essentially means that given any structure $A \in \mathcal{K}$, and two extensions $B_0 \supseteq A$, $B_1 \supseteq A$, such that $B_0 \cap B_1 = A$, we can find bigger $C \in \mathcal{K}$, containing $B_0 \cup B_1$ (often $C = B_0 \cup B_1$). A close relative of the SAP is the *Splitting Property*. The name was coined by Kubiś. Two embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ are *isomorphic*, if there exists an isomorphism $h : B \hookrightarrow C$, such that $h \circ f = g$. The SP is just the SAP for pairs of isomorphic extensions.

Definition 2.1.4. A class $\mathcal{K}$ has the *Splitting Property* (SP) if for any structures $a, b, c \in \mathcal{K}$ and isomorphic embeddings $f : a \hookrightarrow b$, $g : a \hookrightarrow c$, there exists $d \in \mathcal{K}$, together with embeddings $f' : b \hookrightarrow d$, $g' : c \hookrightarrow d$, satisfying $f' \circ f = g' \circ g$, and moreover $\operatorname{rg} f' \cap \operatorname{rg} g' = \operatorname{rg} (f' \circ f)$.

The importance of the notion of Fraïssé class comes from the classical theorem of Fraïssé, dating back to 1954 (see [15]). If it only didn't sound somewhat posh, we could easily call it *the Fundamental Theorem of the Fraïssé Theory*. For an infinite structure A, we denote by $\operatorname{Age} A$ the class of finite substructures of A. We will say that A is *locally finite* if each finite subset of A is contained in a finite substructure. This will be the case for example when we are working with purely relational language. But first, we need some more definitions.

Definition 2.1.5. A countable structure A is

- *$\mathcal{K}$-universal*, if for every structure $a \in \mathcal{K}$, there exists an embedding $a \hookrightarrow A$.
- *injective*, if for any pair of embeddings $f : a \hookrightarrow A$, $g : a \hookrightarrow b$, where $a, b \in \operatorname{Age} A$, there exists an embedding $F : b \hookrightarrow A$, such that $F \circ g = f$.



- *homogeneous*, if any isomorphism between finite substructures of A extends to an automorphism of A.

The celebrated theorem of Fraïssé is:

Theorem 2.1.6 (Fraïssé, [15])**.** *If $\mathcal{K}$ is a Fraïssé class, then there exists a unique up to isomorphism countable, homogeneous structure $\mathbb{K}$ with $\operatorname{Age} \mathbb{K} = \mathcal{K}$.*

Before we proceed with the proof, let us remark, that although homogeneity is stronger than injectivity, in case of locally finite structures these two properties are equivalent.

Lemma 2.1.7. *If two countable, locally finite structures have the same age and are injective, then they are isomorphic. In fact, any isomorphism between finite substructures $a \subseteq A$, $b \subseteq B$ can be extended to an isomorphism between A and B.*

Proof. Fix two structures A and B as in the statement of the Lemma. We can decompose them into increasing chains $A = \bigcup_{n<\omega} a_n$, and $B = \bigcup_{n<\omega} b_n$ of finite substructures, where $a_0 = a$ and $b_0 = b$. Let $f_0 : a_0 \hookrightarrow b_0$ be given. We will inductively define a sequence of embeddings f_n such that $f_n \subseteq f_{n+1}$, and $f = \bigcup_{n<\omega} f_n$ will be an isomorphism from A onto B.

- Suppose n is even and f_n is defined. We find $l > n$ such that $\operatorname{dom} f_n \subseteq a_l$. Since B is injective, we will find f_{n+1} closing the diagram

$$\begin{array}{ccc} \operatorname{dom} f_n & \subseteq & a_l \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \operatorname{rg} f_n & \subseteq & B \end{array}$$

- Suppose n is odd and f_n is defined. We find $l > n$ such that $\operatorname{rg} f_n \subseteq b_l$. Since A is injective, we will find f_{n+1} closing the diagram

$$\begin{array}{ccc} \operatorname{dom} f_n & \subseteq & A \\ {\scriptstyle f_n^{-1}}\uparrow & & \uparrow{\scriptstyle f_{n+1}^{-1}} \\ \operatorname{rg} f_n & \subseteq & b_l \end{array}$$

It is clear that f is the required isomorphism. □

If we take $A = B$, we obtain:

Corollary 2.1.8. *A countable, locally finite structure is homogeneous if and only if it is injective.*

A similar argument shows that injective models satisfy some stronger, "infinitary", universality.

Lemma 2.1.9. *Let A and B be locally finite, countable structures. Assume moreover, that $\operatorname{Age} B \subseteq \operatorname{Age} A$ and A is injective. Then there exists an embedding $B \hookrightarrow A$.*

Proof. Fix A and B as in the statement of the Lemma. We can decompose them into increasing chains $A = \bigcup_{n<\omega} a_n$, and $B = \bigcup_{n<\omega} b_n$ of finite substructures. Let $f_0 : b_0 \hookrightarrow a_0$ be given. We will inductively define a sequence of embeddings f_n such that $f_n \subseteq f_{n+1}$, and $f = \bigcup_{n<\omega} f_n$ will be an embedding from B into A.

Suppose f_n is defined. We find $l > n$ such that $\operatorname{dom} f_n \subseteq b_l$. Since A is injective, we will find f_{n+1} closing the diagram

$$\begin{array}{ccc} \operatorname{dom} f_n & \subseteq & b_l \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \operatorname{rg} f_n & \subseteq & A \end{array}$$

After ω many steps $f = \bigcup_{n<\omega} f_n$ is the required embedding. □

We will present two proofs of the Fraïssé Theorem. It will be useful to identify certain pairs of embeddings.

Definition 2.1.10. Two embeddings $f : a_0 \hookrightarrow a_1$ and $g : b_0 \hookrightarrow b_1$ are *equivalent* if there exist isomorphisms $j_0 : a_0 \to b_0$ and $j_1 : a_1 \to b_1$ such that $j_1 \circ f = g \circ j_0$.

$$\begin{array}{ccc} a_0 & \xrightarrow{f} & a_1 \\ {\scriptstyle j_0}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle j_1} \\ b_0 & \xrightarrow{g} & b_1 \end{array}$$

The first proof. Enumerate as $\{E_n| \ n < \omega\}$ all isomorphism types of structures from $\mathcal{K}$, and as $\{f_n| \ n < \omega\}$ all embeddings between structures from $\mathcal{K}$, up to equivalence. We aim to inductively build an increasing sequence of structures $F_i \in \mathcal{K}$, and set $\mathbb{K} = \bigcup_{n<\omega} F_n$. For bookkeeping purposes we fix a partition of ω into infinite sets $\{\Phi_n| \ n < \omega\}$, such that $\min \Phi_n \geq n$, for $n < \omega$.

Let $F_0 = E_0$, and enumerate as $\{g_i| \ i \in \Phi_0\}$ all (up to equivalence) embeddings g, with $\operatorname{dom} g \subseteq E_0$ and $\operatorname{rg} g \in \mathcal{K}$.

Assume now that the sets F_k are defined for all $k \leq n$, and so is the set $\{g_i| \ i \in \Phi_n\}$. In particular $\operatorname{dom} g_n \subseteq F_n$. Using the AP we can find F'_{n+1}, together with embeddings closing the diagram

$$\begin{array}{ccc} \operatorname{dom} g_n & \longrightarrow & F_n \\ {\scriptstyle g_n}\big\downarrow & & \big\downarrow \\ \operatorname{rg} g_n & \dashrightarrow & F'_{n+1} \end{array}$$

Using the JEP we can enlarge F'_{n+1} to F_{n+1}, so that E_n embeds into F_{n+1}. Finally, use the set Φ_{n+1} to index all $\mathcal{K}$-embeddings starting from substructures of F_{n+1}.

Let $\mathbb{K} = \bigcup_{n<\omega} F_n$. The second step of the construction clearly ensures universality. Why is $\mathbb{K}$ injective? Let $D \subseteq \mathbb{K}$ be a finite substructure, and $f : D \hookrightarrow E$. D is also a substructure of some F_i, and there is $j > i$, such that g_j is equivalent to f. By the definition of F_j, there exists $h : E \hookrightarrow F_j \subseteq \mathbb{K}$ closing the triangle

$$\begin{array}{ccccc} D & \subseteq & F_{j+1} & \subseteq \mathbb{K} \\ {\scriptstyle g_j}\big\downarrow & \nearrow{\scriptstyle h} & & \\ E & & & \end{array}$$

Uniqueness of $\mathbb{K}$ follows from Corollary 2.1.8. $\square$

Second Proof. In this proof we will slightly depart from one of our conventions made at the beginning of the chapter, since now not only isomorphism types of models will be relevant, but also specific sets on which these models are defined. To avoid confusion, we will denote by $\mathcal{K}_*$ the class of all structures from $\mathcal{K}$, that are defined on subsets of ω. Let $\mathcal{K}_*^\omega$ denote the countable product of $\mathcal{K}_*$ with the product topology (we equip $\mathcal{K}_*$ with the discrete topology). Therefore each element $x \in \mathcal{K}_*^\omega$ is a sequence of finite models, and $x(i)$ denotes the i-th model in this sequence. We will be looking at the space

$$\mathrm{Seq}_{\mathcal{K}} = \{x \in \mathcal{K}_*^\omega | \; \forall\, i < \omega \; x(i) \subseteq x(i+1) \text{ is a substructure}\}.$$

First, note that $\mathrm{Seq}_{\mathcal{K}}$ is a closed subspace of $\mathcal{K}_*^\omega$, therefore a Polish space. From the JEP for $\mathcal{K}$, it easily follows that the following sets are open and dense in the space $\mathrm{Seq}_{\mathcal{K}}$:

$$\mathcal{E}_D = \{x \in \mathrm{Seq}_{\mathcal{K}} \mid \exists\, n < \omega \; \exists\, g : D \hookrightarrow x(n)\},$$

for $D \in \mathcal{K}_*$. Also, from the AP it follows that the following sets are open and dense in $\mathrm{Seq}_{\mathcal{K}}$:

$$\mathcal{E}_{f,n}^{D,E} = \{x \in \mathrm{Seq}_{\mathcal{K}} \mid D \subseteq x(n) \text{ is a substructure } \implies$$
$$\exists\, m > n \; \exists\, g : E \hookrightarrow x(m) \; g \circ f = \mathrm{id}_D\},$$

for $D, E \in \mathcal{K}_*$, an embedding $f : D \hookrightarrow E$, and $n < \omega$.

Given that $\mathcal{K}$ has only countably many isomorphism types, $\mathcal{K}_*$ is countable, and so by the Baire Category Theorem, there is a sequence $x \in \mathrm{Seq}_{\mathcal{K}}$, which belongs to sets $\mathcal{E}_D$, $\mathcal{E}_{f,n}^{D,E}$, for all choices of D, E, f, n. It is standard to verify that $\bigcup x$ is a universal, homogeneous structure. □

The first proof is an instance of the "back-and-forth" argument. The main principles of this argument are independent of the class $\mathcal{K}$, and even from model theory in general. They can be phrased in a very abstract category-theoretic setting, as done in [32].

The advantage of the second proof is that it shows more than merely *existence* of the Fraïssé limit. It actually shows that, in a certain sense, the Fraïssé limit is the most typical of all countable, locally finite structures with age included in $\mathcal{K}$. The phrase *in a certain sense* here adheres to the Baire category, as universal homogeneous structures form a residual set in the space $\mathrm{Seq}_{\mathcal{K}}$. When is it true that the Fraïssé limit of $\mathcal{K}$ is also typical in the sense of measure, or probability, theory? This would aim to show that in a certain measure space of all locally finite structures with age contained in $\mathcal{K}$, those structures which are isomorphic to $\mathbb{K}$, form a set of full measure. In the last section we are going to briefly discuss when this is possible.

Theorem 2.1.11. *If $\mathbb{K}$ is a countable, locally finite, homogeneous structure, then $\operatorname{Age}\mathbb{K}$ is a Fraïssé class.*

Proof. The only non-trivial thing to check is the AP. Fix a pair of embeddings

$$\begin{array}{ccc} A & \longrightarrow & B_0 \\ \downarrow & & \\ B_1 & & \end{array}$$

Without loss of generality we may assume that $A \subseteq B_0 \subseteq \mathbb{K}$. By injectivity there exists an embedding $g : B_1 \hookrightarrow \mathbb{K}$, and by local finiteness $B_0 \cup g[B_1]$ is contained in some finite substructure of $\mathbb{K}$, witnessing the AP for $\operatorname{Age}\mathbb{K}$.

$$\begin{array}{ccccc} A & \subseteq & B_0 & \subseteq & \mathbb{K} \\ \downarrow & & & \nearrow_{g} & \\ B_1 & & & & \end{array}$$

This shows the AP. □

The Strong Amalgamation Property for a Fraïssé class $\mathcal{K}$, with the Fraïssé limit $\mathbb{K}$, corresponds to a certain property of $\mathbb{K}$.

Definition 2.1.12. The structure $\mathbb{K}$ *has no algebraicity* if for each finite substructure $F \subseteq \mathbb{K}$, and for each $f \in \mathbb{K} \setminus F$, f has infinite orbit under the action of the pointwise stabilizer of F in $\operatorname{Aut}\mathbb{K}$.

Theorem 2.1.13 (Thm. 7.1.8, [23])**.** *Let $\mathcal{K}$ be a Fraïssé class with the Fraïssé limit $\mathbb{K}$. The following are equivalent.*

1. *$\mathcal{K}$ has the SAP.*

2. *$\mathbb{K}$ has no algebraicity.*

2.2 Examples

We will see some examples illustrating the phenomenon described above. Typically the only non-trivial condition from the definition of a Fraïssé class is the AP, so we will briefly describe why it holds for each of the subsequent classes. Verification of other conditions is easy and can be left to the reader.

2.2.1 Linear Orders

Probably the first example of a Fraïssé class was the class of all finite linear orders.

Proposition 2.2.1. *The class of all finite linear orders has the AP.*

Proof. Take a pair of finite linear orders $(K_0, \leq_0), (K_1, \leq_1)$, such that $\leq_0$ and $\leq_1$ agree on $L = K_0 \cap K_1$. We want to find an ordering $\leq_2$ on $K_0 \cup K_1$ extending both $\leq_0$ and $\leq_1$. This requirement determines $\leq_2$ on all pairs, except for ones of the form $\{x_0, x_1\}$, where $x_i \in K_i \setminus L$, for $i = 0, 1$. We put $x_1 <_2 x_0$ if there is $y \in L$, such that $x_1 <_1 y <_0 x_0$, and $x_0 <_2 x_1$ otherwise. It is routine to check that this defines a linear order on $K_0 \cup K_1$. □

It is easy to see that the corresponding Fraïssé limit is a countable, dense linear order without endpoints. These conditions are satisfied by the ordering of the rationals $(\mathbb{Q}, \leq)$, and since the Fraïssé limit is unique, it follows that it is isomorphic to $(\mathbb{Q}, \leq)$. We have proved the old theorem of Cantor:

Corollary 2.2.2 (Cantor, [8])**.** *Any countable, dense linear order without endpoints is isomorphic to* $(\mathbb{Q}, \leq)$.

2.2.2 Graphs

In the case of (undirected) graphs, verification of the AP is straightforward: we just take the set-theoretic union and add no edges. What is the Fraïssé limit? Clearly, it is a countably infinite graph $\mathcal{R}$, which satisfies the following axiom:

For each pair of disjoint, finite subsets $A, B \subseteq \mathcal{R}$, there exists a point $x \in \mathcal{R} \setminus (A \cup B)$, connected with every point in A, and with no point in B.

An easy argument by induction shows that this property implies injectivity, so by Lemma 1 it determines $\mathcal{R}$ uniquely, up to isomorphism. This graph was first studied by Ackermann [2] in 1937. The set of vertices was the set of all hereditarily finite sets, two of them being connected precisely when one is an element of the other. Erdös and Rényi ([14]) gave a probabilistic construction of $\mathcal{R}$ – assume that for each pair of distinct integers we randomly decide whether they are connected. We do it independently for all pairs of vertices, with a fixed probability $p \in (0, 1)$. It turns out, that with probability 1 the resulting graph will be isomorphic to $\mathcal{R}$. This explains why $\mathcal{R}$ is known as the *random graph*.

Let K_n, $n \geq 3$, denote the complete graph on n vertices. We will say that a graph is *K_n-free*, if it has no induced subgraph isomorphic to K_n. The class of all K_n-free graphs is a Fraïssé class. Let $\mathcal{R}_n$ be the corresponding countable, homogeneous graph. A deep result by Lachlan and Woodrow shows that they essentially exhaust examples of Fraïssé classes of finite graphs. For a graph G, we denote by G^c its complement – the graph obtained by replacing every edge with non-edge, and the other way around.

Theorem 2.2.3 (Lachlan-Woodrow, [34])**.** *Let $\mathcal{U}$ be a countably infinite, homogeneous graph. Then one of the graphs $\mathcal{U}$ and $\mathcal{U}^c$ is isomorphic to either $\mathcal{R}$, $\mathcal{R}_n$, for $n \geq 3$, or a disjoint union of complete graphs of the same size.*

2.2.3 Boolean Algebras

The class of all finite Boolean algebras is a Fraïssé class. The AP follows from the existence of free products with amalgamation in the category of Boolean algebras, which is described in [28] Ch. 11. The corresponding homogeneous algebra is the countable, atomless Boolean algebra.

2.2.4 Partial Orders

The class of all finite partial orders is a Fraïssé class with the resulting homogeneous structure known as the *random partial order*.

Proposition 2.2.4. *The class of all partial orders has the AP.*

Proof. Fix some partial order $(\mathbb{P}, \leq)$ and consider two its extensions $(\mathbb{P}, \leq) \subseteq (\mathbb{P}_0, \leq_0), (\mathbb{P}_1, \leq_1)$, with $\mathbb{P} = \mathbb{P}_0 \cap \mathbb{P}_1$. We define a relation $\leq^*$ on $\mathbb{P}_0 \cup \mathbb{P}_1$ by the conditions

$$x_0 \leq^* x_1 \iff \exists p \in \mathbb{P}\; x_0 \leq_0 p \leq_1 x_1,$$
$$x_1 \leq^* x_0 \iff \exists p \in \mathbb{P}\; x_1 \leq_1 p \leq_0 x_0.$$

Verification of transitivity is straightforward, and so is to check that

$$\forall x, y\; (x \leq^* y \wedge y \leq^* x \implies x = y).$$

Therefore $\leq^*$ is a partial ordering of $\mathbb{P}_0 \cup \mathbb{P}_1$. □

2.2.5 Groups

Somewhat more involved Fraïssé class is the class of finite groups. The amalgamation can be proved using so-called *permutation products* [37]. Resulting group is known as the Hall's universal locally finite group, and was first described by Philip Hall in 1959 [20].

Things are simpler in the case of abelian groups. In this case we can see the AP via reduced products – for two finite abelian groups B_0, B_1 with $B_0 \cap B_1 = A$ let

$$E = B_0 \times B_1 / \langle (a, -a) |\; a \in A \rangle$$

If we identify B_0 and B_1 with their natural copies inside E, then E witnesses the AP for inclusions $A \subseteq B_0$ and $A \subseteq B_1$.

Proposition 2.2.5. *The group* $\mathbb{A} = \bigoplus_{i<\omega} \mathbb{Q}/\mathbb{Z}$ *is the Fraïssé limit of the class of all finite abelian groups.*

Proof. First, see that since each finite abelian group is a direct sum of finite cyclic groups, it can be embedded into $\mathbb{A}$. Moreover, each finitely generated subgroup of $\mathbb{A}$ is finite. Why is that? The only way for a finitely generated abelian group to be infinite, is to have an element of an infinite order, but $\mathbb{A}$ has no elements of infinite order. This shows that $\operatorname{Age}\mathbb{A}$ is exactly the class of finite abelian groups. The group $\mathbb{A}$ is divisible, so it is injective as a $\mathbb{Z}$-module. It is tempting to conclude that since $\mathbb{Z}$-modules are just abelian groups, the proof is completed. However, the standard definition of an injective module refers to all group homomorphisms, while our definition of an injective structure takes into account only 1-1 homomorphisms.

Fix a group monomorphism $f : A \hookrightarrow \mathbb{A}$, and a finite group $B \geq A$. We want to extend f to $\overline{f} : B \hookrightarrow \mathbb{A}$, keeping it 1-1. We can proceed by induction on the number of generators of B, so we can assume that B is generated by the set $A \cup \{b\}$, for some $b \in B$. Let $\overline{f}$ be an extension of f obtained from the fact that $\mathbb{A}$ is injective in the algebraic sense. If $\overline{f}$ is 1-1, we are done, so suppose that for some expression $a + b \neq 0$, $\overline{f}(a + b) = 0$. By replacing b with $a + b$, we can assume that $\overline{f}(b) = 0$. Now notice, that groups $\langle b \rangle$ and A have trivial intersection in B. Indeed, otherwise for some integer k, and $a \in A$, we would have $k \cdot b = a$. Now applying $\overline{f}$ both sides, we obtain $\overline{f}(a) = f(a) = 0$, and so $a = 0$. We may send b to some non-zero element of $\mathbb{A}$, by a homomorphism $g : B \hookrightarrow \mathbb{A}$, which is zero on A. From the remarks above it is clear that $\overline{f} + g : B \hookrightarrow \mathbb{A}$ is the monomorphism we were looking for. □

2.2.6 Metric Spaces

So far we have been looking only at structures in finite languages, but this is not really important. We will call a metric space (X, d) *rational*, if all distances between the points of X are rational numbers. The class of all rational metric spaces is a class of models of a first order theory, in the language consisting of countably many binary relations d_q, for all rationals $q > 0$, where relation $d_q(x, y)$ is interpreted as *distance between* x *and* y *is at least* q. The resulting homogeneous space $(\mathbb{U}, d)$ is known as *the rational Urysohn space*, and its completion $\overline{\mathbb{U}}$, as the *Urysohn space*. The space $\overline{\mathbb{U}}$ is uniquely characterized by the following conditions.

- $\overline{\mathbb{U}}$ contains an isometric copy of any finite metric space.
- Each isometry between between finite subspaces of $\overline{\mathbb{U}}$ extends to a full isometry of $\overline{\mathbb{U}}$ into itself.

Proposition 2.2.6. *The class of all finite, rational metric spaces has the AP.*

Proof. Using induction, we can reduce our task to amalgamating two one-point extensions. Fix a finite, rational metric space (X, d), and two extensions (X_1, d_1), (X_2, d_2), where $X_i = X \cup \{x_i\}$, for $i = 1, 2$, and metrics d_1, d_2 agree with the metric d on X. We want to set the rational distance q between x_1 and x_2, so that the triangle inequality will hold. This reduces to ensuring that

$$\forall\, x \in X\ d_1(x, x_1) + d_2(x, x_2) \geq q,$$

$$\forall\, x \in X\ d_1(x, x_1) + q \geq d_2(x, x_2),$$

and

$$\forall\, x \in X\ d_2(x, x_2) + q \geq d_1(x, x_1).$$

This in turn is just

$$\operatorname{dist}(x_1, X\backslash\{x_1\}) + \operatorname{dist}(x_2, X\backslash\{x_2\}) \geq q \geq |\operatorname{dist}(x_1, X\backslash\{x_1\}) - \operatorname{dist}(x_2, X\backslash\{x_2\})|.$$

Clearly we can find $q > 0$ with this property. □

It makes sense to consider metric spaces with distances restricted to other countable sets. Given any countable subset $D \subseteq [0, \infty)$, let $\mathcal{M}_D$ be the class of finite metric spaces with distances in D. Let us note, that we can code some other classes of models as $\mathcal{M}_D$. For example if $D = \{0, 1, 2\}$, then $\mathcal{M}_D$ is the class of graphs, if we interpret points in distance 1 as connected, and in distance 2 as not connected. It turns out that the AP for $\mathcal{M}_D$ is equivalent to some rather technical condition of D, which was identified by the authors of [9].

Definition 2.2.7. Let $D \subseteq [0, \infty)$ be any countable set. For points $u_1, u_2, v_1, v_2 \in D$, let $\phi(u_1, u_2, v_1, v_2)$ denote the closed interval

$$[\max\{|u_1 - u_2|, |v_1 - v_2|\}, \min\{u_1 + u_2, v_1 + v_2\}].$$

and let $\rho_D(u_1, u_2, v_1, v_2)$ be the assertion that

$$\phi(u_1, u_2, v_1, v_2) \cap D \neq \emptyset.$$

The set D satisfies the *four values condition* if

$$\forall\, u_1, u_2, v_1, v_2 \in D\ \rho_D(u_1, u_2, v_1, v_2) \implies \rho_D(u_1, v_1, u_2, v_2).$$

Theorem 2.2.8 ([9]). *Let $D \subseteq [0, \infty)$ be countable. The class $\mathcal{M}_D$ is a Fraïssé class if and only if D satisfies the four values condition.*

Given that the four values condition is quite involved, it is typically easier to check some stronger, but simpler conditions.

Proposition 2.2.9 ([9]). *The four values condition for a countable set $D \subseteq [0, \infty)$ is implied by any of the following:*

- $\forall\, u_1, u_2, v_1, v_2 \in D\; u_1 - v_1 \leq u_2 + v_2 \implies$
  $\exists\, w \in D\; u_1 - v_1 \leq w \leq u_2 + v_2$,
- $\forall\, u, v \in D\; u + v < \sup D \implies u + v \in D$.

Of course for any D like above, we obtain the corresponding Urysohn space $\mathbb{U}_D$.

2.3 Invariant Measures on the Space of Models

The random graph can be generated by a simple probabilistic procedure, described in 2.2.2. Which other models admit a similar description? Before asking this question in a rigorous manner, we must define a suitable probabilistic space.

Following [1], we denote by Str_L the space of structures in some countable language L, with ω as their underlying set. We introduce the topology on the space Str_L by means of declaring the following sets open.

$$\{M \in \mathrm{Str}_L \mid M \models R^M(n_1, \ldots, n_k)\},$$

for a relational symbol R, and $n_1, \ldots, n_k \in \omega$,

$$\{M \in \mathrm{Str}_L \mid M \models c^M = n\},$$

for a constant symbol c, and $n \in \omega$,

$$\{M \in \mathrm{Str}_L \mid M \models f^M(n_1, \ldots, n_k) = m\},$$

for a function symbol f, and $n_1, \ldots, n_k, m \in \omega$.

We therefore have a σ-algebra of Borel sets, and can speak about Borel measures on the space Str_L. We are interested in models only up to isomorphism, rather than specific representations of these isomorphism types, so we would like to look for measures which are invariant under re-enumerating the universe. More precisely, the group of all permutations of ω, S_∞, acts on the space Str_L by the *logic action*. This means that for a given $M \in \mathrm{Str}_L$, and $g \in S_\infty$, $g \cdot M$ is the structure obtained from M by permuting elements of the universe according to g. A measure μ on Str_L is *invariant*, if for all Borel subsets $B \subseteq \mathrm{Str}_L$, and all permutations $g \in S_\infty$, we have

$$\mu(B) = \mu(g \cdot B).$$

Question. For which models $M \in \mathrm{Str}_L$ does there exist an invariant Borel probability measure on Str_L, concentrated on the set of models isomorphic to M?

A number of necessary conditions can be seen by hand. If the language contains a constant symbol c, then for each natural number $n \in \omega$ the measure of the set

$$\{M \in \mathrm{Str}_L \mid c^M = n\},$$

should be independent of n, because the measure is invariant. Therefore this set would have measure zero, contradicting the fact that the constant c must be somehow interpreted in M. For similar reasons, if f is a function symbol of arity k, then with probability one

$$f^M(n_1 \ldots, n_k) \in \{n_1, \ldots, n_k\}.$$

This essentially means that only purely relational languages are worth considering. One model for which the answer is clearly affirmative is the random graph. This is also the case for countable homogeneous K_n-free graphs, defined in 2.2.2, as proved by Petrov and Vershik [41]. Their method was extended by Ackerman, Freer, and Patel ([1]) to give both necessary and sufficient conditions on the model M.

Theorem 2.3.1 (Thm. 1.1, [1])**.** *If M is a countable structure in a countable language L, then the following conditions are equivalent:*

- *There exists an invariant Borel probability measure on Str_L, concentrated on the set of models isomorphic to M,*
- *M has no algebraicity (see Definition 2.1.12).*

By Theorem 2.1.13, if M is homogeneous, then conditions the Theorem 2.3.1 hold exactly when $\mathrm{Age}(M)$ has the SAP .

Chapter 3

Fraïssé-Jónsson Theory

As the initial example, let us look at three well-known theorems, characterizing certain uncountable linear orders, graphs, and Boolean algebras respectively.

- (Hausdorff, [22]) Assume CH. Then all countably saturated linear orders (see Definition 3.2.1) of cardinality 2^ω are isomorphic.
- Assume CH. There exists a unique up to isomorphism graph G of cardinality 2^ω, with the property that for any countable, disjoint subsets $A, B \subseteq G$, there exists a vertex in $G \setminus (A \cup B)$ connected to every vertex in A and none of the vertices in B.
- (Parovičenko, [40]) Assume CH. Then all Boolean algebras of cardinality 2^ω with the Strong Countable Separation Property (see Definition 3.2.5) are isomorphic.

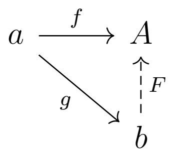
All of the above claims are proved using the "back-and-forth" argument, and all of them fail without CH. The fact that Parovičenko Theorem is equivalent to CH was proved by van Douwen and van Mill in [10]. For the other two, this is much easier to see. These are examples of uncountable Fraïssé limits, which general theory, so called Fraïssé-Jónsson theory, is the natural analog of the classical Fraïssé theory for classes of infinite models.

3.1 Fraïssé-Jónsson Theory

Like before, let us consider a class of structures $\mathcal{K}$ in a countable language, and this time the structures are perhaps infinite. Definitions of the AP, JEP, hereditary class are just like in the finite case, and notions of universal and injective structures admit straightforward analogs. We denote by $\operatorname{Age} A$ the class of substructures of A of size strictly less than $|A|$.

Definition 3.1.1. A structure A of size κ is

- $\mathcal{K}$-*universal*, if for every structure $a \in \mathcal{K}$, there exists an embedding $a \hookrightarrow A$,
- *injective*, if for any pair of embeddings $f : a \hookrightarrow A$, $g : a \hookrightarrow b$, where $a, b \in \operatorname{Age} A$, there exists an embedding $F : b \hookrightarrow A$, such that $F \circ g = f$.



- *homogeneous*, if any isomorphism between substructures of A of size $< \kappa$ extends to an automorphism of A.

In the previous chapter we were building increasing sequences of finite models. Now we would like to construct a transfinite sequence of the length κ consisting of models of cardinality $< \kappa$. At limit stages we will be taking increasing unions, so we need one additional property of the class $\mathcal{K}$. On the other hand, we don't need any version of local finiteness – any infinite subset can be extended to a substructure of the same size (we have assumed that the language is countable), and so any structure of cardinality κ is automatically "locally $< \kappa$". Typically we will be working with cardinals κ satisfying $\kappa^{<\kappa} = \kappa$. Under GCH this is satisfied for all regular cardinals, but it is consistent with ZFC that there is no limit cardinal with this property. Some parts of the theory work under the weaker assumption, that κ is regular.

Definition 3.1.2. Let κ be an uncountable cardinal. $\mathcal{K}$ is κ-closed if an increasing union of $< \kappa$ elements of $\mathcal{K}$ belongs to $\mathcal{K}$.

Assume $\kappa^{<\kappa} = \kappa$. We will say that $\mathcal{K}$ is an uncountable Fraïssé Class (of the length κ), if it is a κ-closed, hereditary class of structures, satisfying the AP, JEP, and having no more than κ many isomorphism types.

Lemma 3.1.3. *Let κ be an uncountable regular cardinal. If two injective structures of size κ have the same* Age*, they are isomorphic. In fact, any isomorphism between substructures of size $< \kappa$ can be extended to an isomorphism between them.*

Proof. Fix two structures A and B as in the statement of the Lemma. We can decompose them into increasing chains $A = \bigcup_{\alpha<\kappa} a_\alpha$, and $B = \bigcup_{\alpha<\kappa} b_\alpha$ of substructures of size $< \kappa$. Let $f_0 : a_0 \hookrightarrow b_0$ be given. We will inductively define an increasing sequence of embeddings f_α such that $f = \bigcup_{\alpha<\kappa} f_\alpha$ will be an isomorphism from A onto B.

- Suppose α is an even successor ordinal and f_α is defined. κ is regular, so there exists $\beta > \alpha$ such that $\operatorname{dom} f_\alpha \subseteq a_\beta$. Since B is injective, we will find $f_{\alpha+1}$ closing the diagram

$$\begin{array}{ccc} \operatorname{dom} f_\alpha & \subseteq & a_\beta \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f_{\alpha+1}} \\ \operatorname{rg} f_\alpha & \subseteq & B \end{array}$$

- Suppose α is an odd successor ordinal and f_α is defined. There exists $\beta > \alpha$ such that $\operatorname{rg} f_\alpha \subseteq b_\beta$. Since A is injective, we will find $f_{\alpha+1}$ closing the diagram

$$\begin{array}{ccc} \operatorname{dom} f_\alpha & \subseteq & A \\ {\scriptstyle f_\alpha^{-1}}\uparrow & & \uparrow{\scriptstyle f_{\alpha+1}^{-1}} \\ \operatorname{rg} f_\alpha & \subseteq & b_\beta \end{array}$$

- Suppose α is a limit ordinal. Let $f_\alpha = \bigcup_{\gamma<\alpha} f_\gamma$.

It is clear that $f = \bigcup_{\alpha<\kappa} f_\alpha$ is the required isomorphism. □

If we take $A = B$ in the Lemma 3.1.3, we obtain:

Corollary 3.1.4. *Any structure of size κ is homogeneous if and only if it is injective.*

Lemma 3.1.5. *Let A and B be structures of size κ, where $\kappa > \omega$ is regular. Assume moreover, that $\operatorname{Age} B \subseteq \operatorname{Age} A$ and A is injective. Then there exists an embedding $B \hookrightarrow A$.*

Proof. Fix A and B as in the statement of the Lemma. We can decompose them into increasing chains $A = \bigcup_{\alpha<\kappa} a_\alpha$, and $B = \bigcup_{\alpha<\kappa} b_\alpha$ of substructures of size $< \kappa$. Let $f_0 : b_0 \hookrightarrow a_0$ be given. We will inductively build an $\subseteq$-increasing sequence of embeddings f_α, such that and $f = \bigcup_{\alpha<\kappa} f_\alpha$ will be an embedding from B into A.

Suppose f_α is defined. Since κ is regular, we will find $\beta > \alpha$ such that $\operatorname{dom} f_\alpha \subseteq b_\beta$. Since A is injective, we will find $f_{\alpha+1}$ closing the diagram

$$\begin{array}{ccc} \operatorname{dom} f_\alpha & \subseteq & b_\beta \\ {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle f_{\alpha+1}} \\ \operatorname{rg} f_\alpha & \subseteq & A \end{array}$$

If α is a limit ordinal, we set $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$. After κ many steps $f = \bigcup_{\alpha<\kappa} f_\alpha$ is the required embedding. □

Theorem 3.1.6. *Let κ be an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. If $\mathcal{K}$ is a Fraïssé class of the length κ, there exists a unique up to isomorphism model $\mathbb{K}$, satisfying the following three properties:*

1. $|\mathbb{K}| = \kappa$,
2. $\operatorname{Age} \mathbb{K} = \mathcal{K}$,
3. $\mathbb{K}$ *is injective.*

In this case $\mathbb{K}$ is also homogeneous.

Proof. Enumerate as $\{E_\alpha |\ \alpha < \kappa\}$ all isomorphism types of structures from $\mathcal{K}$. We aim to inductively build an increasing sequence of structures $F_\alpha \in \mathcal{K}$, and set $\mathbb{K} = \bigcup_{\alpha<\kappa} F_\alpha$. For bookkeeping purposes, we fix a partition of κ, $\{\Phi_\gamma |\ \gamma < \kappa\}$ consisting of sets of cardinality κ, such that $\min \Phi_\gamma \geq \gamma$, for all $\gamma < \kappa$.

Let $F_0 = E_0$, and enumerate as $\{g_\gamma |\ \gamma \in \Phi_0\}$ all (up to equivalence) embeddings g, with $\operatorname{dom} g \subseteq F_0$ and $\operatorname{rg} g \in \mathcal{K}$. We don't mind repetitions.

- Assume we are in a successor step, when F_α is defined, and so is the set $\{g_\gamma |\ \gamma \in \Phi_\alpha\}$. In particular, g_α is defined, and $\operatorname{dom} g_\alpha \subseteq F_\alpha$. Using the AP for the class $\mathcal{K}$, we can find h, and $F'_{\alpha+1} \in \mathcal{K}$, closing the diagram

$$\begin{array}{ccc} \operatorname{dom} g_\alpha & \longrightarrow & F_\alpha \\ {\scriptstyle g_\alpha}\downarrow & & \downarrow \\ \operatorname{rg} g_\alpha & \overset{h}{\dashrightarrow} & F'_{\alpha+1} \end{array}$$

Using the JEP we can enlarge $F'_{\alpha+1}$ to $F_{\alpha+1}$, so that $F_{\alpha+1}$ contains an isomorphic copy of E_α. Finally, we use the set $\Phi_{\alpha+1}$ to index all embeddings starting from substructures of $F_{\alpha+1}$.

- In the limit step, we just define $F_\alpha = \bigcup_{\gamma<\alpha} F_\gamma$, and use the set Φ_α to index all embeddings starting from substructures of F_α.

Let $\mathbb{K} = \bigcup_{\alpha<\kappa} F_\alpha$. The second step of construction clearly ensures universality. Why is $\mathbb{K}$ injective? Fix $D \in \operatorname{Age} \mathbb{K}$ and $f : D \hookrightarrow E$. D is also a substructure of some F_γ, and there is $\delta > \gamma$, such that g_δ is equivalent to f. By the definition of F_δ, there is $h : E \hookrightarrow F_{\delta+1} \subseteq \mathbb{K}$ closing the triangle.

$$\begin{array}{ccccc} D & \subseteq & F_{\delta+1} & \subseteq \mathbb{K} \\ {\scriptstyle g_\delta}\downarrow & \nearrow_{h} & & \\ E & & & \end{array}$$

Uniqueness of $\mathbb{K}$ follows from Lemma 3.1.3, and homogeneity from Corollary 3.1.4. □

All classes of models considered in Chapter 2 have uncountable universal homogeneous models in higher cardinalities (assuming suitable cardinal arithmetic). However the SAP, or even the SP can fail, when passing to the uncountable setting – the class of countable rational metric spaces does not enjoy the SP. This is visible when one tries to amalgamate two copies of the set of rationals $\mathbb{Q}$ with the euclidean metric over $\mathbb{Q} \setminus \{0\}$.

3.2 Examples

3.2.1 Saturated Linear Orders

Definition 3.2.1. A linear order $(L, \leq)$ is λ-*saturated* if for all subsets $A, B \subseteq L$ of size $< \lambda$ such that $A < B$, there exists a point in L strictly between A and B.

In this case L must have cofinality and coinitiality at least λ, since one of the sets A and B can be empty. It can be checked that this definition is equivalent to the model theoretic definition of a λ-saturated linear order. If $|L| = \lambda$, L is λ-saturated if and only if it is an injective structure. The next theorem was known already to Hausdorff. For the proof and some related results we refer the reader to [42] p. 163.

Theorem 3.2.2 ([22])**.** *If $\lambda = \lambda^{<\lambda}$ then there exists a unique up to isomorphism λ-saturated linear order of cardinality λ.*

What does this order look like? In case $\lambda = \omega_1$, $\lambda = \lambda^{<\lambda}$ is exactly CH. One of possible representations of this order is the following:

$$\mathbb{L}^{\omega_1} = \{x \in [-1,1]^{\omega_1} \,|\, |\{\alpha < \omega_1 : x(\alpha) \neq 0\}| \leq \omega\},$$

with the lexicographic ordering – for $x, y \in \mathbb{L}^{\omega_1}$, we set $x < y$ if and only if $x(\alpha) < y(\alpha)$ for the first coordinate α on which x and y differ.

One may wonder, to what extent is CH relevant here? $\mathbb{L}^{\omega_1}$ is ω_1-saturated regardless of CH. If $2^\omega \geq \omega_2$, then both of the following linear orders are ω_1-saturated and have size 2^ω:

$$\{x \in [-1,1]^{\omega_1} \,|\, |\{\alpha < \omega_1 : x(\alpha) \neq 0\}| \leq \omega\},$$

$$\{x \in [-1,1]^{\omega_2} \,|\, |\{\alpha < \omega_2 : x(\alpha) \neq 0\}| \leq \omega\}.$$

They are not isomorphic, since the latter one contains an isomorphic copy of the ordinal ω_2. This was noted already in [17]. We will see in 3.2.4 that some additional condition determines the ordering $\mathbb{L}^{\omega_1}$ uniquely in ZFC.

3.2.2 Saturated Graphs

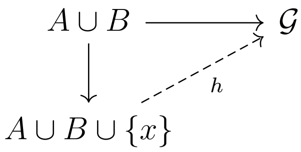
Injective graphs are exactly those that satisfy an axiom similar to the one defining the random graph, which we described in 2.2.2.

Theorem 3.2.3. *Let $\mathcal{G} = (\mathcal{G}, E(\mathcal{G}))$ be a graph of cardinality κ. $\mathcal{G}$ is injective if and only if it satisfies the following axiom*

For each pair of disjoint subsets $A, B \subseteq \mathcal{G}$, $|A|, |B| < \kappa$, there exists a point $x \in \mathcal{G} \setminus (A \cup B)$ connected with every point in A, and with no point in B.

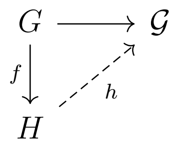
Proof.

"$\Rightarrow$." Take sets A and B like above, and build a graph $A \cup B \cup \{x\}$, where x is connected with each point in A, and with no point in B. By injectivity there exists a diagonal h in the diagram



Clearly $h(x)$ is the point we were looking for.

"$\Leftarrow$." To prove injectivity look at the diagram



where $|H| < \kappa$. We are looking for the diagonal h. Suppose that $H \setminus f[G]$ consists of just one point, say u. By the assumption there exists $y \in \mathcal{G}$ such that

$$\forall x \in G \; \{x, y\} \in E(\mathcal{G}) \iff \{u, f(x)\} \in E(H).$$

We define h by conditions

$$\forall x \in G \; h(f(x)) = x,$$

$$h(u) = y.$$

The general case follows from easy transfinite induction.

□

Corollary 3.2.4. *CH implies that there exists a unique up to isomorphism graph $\mathcal{G}$ of cardinality 2^ω with the following property:*

For each pair of disjoint subsets $A, B \subseteq \mathcal{G}$, $|A|, |B| < 2^\omega$, there exists a point $x \in \mathcal{G} \setminus (A \cup B)$ connected with every point in A, and with no point in B.

3.2.3 Boolean Algebras

Definition 3.2.5 (p. 81 in [28])**.** An infinite Boolean algebra A has the *Strong Countable Separation Property* if the following assertion holds:

Suppose $F, G \subseteq A$ are countable sets with the property that for all nonempty, finite subsets $f \subseteq F$, $g \subseteq G$, $\bigvee f < \bigwedge g$. Then there exist $a \in A$, such that for all $x \in F$, and $y \in G$, $x < a < y$.

This is exactly a characterization of countably saturated Boolean algebras.

Proposition 3.2.6 (5.29 in [28])**.** *Assume that A has the* Strong Countable Separation Property*, $f : B \to A$ is an embedding, and $B \subseteq C$ are countable. Then f extends to an embedding $\tilde{f} : C \to A$.*

Corollary 3.2.7. *If a Boolean algebra A of size ω_1 has the Strong Countable Separation Property , it is injective.*

In the case of Boolean algebras, the structure from Theorem 3.1.6 has a very canonical representation.

Theorem 3.2.8 (Parovičenko's Theorem, 5.30 in [28])**.** *If CH holds, $\mathcal{P}(\omega)/Fin$ is the unique Boolean algebra of cardinality 2^ω with the Strong Countable Separation Property.*

The conclusion of Parovičenko's Theorem is equivalent to CH, as was proved by van Douwen and van Mill [10].

3.2.4 Prime Countably Saturated Linear Orders

Definition 3.2.9. A linear order L is *prime countably saturated* if it is countably saturated, and any countably saturated linear order contains an isomorphic copy of L.

$\mathbb{L}^{\omega_1}$, defined at the beginning of the section, is prime countably saturated. More generally, if D is a compact linear order and $d_0 \in D$ is neither the least, nor the greatest element of D, then we define

$$\mathbb{L}^{\omega_1}_{(D,d_0)} = \{x \in D^{\omega_1} | \, |\{\alpha < \omega_1 : x(\alpha) \neq d_0\}| \leq \omega\}.$$

The case $D = \{-1, 0, 1\}$ and $d_0 = 0$ is the classical construction by Hausdorff, and is described in [21]. Our Theorem 3.2.10 is therefore a slightly more general version of Theorem 3.22 therein, and Theorem 3.2.11 – a version of Theorem 3.13.

Theorem 3.2.10. *$\mathbb{L}^{\omega_1}$ and $\mathbb{L}^{\omega_1}_{(D,d_0)}$ are countably saturated.*

Theorem 3.2.11. *$\mathbb{L}^{\omega_1}$ is prime countably saturated. Likewise, if D is a separable compact line and $d_0 \in D$ is neither the least, nor the greatest element, then $\mathbb{L}^{\omega_1}_{(D,d_0)}$ is prime countably saturated.*

These are however just different representations of the same ordering. This was originally shown by Harzheim [21], and in [30] we gave a simplified proof which uses the notion of I-dimension. This is a convenient tool, introduced by Novák in [38].

Definition 3.2.12 ([38])**.** The I-dimension of a linear order L is defined as

$$\text{I-dim}\, L = \min\{\alpha < \omega_1 |\ L \text{ embedds into } I^\alpha\},$$

where I is a closed unit interval, and I^α stands for its lexicographic power.

Theorem 3.2.13 (Harzheim [21], Kostana [30])**.** *Let $(L, \leq)$ be a countably saturated linear order. The following are equivalent:*

- *L is prime countably saturated;*
- *L is an increasing sum $\bigcup_{\alpha<\omega_1} L_\alpha$, where L_α doesn't contain a copy of ω_1 or ω_1^*, for each $\alpha < \omega_1$ (ω_1^* is the reversed ordering of ω_1);*
- *L is an increasing sum $\bigcup_{\alpha<\omega_1} L_\alpha$, where $\text{I-dim}\, L_\alpha < \omega_1$, for each $\alpha < \omega_1$.*

Unlike Theorem 3.2.2, the next result does not need any assumptions on the cardinal arithmetic.

Theorem 3.2.14 (Harzheim, [21])**.** *All prime countably saturated linear orders are isomorphic.*

The proof of uniqueness of the prime countably saturated linear order is Fraïssé-theoretic in nature. One might guess, that we are just doing "back-and-forth" argument between the sets L_α from Theorem 3.2.13, and this is indeed the idea of the proof. One additional property of $\mathbb{L}^{\omega_1}$, which does not seem to be noticed by Harzheim, is a strong version of homogeneity.

We can use I-dimension to characterize complete suborders of $\mathbb{L}^{\omega_1}$. Recall that a linear order is *Dedekind complete* if each subset has the greatest lower bound and the least upper bound. Equivalently, it is compact in the order topology.

Theorem 3.2.15 (Kostana, Prop. 8, [30])**.** *Any isomorphism between subsets of $\mathbb{L}^{\omega_1}$ of countable I-dim extends to an automorphism of $\mathbb{L}^{\omega_1}$. In particular, any automorphism between countable suborders extends to an automorphism of $\mathbb{L}^{\omega_1}$.*

Corollary 3.2.16. *Any isomorphism between Dedekind complete subsets of $\mathbb{L}^{\omega_1}$ extends to an automorphism of $\mathbb{L}^{\omega_1}$.*

This is an immediate consequence of the next lemma.

Lemma 3.2.17. *Let D be a Dedekind complete linear order. There exists an embedding $i : D \hookrightarrow \mathbb{L}^{\omega_1}$ if and only if $\text{I-dim}\, D < \omega_1$.*

Proof. Assume that $\text{I-dim}\, D < \omega_1$. It is sufficient to show that $D \times \mathbb{L}^{\omega_1}$ is isomorphic to $\mathbb{L}^{\omega_1}$. By Theorem 5 from [30], $D \times \mathbb{L}^{\omega_1}$ is countably saturated. By Theorem 3.2.13

$$\mathbb{L}^{\omega_1} = \bigcup_{\alpha<\omega_1} I^\alpha,$$

and so

$$D \times \mathbb{L}^{\omega_1} = \bigcup_{\alpha<\omega_1} D \times I^\alpha.$$

By Theorems 3.2.13 and 3.2.14, $D \times \mathbb{L}^{\omega_1} \simeq \mathbb{L}^{\omega_1}$.

In the other direction, assume we have an embedding $i : D \hookrightarrow \mathbb{L}^{\omega_1}$. First, note that $\mathbb{L}^{\omega_1}$ is isomorphic to its lexicographic square. Indeed, $\mathbb{L}^{\omega_1} \times \mathbb{L}^{\omega_1}$ can be represented as an increasing union $\bigcup_{\alpha<\omega_1} I^\alpha \times I^\alpha$. This union isomorphic to $\mathbb{L}^{\omega_1}$ by Theorems 3.2.13 and 3.2.14.

We therefore obtains a chain of mappings

$$D \times \mathbb{L}^{\omega_1} \hookrightarrow \mathbb{L}^{\omega_1} \times \mathbb{L}^{\omega_1} \simeq \mathbb{L}^{\omega_1}.$$

And since I-dim is monotone,

$$\omega_1 = \text{I-dim}\, \mathbb{L}^{\omega_1} \geq \text{I-dim}\, D \times \mathbb{L}^{\omega_1} \geq \text{I-dim}\, D \times I > \text{I-dim}\, D.$$

The last inequality is strict by the virtue of Lemma 4 from [30]. □

Chapter 4

Generic Structures

As one looks at the classical construction of a Fraïssé limit, one might notice that it is much in the spirit of the Baire Category Theorem. Recall that we have proved (2.1) that universal homogeneous structures form a residual set in a certain Polish space. Having that in mind, one might try to construct specific instances of universal homogeneous structures, mimicking the definition of a Cohen real from forcing theory. Roughly speaking, a real number is Cohen over some model if it belongs to each residual set from that model. So it is *very generic*, in a sense that for any typical property a real might have, a Cohen real has this property. Of course the same can be said about random numbers, but with different notion of typicality. This is the main idea of this chapter. On one hand, we want to look at the model-theoretic notion of *saturation* as stemming from the forcing language. On the other, we reach to model theory for tools to produce Cohen-like forcing notions (which might often be just different incarnations of the Cohen forcing). It should be mentioned that this topic used to be informally discussed from time to time already, as a kind of folklore idea known to the community. However, up to the author's knowledge, no systematic study of this idea was ever carried out. The closest to it was perhaps a note by Golshani [19], unfortunately not free of significant mistakes.

4.1 Generalizing the Cohen Forcing

Consider the set

$$\mathbb{P} = \{(A, \leq) |\, A \in [\kappa]^{<\omega},\ (A, \leq) \text{ is a linear order}\},$$

where κ is any cardinal, and the ordering is the reversed inclusion of suborders. The following subsets are dense for $\alpha \neq \beta \in \kappa$.

- $D_\alpha = \{(A, \leq) |\, \alpha \in A\}$,

- $D_{\alpha,\beta} = \{(A, \leq) | \exists n < \omega\ n$ is between α and $\beta\}$,

Therefore for $\kappa = \omega$ the generic filter produces an isomorphic copy of the rationals, and for any κ it gives some separable κ-dense order type. We say that a linear order is *κ-dense*, if every open interval has cardinality κ. It is a general phenomenon that for $\kappa = \omega$ this forcing gives the Fraïssé limit of a given class. It is an interesting remark, originally made by Golshani in [19], that every infinite subset of ω from the ground model is dense in the obtained order.

In this chapter $\mathcal{K}$ is a class of structures in some countable, purely relational language. By $\mathcal{K}_\kappa$ we denote the class of structures from $\mathcal{K}$ of cardinality less than κ. Recall that $F(A)$ denotes the universe of a model A. We make the following assumptions about $\mathcal{K}$:

- $\mathcal{K}$ has the Joint Embedding Property (JEP),
- $\mathcal{K}$ has the Amalgamation Property (AP),
- $\mathcal{K}$ is hereditary (so if $A \in \mathcal{K}$, and $B \subseteq A$, then $B \in \mathcal{K}$),
- $\mathcal{K}$ has infinitely many isomorphism types,
- $\mathcal{K}_\kappa$ is closed under increasing unions of the length $< \kappa$.

All of these notions were introduced in Chapter 2. Note, that $\mathcal{K}_\omega$ is a Fraïssé class, and if $\kappa^{<\kappa} = \kappa$, for some uncountable cardinal κ, then $\mathcal{K}_\kappa$ is an uncountable Fraïssé class of the length κ, in the sense of Chapter 3. In the subsequent part of the chapter we are going to make more assumptions on $\mathcal{K}$, like the SAP or the SP.

It will prove convenient to introduce a notation paraphrasing the one for the Cohen forcing in [33].

Definition 4.1.1. Let λ be an infinite cardinal number, and S be any infinite set. Denote by $\mathrm{Fn}(S, \mathcal{K}, \lambda)$ the set

$$\{A \in \mathcal{K} |\ F(A) \in [S]^{<\lambda}\},$$

ordered by the reversed inclusion of substructures.

The following claims hold true, and the proofs are straightforward modifications of arguments for the Cohen forcing [33].

Proposition 4.1.2 (Kubiś). *If $\mathcal{K}$ satisfies the SP, and $\mathcal{K}_\omega$ has at most countably many isomorphism types, then $\mathrm{Fn}(S, \mathcal{K}, \omega)$ satisfies the c.c.c., and even the Knaster condition, for any set S.*

The bound on the number of finite isomorphism types is automatically ensured when the language is finite. When the language is countable, it may or may not be true. Finite metric spaces can be viewed as structures in countable language, and still there are continuum many pairwise non-isomorphic (non-isometric) 2-element structures. If we restrict to finite metric spaces with rational distances, there are clearly only countably many isomorphism types. The relevance of the SP is visible in the example discovered by Kubiś. Let $\mathcal{F}$ be the class of all finite linear graphs, i.e. connected, acyclic, and with degree of every vertex at most 2. It can be easily checked that $\mathcal{F}$ has the AP, but not the SP. If S is any infinite set, then $\mathrm{Fn}(S, \mathcal{F}, \omega)$ forces that S is a linear graph, and each two points of S are in a finite distance. Therefore it collapses $|S|$ to ω.

Proposition 4.1.3. *Let S be any set and assume that $\mathcal{K}_\lambda$ satisfies the SP. We assume moreover, that for any $\delta < \lambda$ there are at most λ many structures from $\mathcal{K}$ with the universe δ. Then $\mathrm{Fn}(S, \mathcal{K}, \lambda)$ is λ-closed, and if $\lambda^{<\lambda} = \lambda$, then $\mathrm{Fn}(S, \mathcal{K}, \lambda)$ is λ^+-c.c.*

It should be stressed that in this proposition we don't count isomorphic types of $\mathcal{K}$-structures of cardinality less than λ. We take into account the number of different, not only non-isomorphic, ways the ordinal δ can be endowed with a first order structure, so that it becomes a member of $\mathcal{K}$. In all but one relevant examples, a bound on this number will be guaranteed by finiteness of the language.

Corollary 4.1.4. *If $\mathcal{K}$ is a class of structures in a finite language and CH holds, then $\mathrm{Fn}(S, \mathcal{K}, \omega_1)$ is ω_2-c.c.*

For a start, we describe models added by $\mathrm{Fn}(S, \mathcal{K}, \omega)$.

Proposition 4.1.5. *Let $\mathbb{P} = \mathrm{Fn}(\omega, \mathcal{K}, \omega)$. Let $G \subseteq \mathbb{P}$ be a generic filter. Then $\bigcup G$ is a structure with the universe ω, isomorphic to the Fraïssé limit $\mathbb{K}$ of the class $\mathcal{K}_\omega$.*

Proof. In order to ensure that $\bigcup G$ is defined on all ω, we must verify density of the sets

$$D_n = \{A \in \mathbb{P} |\; n \in F(A)\},$$

for $n < \omega$, which is straightforward. To see that we obtain the Fraïssé limit we must check that each finite extension of a finite substructure is realized. For this purpose, set

$$E_B^{i,f}, = \{A |\; i : B \hookrightarrow A \text{ is an embedding } \implies$$
$$\exists\, g : B' \hookrightarrow A \; g \text{ is an embedding and } i = g \circ f\},$$

where $B, B' \in \mathcal{K}$, $f : B \hookrightarrow B'$ is an embedding, and $i : B \hookrightarrow \omega$ is any $1-1$ function. We also make a technical assumption that both $F(B)$ and $F(B')$ are disjoint from ω. One could directly apply the AP to show that the sets $E_B^{i,f}$ are dense, however it may be easier to make use of a simple trick. This trick is due to Kubiś.

Fix a structure $A \in \mathbb{P}$, and assume that i, B, B', f are as above. Since $A \subset \omega$, we may extend A to a structure Ω, isomorphic to $\mathbb{K}$, with the universe ω (recall that $\mathbb{K}$ is the Fraïssé limit of $\mathcal{K}$). Then, since this structure is injective, there exists $g : B' \to \Omega$, such that $i = g \circ f$. Clearly if we define $A' = A \cup g[B'] \subseteq \Omega$, then $A' \in E_B^{i,f}$. The proof that the generic structure is universal is left to the reader. $\square$

Note that we used only countably many dense subsets of $\mathbb{P}$, so the Proposition works under the Rasiowa-Sikorski Lemma, without requiring G being "generic" in the sense of forcing theory.

4.2 Results About Rigidity

The generic structure added by $\mathrm{Fn}(\omega, \mathcal{K}, \omega)$ is homogeneous, so it can be of some surprise, that forcing on an uncountable set gives rise to a rigid structure, at least in most of the cases. This is obviously not true for example if $\mathcal{K}$ is the class of all finite sets, but it seems to be true in all sufficiently nontrivial cases. This is proved in 4.2.1. Next, we study linear orders added by forcing with countable support and show that they are not only rigid, but also remain so in any generic extension via a c.c.c. forcing. Note that this is in contrast with the "finite-support-generic" linear orders since, as proved by Baumgartner [7], under CH we can add a nontrivial automorphism to any ω_1-dense separable linear order, using a c.c.c. partial order. Recall that a linear order is ω_1-dense, if every nonempty open interval has cardinality ω_1, and is *separable* if it has a countable dense subset.

4.2.1 $\mathrm{Fn}(\omega_1, \mathcal{K}, \omega)$

We prove that the uncountable partial order and the uncountable undirected graph added by the forcing $\mathrm{Fn}(\omega_1, \mathcal{K}, \omega)$ are rigid. Proofs for linear orders, directed graphs, tournaments or finite rational metric spaces are all easy modifications of either of these.

Theorem 4.2.1. *Let $\mathcal{F}$ be the class of all (undirected) graphs, and S be an uncountable set. Then the generic graph added by $\mathrm{Fn}(S, \mathcal{F}, \omega)$ is rigid.*

Proof. Assume that

$$p \Vdash \text{"}\dot{h} : (S, E(\dot{S})) \to (S, E(\dot{S})) \text{ is a non-identity isomorphism"}.$$

It is easy to check that for every infinite set $F \subseteq S$ from the ground model, and every two different $s, t \in S$, there exists a vertex $e \in F$, with $\{s, e\} \in E(S)$, and $\{t, e\} \notin E(S)$. There are clearly uncountably many pairwise disjoint, infinite subsets of S in the ground model, so h must be non-identity on each of them. Therefore there exists an uncountable set $\{p_s | \, s \in S' \subseteq S\}$ of conditions stronger than p, with

$$p_s \Vdash \dot{h}(s) = \overline{s} \neq s.$$

Without loss of generality we can assume that $\{p_s | \, s \in S'\}$ form a Δ-system with a root R, disjoint from S', and the graph structures of all p_s agree on the root.

Fix two different $s, t \in S'$. We can amalgamate p_s, and p_t over R in such a way, that $\{s, t\} \in E(S)$, and $\{\overline{s}, \overline{t}\} \notin E(S)$, obtaining some stronger condition $q \in \mathrm{Fn}(S, \mathcal{F}, \omega)$. But then q forces, that $\dot{h}$ is not a graph homomorphism. □

Theorem 4.2.2. *Let $\mathcal{F}$ be the class of partial orders, and S be an uncountable set. Then the generic partial order added by* $\mathrm{Fn}(S, \mathcal{F}, \omega)$ *is rigid.*

Proof. Assume that $p \Vdash$ "$\dot{h} : (S, \dot{\leq}) \to (S, \dot{\leq})$ is a non-identity isomorphism". It is easy to check that for every infinite set $E \subseteq S$ from the ground model, $\mathrm{Fn}(S, \mathcal{F}, \omega) \Vdash$ "E is strongly dense". By *strongly dense* we mean that for every $s < t \in S$, there exists $e \in E$, such that $s < e < t$, and for every $s, t \in S$ incomparable, there exists $e_i \in E$, $i = 0, 1, 2, 3, 4$, with $e_0 > s$, incomparable with t; $e_1 < s, t$; $e_2 < s$, incomparable with t; $e_3 > s, t$, and e_4 incomparable with both s and t. Long story short, each type with parameters (not necessarily from E) is realized in E. There are clearly uncountably many pairwise disjoint, infinite subsets of S in the ground model, and h must be non-identity on each of them. Therefore there exists an uncountable set $\{p_s | \, s \in S' \subseteq S\}$ of conditions stronger than p, and

$$p_s \Vdash \dot{h}(s) = \overline{s} \neq s.$$

Without loss of generality we can assume that $\{p_s | \, s \in S'\}$ form a Δ-system with a root R, disjoint from S', and the order structures of all p_s agree on the root. Suppose also, that for each $s \in S'$, $\overline{s} > s$ (the other cases are handled similarily). Since S' is uncountable, we can further thin it out, so that all embeddings of the form $R \subset R \cup \{s\}$ are pairwise isomorphic, and similarly for $\overline{s}$. Recall that two extensions of a given structure R are isomorphic if there is an isomorphism between them, which is identity on R.

Fix two different $s, t \in S'$. There exists an extension $R \subset R \cup \{s, t, \overline{s}, \overline{t}\}$, with $\{s < t < \overline{t} < \overline{s}\}$. We can amalgamate

$$p_s \cup \{t < \overline{t}\}$$

and

$$p_t \cup \{s < \bar{s}\}$$

over

$$R \cup \{s < t < \bar{t} < \bar{s}\},$$

to obtain some condition $q \in \mathrm{Fn}(S, \mathcal{F}, \omega)$. But then $q \Vdash s < t$, and $q \Vdash \dot{h}(s) > \dot{h}(t)$, exhibiting the contradiction. □

It is worth to remark that the uncountable linear ordering added this way satisfies some strong variant of rigidity. Following [6] and [5], we say that an uncountable separable linear order $(L, \leq)$ is k-entangled, for some $k \in \mathbb{N}$, if for every tuple $\bar{t} \in \{T, F\}^k$, and any family $\{(a_0^\xi, \ldots, a_{k-1}^\xi) |\ \xi < \omega_1\}$ of pairwise disjoint k-tuples from L, one can find $\xi \neq \eta < \omega_1$, such that for $i = 0, \ldots, k-1$ $a_i^\xi \leq a_i^\eta$ if and only if $\bar{t}(i) = T$. For $k \geq 2$ this implies that no two uncountable, disjoint subsets of L are isomorphic. Property of being k-entangled for all natural k is featured for example by an uncountable set of Cohen reals, added over some model. Martin's Axiom with negation of CH implies that no uncountable set of reals is k-entangled for all k [6].

4.2.2 $\mathrm{Fn}(\omega_2, \mathcal{LO}, \omega_1)$

We will prove that under CH forcing with countable supports on the set of bigger cardinality gives rise to a rigid linear order, for which we cannot add an automorphism using a c.c.c. forcing. This result holds under CH, however the c.c.c.-absolute rigidity is clearly preserved by any c.c.c. forcing. In effect, existence of a rigid ω_2-dense linear order is consistent with any possible value of 2^ω, and for example $MA +$ "$2^\omega = \kappa$", for any $\kappa = \kappa^{<\kappa}$. Also we can't replace ω_2 with ω_1 in the results of this section. Under CH there exists a unique ω_1-saturated linear order of size ω_1 and as such, it is surely not rigid (see Definition 3.2.1). The partial order $\mathrm{Fn}(\omega_1, \mathcal{LO}, \omega_1)$ forces that the generic order is ω_1-saturated of cardinality ω_1, for the same reasons that $\mathrm{Fn}(\omega, \mathcal{LO}, \omega)$ forces the generic order to be ω-saturated (i.e. dense, without endpoints).

Theorem 4.2.3. *Let $\mathbb{P} = \mathrm{Fn}(\omega_2, \mathcal{LO}, \omega_1)$, where $\mathcal{LO}$ denotes the class of all linear orders. Let $(\omega_2, \leq)$ be a generic order added by $\mathbb{P}$ over a countable, transitive model $\mathbb{V}$, satisfying CH. Denote by $\mathbb{V}[\leq]$ the corresponding generic extension. Let $\mathbb{Q} \in \mathbb{V}[\leq]$ be any forcing notion, such that $\mathbb{V}[\leq] \models$ "$\mathbb{Q}$ is c.c.c.", and H be a $\mathbb{Q}$-generic filter in $\mathbb{V}[\leq]$. Then the linear order $(\omega_2, \leq)$ is rigid in $\mathbb{V}[\leq][H]$.*

We will use a simple lemma assuring that we can amalgamate linear orders in a suitable, asymmetric way.

Lemma 4.2.4. *Let $(L_1, \leq_1)$, $(L_2, \leq_2)$ be any linear orders, $R = L_1 \cap L_2$, and $(R, \leq_1) = (R, \leq_2)$. There exists a linear order $\leq$ on $L_1 \cup L_2$, extending both $\leq_1$ and $\leq_2$ and satisfying*

$$\forall l_1 \in L_1 \setminus R \quad \forall l_2 \in L_2 \setminus R \quad (l_1 < l_2 \iff \exists r \in R \quad l_1 <_1 r <_2 l_2).$$

Proof. We take the above formula as the definition. □

Lemma 4.2.5. *Let $\mathbb{P} = \mathrm{Fn}(\omega_2, \mathcal{K}, \omega_1)$, $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is a c.c.c. forcing notion", and assume that*

$$\mathbb{P} * \dot{\mathbb{Q}} \Vdash \overline{h} : \omega_2 \to \omega_2 \textit{ is a bijection}.$$

Then for every $p \in \mathbb{P}$ exists $p_c \leq p$ with the property that $(p_c, \dot{\mathbb{Q}}) \Vdash \overline{h}[p_c] = p_c$.

Proof. Let $\{F_n\}_{n<\omega}$ be a partition of ω into infinite sets, such that $\forall n < \omega \ n \leq \min F_n$. We define a sequence of conditions $p_n \in \mathbb{P}$ by induction, starting with $p_0 = p$. Enumerate $p_0 = \{r_n | \ n \in F_0\}$. Suppose we have defined p_n. We may take a sequence of $\mathbb{P}$-names with the property

$$p_n \Vdash \text{" } \{\dot{q}^k_{n+1}\}_{k<\omega} \text{ is a maximal antichain deciding } \overline{h}(r_n)\text{"}.$$

Since $\mathbb{P}$ is σ-closed, we will find $p'_n \leq p_n$ deciding all the names $\dot{q}^k_{n+1}$ for $k < \omega$. Therefore the set $A = \{\beta < \omega_2 | \ \exists k < \omega \ (p'_n, \dot{q}^k_{n+1}) \Vdash \overline{h}(r_n) = \beta\}$ is at most countable. Let $p_{n+1} = p'_n \cup A$ (with relations defined arbitrarily), and enumerate $p_{n+1} = \{r_k | \ k \in F_{n+1}\}$. The inductive step is completed.

Take $p_c = \bigcup_{n<\omega} p_n$. We will show that for any $\dot{q}$, with $p_c \Vdash \dot{q} \in \dot{\mathbb{Q}}$, and any $\alpha \in p_c$, $(p_c, \dot{q}) \Vdash \overline{h}(\alpha) \in p_c$. Indeed, in this situation there is some $n < \omega$ such that $\alpha \in p_n$. Therefore we will find $k < \omega$ with $\alpha = r_k$, $k \in F_n$. In the k-th indutive step we ensure that $(p_{k+1}, \dot{q}) \Vdash \overline{h}(r_k) \in p_{k+1}$. It follows that $(p_c, \dot{q}) \Vdash \overline{h}(r_k) \in p_c$. □

Proof of Theorem 4.2.3. We work in $\mathbb{V}$. Let $\dot{\leq}$ be a $\mathbb{P}$-name for $\leq$. Suppose that

$$\mathbb{P} \Vdash \text{ "}\dot{\mathbb{Q}} \text{ is c.c.c." },$$

and

$$\mathbb{P} * \dot{\mathbb{Q}} \Vdash \text{"}\overline{h} : (\omega_2, \dot{\leq}) \to (\omega_2, \dot{\leq}) \text{ is a non-identity isomorphism"}.$$

Step 0 It can be easily verified, that if $\overline{h}$ was identity on a dense set, then it would be identity everywhere. Therefore there exist $\mathbb{P} * \dot{\mathbb{Q}}$-names $\overline{\delta}_0, \overline{\delta}_1$, such that

$$\mathbb{P} * \dot{\mathbb{Q}} \Vdash \overline{\delta}_0 \dot{<} \overline{\delta}_1, \ \forall x \in (\overline{\delta}_0, \overline{\delta}_1) \ \overline{h}(x) \neq x,$$

where $(\overline{\delta}_0, \overline{\delta}_1)$ is the interval with respect to the ordering with the name $\dot{\leq}$. Fix $(p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}$ deciding $\overline{\delta}_0$ and $\overline{\delta}_1$, i.e. $(p, \dot{q}) \Vdash \forall i \in \{0, 1\}\ \overline{\delta}_i = \delta_i$, for some $\delta_i \in \omega_2$. Without loss of generality, we can assume that $\dot{q}$ is the greatest element of $\dot{\mathbb{Q}}$, so that

$$(p, \dot{\mathbb{Q}}) \Vdash \delta_0 \dot{<} \delta_1,\ \forall x \in (\delta_0, \delta_1)\ \overline{h}(x) \neq x.$$

Step 1 For $\alpha \in \omega_2 \setminus (\{\delta_0, \delta_1\} \cup \operatorname{supp} p)$ we fix a condition $p_\alpha = (p_\alpha, \leq_\alpha) \leq p$, with

$$\delta_0 <_\alpha \alpha <_\alpha \delta_1.$$

Take a sequence of names satisfying

$$p_\alpha \Vdash \text{"}\{\dot{q}^n_\alpha\}_{n<\omega} \text{ is a maximal antichain deciding } \overline{h}(\alpha)\text{"}.$$

Since $\mathbb{P}$ is σ-closed, we can assume that p_α decides all the names $\dot{q}^n_\alpha$, so the set $F(\alpha) = \{\beta < \omega_2 |\ \exists n < \omega\ \ (p_\alpha, \dot{q}^n_\alpha) \Vdash \overline{h}(\alpha) = \beta\}$ is countable (recall that $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is c.c.c."). Note, that since $p_\alpha \leq p$, $\alpha \notin F(\alpha)$. Finally, we can assume that $F(\alpha) \subseteq p_\alpha$, and, due to Lemma 4.2.5, that $(p_\alpha, \dot{\mathbb{Q}}) \Vdash \overline{h}[p_\alpha] = p_\alpha$.

Step 2 Using Δ-Lemma for countable sets, we can find $I \subseteq \omega_2$ of cardinality ω_2, with the following conditions satisfied

- $\forall \alpha \in I\ \forall \beta \in I \quad \beta \neq \alpha \implies p_\alpha \cap p_\beta = R$, for some fixed countable $R \subseteq \omega_2$,
- $\forall \alpha \in I\ \forall \beta \in I \quad \leq_\alpha \restriction R \times R = \leq_\beta \restriction R \times R$,
- extensions $R \subset R \cup \{\alpha\}$, for $\alpha \in I$, are pairwise isomorphic,
- $\forall \alpha \in I \quad (p_\alpha, \dot{\mathbb{Q}}) \Vdash \overline{h}[R] = R$.

All these conditions, perhaps excluding the last one, are direct consequences of CH. To justify the last claim, notice that $\mathbb{P} * \dot{\mathbb{Q}}$ is ω_2-c.c. and so the set

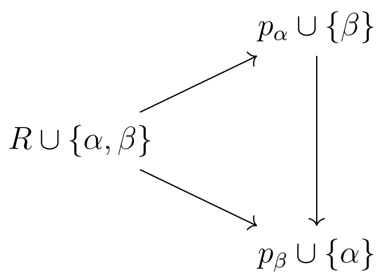
$$A = \{\beta < \omega_2 |\ \exists (p, \dot{q}) \in \mathbb{P} * \dot{\mathbb{Q}}\ \exists r \in R\ (p, \dot{q}) \Vdash \overline{h}(r) = \beta\}$$

has cardinality at most ω_1. We choose to $\{p_\alpha |\ \alpha \in I\}$ only those conditions, which satisfy $(p_\alpha \setminus R) \cap A = \emptyset$. Take $r \in R$. Then $(p_\alpha, \dot{\mathbb{Q}}) \Vdash \overline{h}(r) \in p_\alpha \cap A \subseteq R$.

Step 3 Take $\alpha, \beta \in I$, $\alpha \neq \beta$. Using the fact that the extensions $R \subseteq R \cup \{\alpha\}$ and $R \subseteq R \cup \{\beta\}$ are isomorphic, we can extend the ordering $\leq_\alpha = \leq_\beta$ on R to

$$(R \cup \{\alpha, \beta\}, \leq_{\alpha,\beta})$$

in such a way that there is no element from R between α and β. We can of course decide that $\alpha <_{\alpha,\beta} \beta$. We now apply Lemma 4.2.4 to the pair of isomorphic extensions



where the vertical arrow maps β to α.

Extend $\leq_{\alpha,\beta}$ to $p_\alpha \cup p_\beta$, ensuring that

- $\neg \exists r \in R \quad \alpha <_{\alpha,\beta} r <_{\alpha,\beta} \beta$;
- $\forall \gamma \in p_\alpha \setminus (R \cup \{\alpha\}) \quad \forall \eta \in p_\beta \setminus (R \cup \{\beta\})$
  $\gamma <_{\alpha,\beta} \eta \iff \exists r \in R \quad \gamma <_{\alpha,\beta} r <_{\alpha,\beta} \eta$.

Take some condition $r \leq p_{\alpha,\beta}$ and $\dot{q}$ deciding the values of $\overline{h}(\alpha)$ and $\overline{h}(\beta)$. Then $(r, \dot{q}) \Vdash \overline{h}(\alpha) = h(\alpha), \ \overline{h}(\beta) = h(\beta)$. Since there is no element from R between α and β, and R is $\overline{h}$ invariant, there is also no element from R between $h(\alpha)$ and $h(\beta)$. But since $h(\alpha) \in p_\alpha \setminus \{\alpha\}$, and $h(\beta) \in p_\beta \setminus \{\beta\}$, $h(\beta) <_{\alpha,\beta} h(\alpha)$. Therefore $(r, \dot{q}) \Vdash h(\alpha) > h(\beta)$, giving rise to a contradiction. This finishes the proof. □

The following argument, suggested to us by Shelah, shows that it is possible to have a separable rigid linear order, whose rigidity is c.c.c.-absolute. By Theorem 24 from [6], MA_{ω_1} is consistent with the existence of a rigid set of reals of cardinality ω_1.

Theorem 4.2.6. *Assume MA_{ω_1}, and let $A \subseteq \mathbb{R}$ be a rigid linear order of cardinality ω_1. Then A remains rigid in any generic extension by a c.c.c. forcing.*

Proof. Without loss of generality we may assume that $A = (\omega_1, \leq)$. Let $\mathbb{S}$ be any c.c.c. forcing, and suppose towards contradiction that

$$\mathbb{S} \Vdash \text{"}\dot{f} : (\omega_1, \leq) \hookrightarrow (\omega_1, \leq) \text{ is a non-identity isomorphism."}$$

For all $\gamma < \omega_1$, let $A_\gamma \subseteq \mathbb{S}$ be some maximal antichain deciding $\dot{f}(\gamma)$. By Martin's Axiom there is a filter $H \subseteq \mathbb{S}$ intersecting all of the sets A_γ. Therefore $\dot{f}[H]$ is well defined, and is a non-trivial automorphism of $(\omega_1, \leq)$, contrary to the fact that A was rigid. □

4.3 Linear Orders with Few Automorphisms

Ohkuma proved in [39] that there exist 2^{2^ω} pairwise non-isomorphic subgroups $(G, +) \leq (\mathbb{R}, +)$, with the property that $\mathrm{Aut}\,(G, \leq) \simeq (G, +)$, meaning that G has no order-automorphisms other than translations. These groups all have cardinality 2^ω, however the authors of [18] have shown that consistently there are uncountable groups of cardinality less than 2^ω with this property. These are examples of separable, uncountable linear orders, with few, but more than one, automorphisms. We are going to provide one more construction in this spirit.

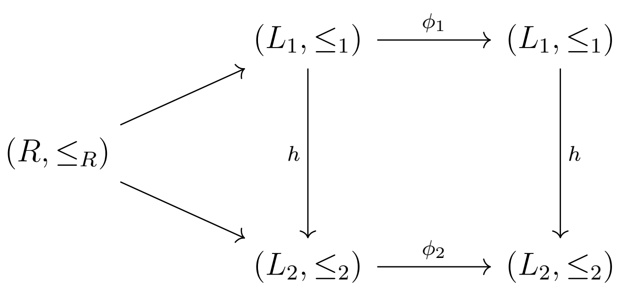
Theorem 4.3.1. *It is consistent with ZFC, that there exists an ω_1-dense, separable linear order $(A, \leq)$, together with a non-identity automorphism ϕ, such that $\mathrm{Aut}\,(A, \leq) = \{\phi^k |\ k \in \mathbb{Z}\}$. Moreover, ϕ satisfies $\phi(x) > x$ for all $x \in A$.*

Let $<_{ord}$ denote the usual order on ω_1. The required modification of $\mathrm{Fn}(\omega_1, \mathcal{LO}, \omega)$ is the forcing $\mathbb{P}$ consisting of triples $p = (p, \leq_p, \phi_p)$ satisfying the conditions:

1. $\leq_p$ is a linear ordering of $p \in [\omega_1]^{<\omega}$,
2. ϕ_p is an $\leq_p$-increasing bijection between two subsets of p,
3. $\forall x \in \mathrm{dom}\, p \quad (x <_p \phi_p(x))$,
4. $\forall x \in \mathrm{dom}\, p \quad (\phi_p(x) <_{ord} x + \omega)$, with respect to the ordinal addition on ω_1,
5. $\forall x \in \mathrm{rg}\, p \quad (\phi_p^{-1}(x) <_{ord} x + \omega)$, with respect to the ordinal addition on ω_1.

We denote by $(\omega_1, \leq)$ the ordering added by $\mathbb{P}$, and by ϕ the corresponding automorphism. Before proceeding with the main proof, we will see that it is possible to amalgamate finite linear orders together with partial automorphisms in a suitable way. It will be convenient to denote by $\mathrm{Part}\,(L, \leq)$ the set of finite, partial automorphisms of a linear order $(L, \leq)$.

Lemma 4.3.2. *Let $(L_1, \leq_1)$, $(L_2, \leq_2)$ and $(R, \leq_R) = (L_1, \leq_1) \cap (L_2, \leq_2)$ be finite linear orders. Fix partial automorphisms $\phi_1 \in \mathrm{Part}\,(L_1, \leq_2)$, $\phi_2 \in \mathrm{Part}\,(L_2, \leq_2)$. Assume that R is invariant for ϕ_1 and ϕ_2, and that (L_1, ϕ_1) and (L_2, ϕ_2) are extensions of R isomorphic via $h : L_1 \to L_2$, in a sense that they make the diagram below commutative.*



Take $a, b \in L_1 \setminus R$ lying in different orbits of ϕ_1. There exists a linear order $\leq_c$ on $L_1 \cup L_2$ extending $\leq_1$ and $\leq_2$, and such that $\phi_1 \cup \phi_2 \in \mathrm{Part}\,(L_1 \cup L_2, \leq_c)$, and moreover $a <_c h(a)$, and $h(b) <_c b$.

Proof. We can assume that $R \subseteq L_1 \subseteq \mathbb{Q}$, and the usual ordering of $(\mathbb{Q}, \leq)$ extends $\leq_1$. We look for an increasing function $f : (L_2, \leq_2) \to (\mathbb{Q}, \leq)$ such that $f \restriction R = \mathrm{id}_R$, $f[L_2 \setminus R] \cap (L_1 \setminus R) = \emptyset$, and

$$f \circ h(a) > a,$$

$$f \circ h(b) < b.$$

Indeed, having f as above we will define

$$x <_c y \iff x < f(y),$$

for $x \in L_1$ and $y \in L_2$.

It can be seen that the only reason why we can't take $f = h^{-1}$ is the disjointness requirement. So we should expect that f will be just a slight distortion of h^{-1}. We must also ensure that $\phi_1 \cup \phi_2$ will be order-preserving.

Let $\{x_1, \ldots, x_n\}$ be an $\leq_1$-increasing enumeration of L_1. For $k = 1, \ldots, n$ choose an open interval I_k around x_k in such a way that all intervals obtained this way are pairwise disjoint, and for $l \neq k = 1, \ldots, n$ if $x_l = \phi_1^m(x_k)$, then $\overline{\phi}_1^m[I_l] = I_k$, where $\overline{\phi}_1 : (\mathbb{Q}, \leq) \to (\mathbb{Q}, \leq)$ is an extension of ϕ_1.

For each k we choose $f(h(x_k)) \in I_k \setminus \{x_k\}$, so that $\phi_1^m(f \circ h(x_k)) = f \circ h \circ \phi_1^m(x_k)$, for $m \in \mathbb{Z}$, whenever this expression makes sense. Since a and b are in distinct orbits of ψ_1, we can ensure inequalities $f \circ h(a) > a$ and $f \circ h(b) < b$.

Why is it the case that $\phi_1 \cup \phi_2 \in \mathrm{Part}\,(L_1 \cup L_2, \leq_c)$? Note, that for all pairs of integers $k, l = 1, \ldots, n$ we have

$$x_k <_c x_l \iff x_k < f \circ h(x_l)$$

by the choice of f, and so

$$x_k <_c h(x_l) \iff x_k < f \circ h(x_l) \iff x_k < x_l$$
$$\iff \phi_1(x_k) < \phi_1(x_l) = \phi_2 \circ h(x_l).$$

□

Proposition 4.3.3. *The forcing $\mathbb{P}$ satisfies the Knaster condition.*

Proof. Let $\{p_\alpha = (p_\alpha, \leq_\alpha, \phi_\alpha) | \ \alpha < \omega_1\} \subseteq \mathbb{P}$. We fix a Δ-system $\{p_\alpha | \ \alpha \in S\}$, with some additional properties:

- $\forall \alpha \in S \ \forall \beta \in S \quad \alpha \neq \beta \implies (p_\alpha, \leq_\alpha) \cap (p_\beta, \leq_\beta) = (R, \leq_R)$, for some fixed ordering $\leq_R$ of R,
- $\forall \alpha \in S \quad \phi_\alpha[R] \subseteq R$,
- $\forall \alpha \in S \quad \phi_\alpha^{-1}[R] \subseteq R$,
- $\phi_\alpha \restriction R$ is independent from the choice of α.

For ensuring the last three properties we use 4. and 5. from the definition of $\mathbb{P}$. Thanks to the Lemma 4.3.2, we obtained an uncountable set of pairwise comparable conditions. □

Lemma 4.3.4. *For every $\alpha_0 \in \omega_1$, the orbit of α_0 under ϕ is cofinal and coinitial in $(\omega_1, \leq)$*

Proof. It is easy to see that the required family of dense sets is

$$E_\beta = \{p = (p, \leq_p, \phi_p) \in \mathbb{P} | \ \{\alpha_0, \beta\} \subseteq p, \ \exists k \geq 0 \ \beta <_p \phi_p^k(\alpha_0), \ \phi_p^{-k}(\alpha_0) <_p \beta\},$$

for $\beta \in \omega_1$.

In order to check, that E_β is dense, fix some condition $p = (p, \leq_p, \phi_p) \in \mathbb{P}$ and $\beta < \omega_1$. We can assume that $\{\alpha_0, \beta\} \subseteq p$. In order to extend p so that it belongs to E_β, we embed $(p, \leq_p)$ into the set of algebraic numbers A. Now we can extend ϕ_p to an increasing function $\overline{\phi} : A \to A$, such that for some rational $\epsilon > 0 \ \forall a \in A \ \overline{\phi}(a) > a + \epsilon$. It is clear that the orbit of α_0 under $\overline{\phi}$ is both cofinal and coinitial in A. Finally we just cut out a suitable finite fragment of $\overline{\phi}$, and extend p accordingly. □

Lemma 4.3.5. *For each isomorphism $h : (\omega_1, \leq) \to (\omega_1, \leq)$, and for every uncountable set $F \subseteq \omega_1$, there exist $\alpha \in F$ and $k \in \mathbb{Z}$, such that $h(\alpha) = \phi^k(\alpha)$.*

Proof. Fix a sequence of names for elements of F, $\{\dot{x}_\alpha | \ \alpha < \omega_1\}$. Let

$$p \Vdash \dot{h} : (\omega_1, \dot{\leq}) \to (\omega_1, \dot{\leq}) \text{ is an isomorphism.}$$

For every $\alpha < \omega_1$ we fix a condition $p_\alpha = (p_\alpha, \leq_\alpha, \phi_\alpha) \leq p$, so that $p_\alpha \Vdash \dot{x}_\alpha = x_\alpha, \ \dot{h}(x_\alpha) = \overline{x}_\alpha$, for some ordinals $x_\alpha, \overline{x}_\alpha \in \omega_1$. We can also assume that for each α, $x_\alpha \neq \overline{x}_\alpha$, for otherwise we just take $k = 0$.

We choose an uncountable Δ-system $\{p_\alpha | \ \alpha \in S\}$, and make it as uniform as possible:

- $\forall \alpha \in S\ \forall \beta \in S\ \alpha \neq \beta \implies (p_\alpha, \leq_\alpha) \cap (p_\beta, \leq_\beta) = (R, \leq_R)$, for some fixed ordering $\leq_R$ of R,
- $\forall \alpha \in S\ \phi_\alpha[R] \subseteq R$,
- $\forall \alpha \in S\ \phi_\alpha^{-1}[R] \subseteq R$,
- extensions $(R, \leq_R) \subseteq (R \cup \{x_\alpha\}, \leq_\alpha)$ are pairwise isomorphic,
- extensions $(R, \leq_R) \subseteq (R \cup \{\overline{x}_\alpha\}, \leq_\alpha)$ are pairwise isomorphic,
- extensions $(R, \leq_R) \subseteq (p_\alpha, \leq_\alpha)$ are pairwise isomorphic,
- The way ϕ_α acts on p_α is independent from the choice of $\alpha \in S$. More precisely, for all distinct $\alpha, \beta \in S$ the following diagram commutes

$$\begin{array}{ccc} p_\alpha & \xrightarrow{\phi_\alpha} & p_\alpha \\ h\big\downarrow & & \big\downarrow h \\ p_\beta & \xrightarrow{\phi_\beta} & p_\beta \end{array}$$

  where h is the unique isomorphism between $(p_\alpha, \leq_\alpha)$ and $(p_\beta. \leq_\beta)$.

In particular, this unique isomorphism h maps x_α to x_β, and $\overline{x}_\alpha$ to $\overline{x}_\beta$. Fix $\alpha \in S$. We claim that x_α and $\overline{x}_\alpha$ are in the same orbit of ϕ_α. Suppose otherwise. We fix $\beta \in S \setminus \{\alpha\}$, and apply Lemma 4.3.2 for $a = x_\alpha$ and $b = \overline{x}_\alpha$. This way we obtain a condition

$$q = (p_\alpha \cup p_\beta, \leq_q, \phi_\alpha \cup \phi_\beta) \leq p_\alpha, p_\beta,$$

satisfying $x_\alpha <_q x_\beta$, and $\overline{x}_\beta <_q \overline{x}_\alpha$. But then

$$q \Vdash \dot{x}_\alpha \dot{<} \dot{x}_\beta,\ \dot{h}(\dot{x}_\beta) \dot{<} \dot{h}(\dot{x}_\alpha),$$

contrary to the choice of p. It follows that $\overline{x}_\alpha$ and x_α must be in the same orbit of ϕ_1, and so

$$\exists k \in \mathbb{Z}\ \overline{x}_\alpha = \phi_\alpha^k(x_\alpha).$$

By the definition of $\overline{x}_\alpha$, this shows that

$$p_\alpha \Vdash \dot{h}(\dot{x}_\alpha) = \dot{\phi}_\alpha{}^k(\dot{x}_\alpha).$$

Given that p was arbitrary, and $p_\alpha \leq p$, this finishes the proof. $\square$

Now we are in a position to prove Theorem 4.3.1.

Proof of Theorem 4.3.1. Since $(\omega_1, \leq)$ is separable, we can replace it by an isomorphic copy $A \subseteq \mathbb{R}$, A being ω_1-dense, and dense in $\mathbb{R}$. Then $\phi : A \to A$ is an increasing bijection, strictly above the diagonal. Let $h : A \to A$ be any increasing bijection. Both h and ϕ extend uniquely to the whole real line, so we can assume that $\phi, h : \mathbb{R} \to \mathbb{R}$ are continuous, increasing bijections.

For $k \in \mathbb{Z}$, let $F_k = \{x \in \mathbb{R} |\ h(x) = \phi^k(x)\}$. By continuity, the sets F_k are closed, and by Lemma 4.3.5 $\bigcup_{i\in\mathbb{Z}} F_i$ is dense. Fix some $k \in \mathbb{Z}$ for which the set F_k is nonempty. We aim to prove that $F_k = \mathbb{R}$. If not, there exists $x \in F_k$, and $\delta > 0$ satisfying at least one of conditions

$$(x, x + \delta) \cap F_k = \emptyset,$$

and

$$(x - \delta, x) \cap F_k = \emptyset.$$

Assume the first case, the other being similar. Since the union of the sets F_i is dense, we can find a decreasing sequence $\{x_n\}_{n<\omega}$, converging to x, and integers k_n, for which $h(x_n) = \phi^{k_n}(x_n)$.

Suppose that for infinitely many n, the inequality $k_n > k$ holds. By replacing $\{x_n\}_{n<\omega}$ with a subsequence, we may assume that this is the case for all $n < \omega$. Then

$$\phi^{k_n}(x_n) \geq \phi^{k+1}(x_n) \underset{n\to\infty}{\longrightarrow} \phi^{k+1}(x) > \phi^k(x) = h(x),$$

which contradicts $\lim\limits_{n\to\infty} \phi^{k_n}(x_n) = h(x)$.

If for infinitely many n the inequality $k_n < k$ holds, we proceed in analogous manner. The only way out is $k_n = k$ for all but finitely many n, but this in turn contradicts $(x, x+\delta) \cap F_k = \emptyset$. Therefore $F_k = \mathbb{R}$, and the theorem is proved. □

Chapter 5

Generic Structures and Martin's Axiom

Models of size ω_1 arising from the Fraïssé-Jónsson theory, relying on the Continuum Hypothesis, are homogeneous with respect to countable substructures, but since we have plenty of countable submodels, they are highly "non-separable". The question arises whether we can build uncountable models using finite submodels? One indication that such theory should be possible to formulate is the famous theorem of Baumgartner, concerning separable ω_1-dense linear orders. Recall that a linear order is ω_1-dense if each non-empty open interval has cardinality ω_1.

Theorem (Baumgartner, [7]). *It is consistent with ZFC that there exists a unique up to isomorphism separable ω_1-dense linear order.*

Once the conclusion of this theorem holds, we can take any subfield of the reals of size ω_1 as a model of this ordering. This shows that it is homogeneous with respect to finite subsets (as is any subfield of reals). Ideas of Baumgartner were extended by Avraham, Rubin, and Shelah in [6] and [5]. Among other things, they show that Baumgartner's Theorem does not follow from $MA+\neg CH$, but the latter axiom is a significant step in the direction of Baumgartner's Theorem. They introduce another axiom, known as OCA_{ARS}, and show that $MA_{\omega_1} + OCA_{ARS}$ implies that either the conclusion of Baumgartner's Theorem holds, or there are up to isomorphism exactly three homogeneous separable ω_1-dense linear orders ([5], Sec. 6). This gives some insight that MA_{ω_1} might be used in place of induction, in some uncountable variant of the Fraïssé theory.

We take ideas from papers [6] and [5], and apply them to other structures beyond linear orders – mostly metric spaces and graphs. Specifically, we take generic models described in Chapter 4, and show that, although initially rigid, they become highly homogeneous in suitable forcing extensions satisfying MA_{ω_1}.

Inductive arguments from the classical theory are replaced by Martin's Axiom, and in some cases it is enough to imply certain uniqueness.

5.1 Following Avraham, Rubin, and Shelah

We adapt the technology from [6] and [5], to prove that it is relatively consistent with ZFC that there exists a separable rational metric space (X, d) of size ω_1, such that any uncountable 1-1 function from X to itself is an isometry on an uncountable subset.

We first introduce a metric analog of a k-increasing linear order, introduced in [6].

Definition 5.1.1. Let (X, d) be a metric space.

- We call a pair of tuples $\overline{x} = (x_1, \ldots, x_n), \overline{y} = (y_1, \ldots, y_n) \in X^n$ *alike* if they satisfy the following axioms:

  A1 $\forall\, i, j = 1, \ldots, n\ (d(x_i, y_i) = d(x_j, y_j))$

  A2 $\forall\, i, j = 1, \ldots, n\ (d(x_i, x_j) = d(y_i, y_j))$

  A3 $\forall\, i, j = 1, \ldots, n\ (x_i \neq x_j \implies d(x_i, x_j) = d(x_i, y_j))$

  We then write $\overline{x} \circledast \overline{y}$.

- We call (X, d) *rectangular* if it is uncountable, and for any sequence of pairwise disjoint tuples $\{(x_1^\xi, \ldots, x_n^\xi) | \xi < \omega_1\} \subseteq X^n$, there are $\xi \neq \eta < \omega_1$, such that $(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta)$.
  (by "disjoint tuples", we mean that

  $$\{x_1^\xi, \ldots, x_n^\xi\} \cap \{x_1^\eta, \ldots, x_n^\eta\} = \emptyset$$

  whenever $\xi \neq \eta$.)

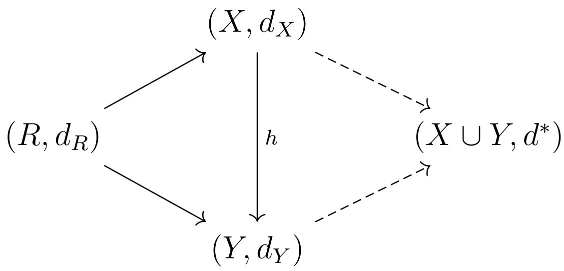
Denote by $Metr$ the class of rational metric spaces. Keeping up with the general notation, $\mathrm{Fn}(\omega_1, Metr, \omega)$ is the partial order

$$\{(Y, d) |\ Y \in [\omega_1]^{<\omega},\ \text{and } (Y, d) \text{ is a rational metric space}\},$$

with the ordering relation being the reversed inclusion preserving the metric.

We begin with a technical lemma to ensure that we can amalgamate metric spaces in a specific way.

Lemma 5.1.2. *Let* $(R, d_R), (X, d_X), (Y, d_Y)$ *be finite metric spaces, such that* $(X, d_X) \cap (Y, d_Y) = (R, d_R)$*, and suppose* $h : (X, d_X) \to (Y, d_Y)$ *is an isometric bijection, which is identity on* R*.*



Then there exists a metric d^* *on* $X \cup Y$ *extending both* d_X *and* d_Y*, such that if* $(x_1, \ldots, x_n)$ *is a bijective enumeration of* $X \setminus R$*, then* $(x_1, \ldots, x_n) \circledast (h(x_1), \ldots, h(x_n))$*.*

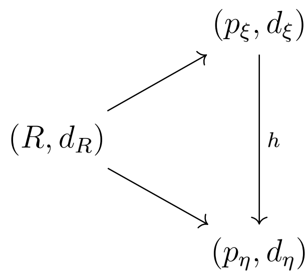
Proof. Let $s = \min\{d_X(x, x')|\ x \neq x' \in X\}$. Given that d^* must extend the metrics of X and Y, we must set the distances between elements from $X \setminus R$ and $Y \setminus R$. Therefore we set $d^*(x, h(x)) = s$ and $d^*(x, h(x')) = d_X(x, x')$, for all $x \neq x' \in X \setminus R$. A standard computation shows that this definition gives a well-defined metric structure on $X \cup Y$, satisfying the required conditions. □

In the light of Proposition 4.1.2, we have the c.c.c. property for $\mathrm{Fn}(\omega_1, Metr, \omega)$. Even more generally, for any countable set $K \subseteq [0, \infty)$ the proof of Lemma 5.1.2 shows that the class of finite metric spaces with distances in K has the SP.

Proposition 5.1.3. $\mathrm{Fn}(\omega_1, Metr, \omega) \Vdash$ *"*$(\omega_1, \dot{d})$ *is rectangular".*

Proof. Let $\{(\dot{x}_1^\xi, \ldots, \dot{x}_n^\xi)|\ \xi < \omega_1\}$ be a sequence of $\mathrm{Fn}(\omega_1, Metr, \omega)$-names for pairwise disjoint n-tuples from $(\omega_1, \dot{d})$. Fix a condition p. For every $\xi < \omega_1$, we find a condition $p_\xi = (p_\xi, d_\xi) \leq p$, deciding the values of $\dot{x}_i^\xi$ and $\dot{d} \restriction \{x_1^\xi, \ldots, x_n^\xi\} \times \{x_1^\xi, \ldots, x_n^\xi\}$. We choose an uncountable set $S \subseteq \omega_1$, satisfying the following conditions:

- $\{p_\xi|\ \xi \in S\}$ is a Δ-system with the root $R = (R, d_R)$,
- $\forall\ \xi, \eta \in S$ there exists an isometry $h : p_\xi \to p_\eta$, which is identity on R, and $h(x_i^\xi) = x_i^\eta$, for $i = 1, \ldots, n$, as shown in the diagram.



This can be easily done, since given any point $\alpha \in \omega_1$ outside of R, there are only countably many possible configurations of distances between this point and R. Now we choose $\xi \neq \eta \in S$, and apply Lemma 5.1.2 to obtain $q \leq p_\xi, p_\eta$, which forces that $(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta)$. □

It is easy to verify that any infinite subset of ω_1 from the ground model is forced by $\mathrm{Fn}(\omega_1, Metr, \omega)$ to be dense. It has even a much stronger property, which we isolate, since it will be important later.

Definition 5.1.4. Let (E, d_E) be a metric space with distances in some countable set $K \subseteq [0, \infty)$. A subset $D \subseteq E$ is a *saturated subset* of E if for any finite subset $E_0 \subseteq E$, for any single-point extension $F = (E_0 \cup \{f\}, d_F)$, with distances in K, there exists $d \in D$ such that for all $e \in E_0$, we have $d_E(e, d) = d_F(e, f)$.

To put it shortly, every possible configuration of distances from a tuple of points in E can be realized by a point in D. The rational Urysohn space is a saturated subspace of itself, in the class of rational metric spaces.

Definition 5.1.5. Let (X, d) be any metric space, with distances in a given countable set.

1. (X, d) is *separably saturated* if it has a countable saturated subset.
2. (X, d) is *hereditarily separably saturated* (HSS) if for any countable subset $A \subseteq X$, the space $X \setminus A$ is separably saturated.

Proposition 5.1.6. $\mathrm{Fn}(\omega_1, Metr, \omega) \Vdash$ *"$(\omega_1, \dot{d})$ is HSS (with distances in $\mathbb{Q}$)".*

Proof. Each infinite subset of ω_1 which belongs to the ground model is saturated. □

Let $(\mathcal{M}, d)$ be the metric space we added to our model by $\mathrm{Fn}(\omega_1, Metr, \omega)$. Our next task is to force MA_{ω_1}, while preserving $(\mathcal{M}, d)$ being rectangular. Following the ideas from [6], we distinguish the special class of c.c.c. partial orders.

Definition 5.1.7. A partial order $\mathbb{P}$ is *appropriate* if given any natural number $n > 0$, for each family consisting of pairwise disjoint tuples

$$\{(p_\xi, x_1^\xi, \ldots, x_n^\xi)|\ \xi < \omega_1\} \subseteq \mathbb{P} \times (\mathcal{M}, d)^n,$$

there exist $\xi \neq \eta < \omega_1$, such that p_ξ and p_η are comparable, and

$$(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta).$$

Proposition 5.1.8. *If $\mathbb{P}$ is appropriate then $\mathbb{P} \Vdash$ "$(\mathcal{M}, d)$ is rectangular".*

Proof. Fix a sequence of $\mathbb{P}$-names $\{(\dot{x}_1^\xi, \ldots, \dot{x}_n^\xi)|\ \xi < \omega_1\} \subseteq (\mathcal{M}, d)^n$ for pairwise disjoint n-tuples. For a given condition $p \in \mathbb{P}$, and $\xi < \omega_1$, we fix a condition $p_\xi \leq p$ deciding $(\dot{x}_1^\xi, \ldots, \dot{x}_n^\xi)$. Then we apply appropriateness for the family $\{(p_\xi, x_1^\xi, \ldots, x_n^\xi)|\ \xi < \omega_1\}$. This way we obtain $\xi \neq \eta < \omega_1$, and $q \leq p_\xi, p_\eta$, such that $q \Vdash (\dot{x}_1^\xi, \ldots, \dot{x}_n^\xi) \circledast (\dot{x}_1^\eta, \ldots, \dot{x}_n^\eta)$. □

The argument showing that appropriateness is preserved under iterations is really not different from the one showing that the c.c.c. is preserved, applied for example in [33]. We include it for completeness.

Proposition 5.1.9. *Finite support iterations of appropriate posets are appropriate.*

Proof. Assume that $\mathbb{P}$ is appropriate, and $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is appropriate". Take a sequence

$$\{(p_\xi, \dot{q}_\xi, x_1^\xi, \ldots, x_n^\xi)|\ \xi < \omega_1\},$$

and towards contradiction assume that it witnesses $\mathbb{P} * \dot{\mathbb{Q}}$ not being appropriate. Let $\dot{\sigma}$ be a $\mathbb{P}$-name defined $\dot{\sigma} = \{(\xi, p_\xi)|\ \xi < \omega_1\}$. If $G \subseteq \mathbb{P}$ is an M-generic filter, then

$$M[G] \models (\xi \in \sigma \iff p_\xi \in G).$$

We claim that in $M[G]$, for any two conditions $\eta, \xi \in \sigma$, if $(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta)$, then q_ξ and q_η are inconsistent. For otherwise, there exist $q \leq q_\eta, q_\xi$, and $p \in G$, which forces it. Since p_ξ and p_η are in G, which is a filter, we may choose $p \leq p_\xi, p_\eta$. Then $(p, \dot{q}) \leq (p_\xi, \dot{q}_\xi), (p_\eta, \dot{q}_\eta)$, and $(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta)$ contrary to the choice of the sequence $\{(p_\xi, \dot{q}_\xi, x_1^\xi, \ldots, x_n^\xi)|\ \xi < \omega_1\}$.

G was an arbitrary generic filter, and remember that $\mathbb{P} \Vdash$ "$\dot{\mathbb{Q}}$ is appropriate". The conclusion of this is that $\mathbb{P} \Vdash |\dot{\sigma}| < \omega_1$. Then there exists a $\mathbb{P}$-name for a countable ordinal $\dot{a}$, for which $\mathbb{P} \Vdash \dot{\sigma} \subseteq \dot{a}$. Since $\mathbb{P}$ is c.c.c. there are only countably many possible values of $\dot{a}$, and by taking supremum of them, we can replace $\dot{a}$ by a canonical name a. But note that $p_a \Vdash a \in \dot{\sigma}$. This is a contradiction, and it finishes the proof for $\mathbb{P} * \dot{\mathbb{Q}}$.

Consider now a finite support iteration of an infinite length κ of appropriate forcings $\overline{\mathbb{P}} = \{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha | \ \alpha < \kappa\}$. We prove by induction on κ, that $\overline{\mathbb{P}}$ is appropriate. The successor step has just been taken care of, so suppose that the conclusion holds for any ordinal less than κ, and κ is limit. Take any disjoint sequence $\{(\overline{p}_\alpha, \overline{x}_\alpha) | \ \alpha < \omega_1\} \subseteq \overline{\mathbb{P}} \times (\mathcal{M}, d)^n$. We can assume that the supports of conditions $\overline{p}_\alpha$ form a Δ-system with the root R, and that for any α, $\operatorname{supp} \overline{p}_\alpha \setminus R$ is above R. Let $\delta = \max R$. There exist two different $\alpha, \beta < \omega_1$, for which the relation $\overline{x}_\alpha \circledast \overline{x}_\beta$ holds and $\overline{p}_\alpha \restriction \delta$ is comparable with $\overline{p}_\beta \restriction \delta$. From the general theory of finite support iterations it follows that $\overline{p}_\alpha$ and $\overline{p}_\beta$ are comparable. □

It may look suspicious that in the proof above we never actually used $\mathbb{P}$ being appropriate, only c.c.c. However one can verify that if some c.c.c. forcing forces a poset to be appropriate, then it must be appropriate itself. So equally good we could have assumed that $\mathbb{P}$ is c.c.c. The immediate consequence of this proposition is

Lemma 5.1.10. *It is consistent with $ZFC + MA($appropriate$) +$ "$2^\omega = \omega_2$" that $(\mathcal{M}, d)$ is rectangular.*

Conveniently, the full Martin's Axiom will hold in such model. Recall that if $\mathbb{P}$ is a forcing notion, a set $D \subseteq \mathbb{P}$ is *predense* if each element of $\mathbb{P}$ is comparable with an element of D. A set D is *predense below* p if each element of $\mathbb{P}$ stronger that p is comparable with an element of D.

Lemma 5.1.11. *Let $\mathbb{P}$ be a c.c.c. forcing notion of size ω_1. There exists $p \in \mathbb{P}$ and a family of ω_1 many subsets of $\mathbb{P}$, predense below p, such that any filter $G \subseteq \mathbb{P}$ containing p and intersecting all of them is uncountable.*

Proof. Enumerate bijectively $\mathbb{P}$ as $\{p_\gamma | \ \gamma < \omega_1\}$. Let $D_\alpha = \{p_\gamma | \ \alpha < \gamma < \omega_1\}$. We aim to find $p \in \mathbb{P}$ such that uncountably many of sets D_α are predense below p – clearly this will finish the proof. If p with this property doesn't exist, then the following assertion holds:

$$\forall \gamma < \omega_1 \ \exists \gamma' > \gamma \ \text{"}D_{\gamma'} \text{ is not predense below } p_\gamma\text{"},$$

and consequently

$$\forall \gamma < \omega_1 \ \exists p'_\gamma \leq p_\gamma \ \exists \gamma' > \gamma \ \forall \eta > \gamma' \ p'_\gamma \perp p_\eta.$$

Using this, we can easily define an uncountable antichain $\{p_\gamma | \ \gamma \in E\}$, ensuring at each step of induction that

$$\forall \gamma \in E \ \exists \gamma' > \gamma \ \forall \eta > \gamma' \ p_\gamma \perp p_\eta.$$

□

Lemma 5.1.12. *MA_{ω_1}(appropriate) implies that any family of pairwise disjoint tuples $\{(x_1^\xi, \ldots, x_n^\xi) \mid \xi < \omega_1\} \subseteq (\mathcal{M}, d)^n$ contains an uncountable subfamily of pairwise alike tuples.*

Proof. Let $\mathbb{P} = \{F \in [\omega_1]^{<\omega} \mid \forall \xi \neq \eta \in F\ \overline{x}_\xi \circledast \overline{x}_\eta\}$, where $\overline{x}_\xi = (x_1^\xi, \ldots, x_n^\xi)$. The ordering is given by the reversed inclusion. We claim that $\mathbb{P}$ is appropriate. Fix an uncountable family $\{F_\alpha \mid \alpha < \omega_1\} \subseteq \mathbb{P}$, and a family of tuples $\{(v_1^\alpha, \ldots, v_s^\alpha) \mid \alpha < \omega_1\} \subseteq (\mathcal{M}, d)^s$. Without loss of generality we may assume that

$$F_\alpha = \{e_1, \ldots, e_k, e_{k+1}^\alpha, \ldots, e_{k+m}^\alpha\},$$

where sets $\{e_{k+1}^\alpha, \ldots, e_{k+m}^\alpha\}$ are pairwise disjoint for different α. For each α, let $\overline{y}_\alpha \in (\mathcal{M}, d)^{n \cdot m + s}$ be a concatenation of all tuples $\overline{x}_{e_{k+1}^\alpha}, \ldots, \overline{x}_{e_{k+m}^\alpha}$, and $(v_1^\alpha, \ldots, v_s^\alpha)$. There exists $\alpha \neq \beta < \omega_1$, such that $\overline{y}_\alpha \circledast \overline{y}_\beta$, and we claim that they witness the fact that $\mathbb{P}$ is appropriate. Let us write:

$$\overline{y}_\alpha = (x_1^1, \ldots, x_n^1, x_1^2, \ldots, x_n^2, \ldots, x_1^m, \ldots, x_n^m, v_1, \ldots, v_s),$$
$$\overline{y}_\beta = (y_1^1, \ldots, y_n^1, y_1^2, \ldots, y_n^2, \ldots, y_1^m, \ldots, y_n^m, u_1, \ldots, u_s).$$

What is clear, is that $(v_1, \ldots, v_s) \circledast (u_1, \ldots, u_s)$, and for all $k \leq m$, $(x_1^k, \ldots, x_n^k) \circledast (y_1^k, \ldots, y_n^k)$. What is not clear, is that in this case the $\circledast$ relation is "shift-invariant", i.e. for all $1 \leq p \neq r \leq m$ we have $(x_1^p, \ldots, x_n^p) \circledast (y_1^r, \ldots, y_n^r)$. We will check that this is the case.

A1 For all $1 \leq r \neq p \leq m$, and $1 \leq i, j \leq n$, we have

$d(x_i^p, y_i^r) = d(x_i^p, x_i^r)$ by A3 for $\overline{y}_\alpha$ and $\overline{y}_\beta$,
$d(x_i^p, x_i^r) = d(x_j^p, x_j^r)$ by A1 for $(x_1^p, \ldots, x_n^p)$ and $(x_1^r, \ldots, x_n^r)$,
$d(x_j^p, x_j^r) = d(x_j^p, y_j^r)$ by A3 for $\overline{y}_\alpha$ and $\overline{y}_\beta$.

A2 For all $1 \leq r \neq p \leq m$, and $1 \leq i, j \leq n$, we have

$d(x_i^p, x_j^p) = d(x_i^r, x_j^r)$ by A2 for $(x_1^p, \ldots, x_n^p)$, $(x_1^r, \ldots, x_n^r)$,
$d(x_i^r, x_j^r) = d(y_i^r, y_j^r)$ by A2 for $\overline{y}_\alpha$, $\overline{y}_\beta$.

A3 If $x_i^r \neq x_j^r$ then

$d(x_i^r, x_j^r) = d(x_i^r, x_j^p)$ by A3 for $(x_1^p, \ldots, x_n^p)$, $(x_1^r, \ldots, x_n^r)$,
$d(x_i^r, x_j^p) = d(x_i^r, y_j^p)$ by A3 for $\overline{y}_\alpha$, $\overline{y}_\beta$.

Therefore $F_\alpha \cup F_\beta \in \mathbb{P}$, and this concludes the proof that $\mathbb{P}$ is appropriate. Notice that all singletons belong to $\mathbb{P}$, so $|\mathbb{P}| = \omega_1$. Applying Martin's Axiom to the family of predense sets given by the Lemma 5.1.11, we find an uncountable filter $G \subseteq \mathbb{P}$. The set $\{\overline{x}_\alpha |\ \alpha \in \bigcup G\}$ is an uncountable family of tuples, and each two of them are alike.

□

Proposition 5.1.13. *MA_{ω_1}(appropriate) implies that any c.c.c. partial order of size ω_1 is appropriate.*

Proof. Suppose that $\mathbb{P}$ is a c.c.c. partial order of cardinality ω_1, and fix some disjoint family $\{(p_\xi, x_1^\xi, \ldots, x_n^\xi)|\ \xi < \omega_1\} \subseteq \mathbb{P} \times (\mathcal{M}, d)^n$. If MA_{ω_1}(appropriate) holds, we can assume that all tuples $(p_\xi, x_1^\xi, \ldots, x_n^\xi)$ are pairwise alike, and since $\mathbb{P}$ is c.c.c. we will find two p_ξ and p_η, which are comparable. □

The immediate consequence is

Theorem 5.1.14. *It is consistent with $ZFC + MA +$ "$2^\omega = \omega_2$" that $(\mathcal{M}, d)$ is rectangular.*

We stress that the notion of appropriate poset did rely on the specific model $(\mathcal{M}, d)$. But since we now have access to full Martin's Axiom, we don't have to think of $(\mathcal{M}, d)$ as any distinguished structure. It just witnesses the fact that Martin's Axiom is consistent with the existence of a metric space, which is rectangular and HSS. For this reason, further results apply to any space with these properties, not necessarily $(\mathcal{M}, d)$.

Theorem 5.1.15. *Let (X, d) be a rectangular rational metric space of size ω_1. MA_{ω_1} implies that any uncountable 1-1 function $f \subseteq X \times X$ is an isometry on an uncountable set.*

Proof. Let $f \subseteq X \times X$ be an uncountable 1-1 function. Consider the partial order

$$\mathbb{P}_f = (\{E \in [\text{dom } f]^{<\omega}|\ f \restriction E \text{ is an isometry}\}, \subseteq).$$

We will check that $\mathbb{P}_f$ is c.c.c. Take a sequence $\{e_\xi|\ \xi < \omega_1\} \subseteq \mathbb{P}_f$. Applying Δ-system Lemma we may assume that

- $\forall\, \xi < \omega_1\ |e_\xi| = m$,
- $\forall\, \xi \neq \eta < \omega_1\ e_\xi \cap e_\eta = r$,
- $e_\xi = (e_1^\xi, \ldots, e_l^\xi, e_{l+1}^\xi, \ldots, e_m^\xi)$, where $\{e_1^\xi, \ldots, e_l^\xi\} = r$.

Look at the family of tuples

$$\{(e^\xi_{l+1}, \ldots, e^\xi_m, f(e^\xi_{l+1}), \ldots, f(e^\xi_m))|\ \xi < \omega_1\}.$$

Using for example Δ-system Lemma one can easily trim this sequence so that all tuples are pairwise disjoint. Now, given that (X, d) is rectangular, we find $\xi \neq \eta < \omega_1$, such that

$$(e^\xi_{l+1}, \ldots, e^\xi_m, f(e^\xi_{l+1}), \ldots, f(e^\xi_m)) \circledast (e^\eta_{l+1}, \ldots, e^\eta_m, f(e^\eta_{l+1}), \ldots, f(e^\eta_m)).$$

We must check that e_ξ and e_η are comparable, that is $f \restriction (e_\xi \cup e_\eta)$ is an isometry. But notice that for $i \neq j = l+1, \ldots, m$,

$$d(e^\xi_i, e^\eta_j) = d(e^\xi_i, e^\xi_j) = d(f(e^\xi_i), f(e^\xi_j)) = d(f(e^\xi_i), f(e^\eta_j)).$$

This proves that $e_\xi \cup e_\eta \in \mathbb{P}_f$.

Notice that all singletons belong to $\mathbb{P}_f$, so $|\mathbb{P}| = \omega_1$. Applying Martin's Axiom to the family of predense sets given by the Lemma 5.1.11, we find an uncountable filter $G \subseteq \mathbb{P}_f$. The set $\bigcup\{E|\ E \in G\}$ is an uncountable set on which f is an isometry. □

Corollary 5.1.16. *It is consistent with $ZFC + MA +$ "$2^\omega = \omega_2$" that there exists an uncountable, separable (even hereditarily separably saturated) rational metric space (X, d) such that each uncountable function $f \subseteq X \times X$ is an isometry on an uncountable set.*

5.2 Rectangular Models

The reader might have noticed that in the previous section the triangle inequality for the space $(\mathcal{M}, d)$ was never applied. The crucial property we used was a variant of the SP, which ensures that "remainders" will be alike (see Lemma 5.1.2). In this section we will define a version of the SP which allows to proceed with the proof of Theorem 5.1.15 in case of other classes of structures. One assumption which seems hard to be removed is that the language consists of binary relational symbols only.

Each language consisting of finitely many binary relational symbols can be identified with a finite coloring of ordered pairs – given a model A, we assign to each element of A^2 its isomorphism type. There are only finitely many symbols in the language, so this coloring is indeed finite, and moreover, it determines the model A completely. Also, any function is a homomorphism precisely when it preserves this coloring. This observation allows to generalize results from Section 2 to other classes, besides metric spaces. In fact, the finiteness of language is not relevant as long as there are only countably many isomorphism types of 2-element models. Let $\mathcal{K}$ be some class of structures in a countable language $\{\mathcal{R}_i\}_{i<\omega}$ consisting of binary relational symbols. Assume also that $\mathcal{K}$ has only countably many isomorphism types of finite models. Let c be the corresponding coloring of pairs in models from $\mathcal{K}$ – by the remark above we may forget about the relational symbols, and think of models from $\mathcal{K}$ as having only one "relation", namely c. We introduce the $\circledast$ relation by axioms similar to the metric case, but unlike metrics, the coloring c might not be symmetric. This is the only significant difference.

Definition 5.2.1. Let $X \in \mathcal{K}$, and $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X^n$ be disjoint. We will say that they are *alike*, and write $(x_1, \ldots, x_n) \circledast (y_1, \ldots, y_n)$, if the following axioms are satisfied

A1a $\forall\, i, j = 1, \ldots, n\ c(x_i, y_i) = c(x_j, y_j)$

A1b $\forall\, i, j = 1, \ldots, n\ c(y_i, x_i) = c(y_j, x_j)$

A2a $\forall\, i, j = 1, \ldots, n\ c(x_i, x_j) = c(y_i, y_j)$

A3a $\forall\, i, j = 1, \ldots, n\ (x_i \neq x_j \implies c(x_i, x_j) = c(x_i, y_j) = c(y_i, x_j))$

If all relations $\mathcal{R}_k$ are anti-reflexive ($\forall x\ \neg\mathcal{R}_k(x, x)$), then we can omit the clause $(x_i \neq x_j) \implies$ in A3a. It is standard to check that

$$(x_1, \ldots, x_n) \circledast (y_1, \ldots, y_n) \iff (y_1, \ldots, y_n) \circledast (x_1, \ldots, x_n).$$

Definition 5.2.2. $X \in \mathcal{K}$ is *rectangular* if $|X| > \omega$, and for any family of pairwise disjoint tuples $\{(x_1^\xi, \ldots, x_n^\xi) |\ \xi < \omega_1\} \subseteq X^n$, there exist $\xi \neq \eta < \omega_1$, such that $(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta)$.

Definition 5.2.3. $\mathcal{K}$ has the *Rectangular Splitting Property* (RSP) if for all $R, X, Y \in \mathcal{K}$, for all pairs of isomorphic extensions $R \subseteq X$, $R \subseteq Y$, with the corresponding isomorphism $h : X \to Y$, there exists $Z \in \mathcal{K}$, with the universe $X \cup Y$, such that for any sequence $(x_1, \ldots, x_n)$ enumerating bijectively $X \setminus R$, Z satisfies $(x_1, \ldots, x_n) \circledast (h(x_1), \ldots, h(x_n))$.

The RSP ensures the conclusion of Lemma 5.1.2. Moreover, since the definition of $\circledast$ relation is independent of the ordering of the tuples, if there exists an enumeration $(x_1, \ldots, x_n)$ like above, it can be replaced by any other enumeration, even not 1-1.

Theorem 5.2.4. *If $\mathcal{K}$ has the RSP then* $\mathrm{Fn}(\omega_1, \mathcal{K}, \omega)$ *forces the generic structure to be rectangular.*

Proof. Exactly like the proof of Proposition 5.1.3. □

Let us fix a rectangular model $(\mathcal{X}, c)$. The appropriate forcings are defined in the same way as in Definition 5.1.7:

Definition 5.2.5. A partial order $\mathbb{P}$ is appropriate if given any natural number $n > 0$, for each family consisting of pairwise disjoint tuples

$$\{(p_\xi, x_1^\xi, \dots, x_n^\xi) |\ \xi < \omega_1\} \subseteq \mathbb{P} \times (\mathcal{X}, c)^n,$$

there exist $\xi \neq \eta < \omega_1$, such that p_ξ and p_η are comparable, and

$$(x_1^\xi, \dots, x_n^\xi) \circledast (x_1^\eta, \dots, x_n^\eta).$$

Proposition 5.2.6. *If $\mathbb{P}$ is appropriate, then $\mathbb{P} \Vdash$ "$(\mathcal{X}, c)$ is rectangular".*

Proof. Exactly like the proof of Proposition 5.1.8. □

Proposition 5.2.7. $MA_{\omega_1}(appropriate)$ *implies that any family of pairwise disjoint tuples*$\{(x_1^\xi, \dots, x_n^\xi) |\ \xi < \omega_1\} \subseteq (\mathcal{X}, c)^n$ *contains an uncountable subfamily of pairwise alike tuples.*

Proof. Same as Lemma 5.1.12. □

What follows, is a generalization of Theorem 5.1.14.

Theorem 5.2.8. *It is consistent with $ZFC + MA +$ "$2^\omega = \omega_2$" that $(\mathcal{X}, c)$ is rectangular.*

The proof that appropriateness is preserved under finite support iterations is also the same. $(\mathcal{X}, c)$ was a specific model providing us with the notion of appropriate partial order, but once Theorem 5.2.8 is proved, we may really forget about it. It just witnesses the fact, that *some* rectangular model can exist in the presence of Martin's Axiom.

Proposition 5.2.9. *Let $(X, c) \in \mathcal{K}$ be rectangular. MA_{ω_1} implies that any uncountable 1-1 function $f \subseteq X \times X$ is a homomorphism on an uncountable set.*

Proof. Almost exactly like the proof of Theorem 5.1.15 – the only difference is that we write c instead of d, *isomorphism* instead of *isometry*, and in the end use A1a and A1b instead of A1, since c might not be symmetric. □

Proposition 5.2.10. *The classes of graphs, directed graphs, tournaments, linear orders, and partial orders have the RSP.*

Proof. We will prove the RSP for linear and partial orders. Arguments for other classes are easy and left to the reader. For each partial order $\leq$ there exists a corresponding quasi-ordering relation $<$, which is anti-reflexive.

Suppose we have a diagram of linear quasi-orders R, X, Y, and $h : X \to Y$, like in the definition of RSP. We define a quasi-ordering on $Z = X \cup Y$, extending both $<_X$ and $<_Y$, by conditions:

- $\forall i < \omega\ (x_i < h(x_i))$,
- $\forall i \neq j < \omega\ (x_i < h(x_j) \iff x_i <_X x_j)$.

Clearly $<$ is an anti-reflexive relation on Z. For checking transitivity we must go through several (somewhat boring) cases.

1. $x_i < h(x_j),\ h(x_j) < x_k$. Either $x_i = x_j$ or $x_i < x_j$. In the first case $h(x_i) < x_k$, and so $x_i < x_k$. In the second, $x_i < x_k$, and $x_i < x_k$ follows from transitivity of $<$ on X.
2. $h(x_i) < x_j,\ x_j < x_k$. In this case $x_i < x_j$, so $x_i < x_k$, and $h(x_i) < x_k$.
3. $x_i < x_j,\ x_j < h(x_k)$. Either $x_j = x_k$ or $x_j < x_k$. In both cases $x_i < x_k$, so $x_i < h(x_k)$.
4. $h(x_i) < h(x_j),\ h(x_j) < x_k$. In this case $x_i < x_j$, and $x_j < x_k$. By transitivity $x_i < x_k$, and $h(x_i) < x_k$ follows.
5. $x_i < h(x_j),\ h(x_j) < h(x_k)$. If $x_i < x_j$, then we proceed like before. If $x_i = x_j$, $h(x_i) < h(x_k)$. It follows that $x_i < x_k$, and $x_i < h(x_k)$.
6. $h(x_i) < x_j,\ x_j < h(x_k)$. We see that $x_i < x_j$. If $x_j < x_k$, we use transitivity of $<$ on X. If $x_j = x_k$, then $h(x_i) < x_k$, and so $x_i < x_k$. It follows that $h(x_i) < h(x_k)$.

The proof for partial orders is strictly simpler – we define the quasi-ordering by conditions:

- $\forall i < \omega \quad x_i$ is incomparable with $h(x_i)$,
- $\forall i, j < \omega \quad x_i < h(x_j) \iff x_i < x_j$,
- $\forall i, j < \omega \quad \neg h(x_i) < x_j$.

Verification of transitivity is a run through the same cases, except this time we do not have to care if x_i, x_j, x_k are distinct. □

For each class $\mathcal{K}$ having the RSP, one can prove a variant of Corollary 5.1.16. We could of course state a general theorem, after introducing a notion of "separable model" for arbitrary binary relational class. We will refrain from doing so, and provide just two such variants as an illustration. Interested reader will easily formulate corresponding results for tournaments, directed graphs, etc.

Theorem 5.2.11. *Each of the following is consistent with* $ZFC + MA +$ "$2^\omega = \omega_2$"*:*

1. *There exists a graph* G *of size* ω_1*, with a countable subset* $D \subseteq G$*, satisfying the following properties:*
   - *For all pairs of disjoint finite subsets* $A, B \subseteq G$*, there exists a vertex* $d \in D$*, connected with each point in* A*, and with no point in* B*.*
   - *Each uncountable 1-1 function* $f \subseteq G \times G$ *is a graph homomorphism on an uncountable set.*
2. *(Avraham-Shelah, [6]) There exists a separable,* ω_1*-dense linear order* $(L, \leq)$*, such that each uncountable 1-1 function* $f \subseteq L \times L$ *is order preserving on an uncountable subset.*

The first point of this theorem is also a consequence of Theorem 5.1.15 – we can turn a metric space into a graph by connecting two vertices iff the distance between them is ≥ 1.

5.3 Classification Results

If the uncountable models we are considering are to resemble Fraïssé limits, we should be able to do some kind of "back-and-forth" arguments, like in the classical theory. There is no way inductive arguments can work, but thanks to Theorem 5.2.8 we can rely on Martin's Axiom instead of induction.

5.3.1 HSS Rectangular Metric Spaces

We will be looking at metric spaces with distances in a given countable set $K \subseteq [0, \infty)$.

Theorem 5.3.1. *Assume* MA_{ω_1}*. Let* (X, d) *be any rectangular HSS metric space of size* ω_1*, with distances in* K*. Let* $Y \subseteq X$ *be any HSS uncountable subspace. Then* X *and* Y *are isometric.*

Proof. Since X is hereditarily separably saturated, we can decompose it into a disjoint union of countable saturated subsets $\{X_\alpha\}_{\alpha<\omega_1}$. Of course we can do the same with Y, so let us write $Y = \bigcup_{\alpha<\omega_1} Y_\alpha$. Let $\mathbb{P}$ consist of finite partial isometries between X and Y, which map elements from X_α to Y_α for all $\alpha < \omega_1$. It is standard to check that the following sets are dense for $x \in X$, $y \in Y$:

$$D_x = \{p \in \mathbb{P} |\ x \in \operatorname{dom} p\},$$

$$E_y = \{p \in \mathbb{P} |\ y \in \operatorname{rg} p\}.$$

We are left with the task of verifying the c.c.c. property. Fix any uncountable subset $\{p_\gamma |\ \gamma < \omega_1\} \subseteq \mathbb{P}$. Using Δ-system Lemma we can write

$$\operatorname{dom} p_\gamma = (x_1, \ldots, x_k, x^\gamma_{k+1}, \ldots, x^\gamma_m),$$

$$\operatorname{rg} p_\gamma = (y_1, \ldots, y_k, y^\gamma_{k+1}, \ldots, y^\gamma_m),$$

where tuples $(x^\gamma_{k+1}, \ldots, x^\gamma_m)$ are pairwise disjoint, and moreover $p_\gamma(x_i) = y_i$, and $p_\gamma(x^\gamma_i) = y^\gamma_i$ for each $\gamma < \omega_1$. Recall that X is rectangular, so we can find $\xi \neq \eta < \omega_1$, such that

$$(x^\xi_{k+1}, \ldots, x^\xi_m, y^\xi_{k+1}, \ldots, y^\xi_m) \circledast (x^\eta_{k+1}, \ldots, x^\eta_m, y^\eta_{k+1}, \ldots, y^\eta_m).$$

If $k < i \neq j \leq m$, then

$$\begin{aligned} d(x^\eta_i, x^\xi_j) =& d(x^\eta_i, x^\eta_j) = \\ d(y^\eta_i, y^\eta_j) =& d(y^\eta_i, y^\xi_j) \end{aligned}$$

Also for $k < i \leq m$

$$d(x^\eta_i, x^\xi_i) = d(y^\eta_i, y^\xi_i)$$

Clearly $p_\xi \cup p_\eta \in \mathbb{P}$. □

Corollary 5.3.2. *Assume MA_{ω_1}. Let (X, d) be any rectangular HSS metric space of size ω_1, with distances in K. X can be decomposed into a disjoint union of λ many its isometric copies for any $\lambda \in \{2, \ldots, \omega_1\}$.*

Proof. Let $X = \bigcup_{\alpha<\omega_1} X_\alpha$ be a decomposition of X into countable saturated subsets. Let $\omega_1 = \bigcup_{\alpha<\lambda} A_\alpha$ be a decomposition of ω_1 into pairwise disjoint uncountable subsets. For any $\gamma < \lambda$ the space

$$X_\gamma = \bigcup_{\alpha \in A_\gamma} X_\alpha$$

is isometric to (X, d). □

After taking $X = Y$ and obvious adjustments to the forcing used, we obtain the classical homogeneity.

Corollary 5.3.3. *Assume MA_{ω_1}. If (X, d) is any rectangular HSS metric space of size ω_1, with distances in K, then any finite partial isometry of (X, d) extends to a full isometry.*

Still, this kind of homogeneity is not sufficient to prove uniqueness, like for the rational Urysohn space. Recall, that by Proposition 5.1.6

$$\mathrm{Fn}(\omega_1, Metr, \omega) \Vdash \text{"}(\omega_1, \dot{d}) \text{ is HSS (with distances in } \mathbb{Q})\text{"}.$$

Theorem 5.3.4. *Assume MA_{ω_1}. If there exists a rectangular HSS rational metric space of size ω_1, then there exist infinitely many pairwise non-isometric such spaces.*

Proof. If (X, d) is a rectangular, hereditarily separably saturated rational metric space of size ω_1, we introduce a family of metrics on X

$$d_k(x, y) = k \cdot d(x, y),$$

for positive integers k. If $k < l$ are positive integers, the spaces (X, d_k) and (X, d_l) are not isometric. Indeed, if $f : X \hookrightarrow X$ is a bijection, then by Theorem 5.1.15, f is an isometry of the space (X, d) on some pair of distinct points $x, y \in X$. Then $d_k(x, y) = k \cdot d(x, y) = k \cdot d(f(x), f(y)) < l \cdot d(f(x), f(y)) = d_l(f(x), f(y))$. □

5.3.2 HSS Rectangular Graphs

If we take $K = \{0, 1, 2\}$, metric spaces with distances in K are graphs – think of two points in distance 1 as connected, and two points in distance 2 as not connected. Notions of saturated set and separably saturated space translate to the following.

Definition 5.3.5. Let G be a graph. A subset $D \subseteq G$ is a *saturated subset* of G if for any pair of disjoint finite subsets $A, B \subseteq G$, there exists $d \in D \setminus (A \cup B)$ which is connected with each vertex in A and with no vertex in B.

Definition 5.3.6. If G is any graph, then

1. G is *separably saturated* if it has a countable saturated subset.

2. G is *hereditarily separably saturated* (HSS) if for any countable subset $E \subseteq G$, the graph $G \setminus E$ is separably saturated.

Proposition 5.3.7. $\mathrm{Fn}(\omega_1, Graphs, \omega) \Vdash$ *"$(\omega_1, \dot{E})$ is a rectangular HSS graph".*

Proof. Each infinite subset of ω_1, which belongs to the ground model, is saturated, what is clearly sufficient for being hereditarily separably saturated. Rectangularity is a consequence of Theorem 5.2.4. □

From Theorem 5.3.1, with $K = \{0, 1, 2\}$, it follows that HSS rectangular graphs are in certain sense minimal.

Theorem 5.3.8. *Assume MA_{ω_1}. If G is a HSS rectangular graph of size ω_1, then each uncountable HSS subgraph of G is isomorphic with G.*

Theorem 5.3.9. *Assume MA_{ω_1}. If G and H are two HSS rectangular graphs of size ω_1, then $G \simeq H$ if and only if G and H^c do not contain a common uncountable subgraph (H^c denotes the complement of H).*

Proof.

"$\Rightarrow$." We must show that G and G^c do not contain a common uncountable subgraph. Fix arbitrary graph F of size ω_1, and suppose towards contradiction that there exists a pair of embeddings $i : F \hookrightarrow G$, $j : F \hookrightarrow G^c$. The partial function given by $i(\alpha) \mapsto j(\alpha)$ is a bijection between uncountable subsets of G. By Proposition 5.2.9 it is a homomorphism on some pair of points $\alpha \neq \beta$. This contradicts the choice of i and j.

"$\Leftarrow$." Assume that there is no uncountable graph which embeds both into G and H^c. We proceed like in the proof of Theorem 5.3.1. G and H are HSS, so we can decompose them into disjoint unions of countable saturated subsets $\{G_\alpha\}_{\alpha<\omega_1}$, and $\{H_\alpha\}_{\alpha<\omega_1}$ respectively. Let $\mathbb{P}$ consist of finite partial isomorphisms between G and H, which map elements from G_α to H_α for all $\alpha < \omega_1$. The following sets are dense for $g \in G$, $h \in H$:

$$D_g = \{p \in \mathbb{P} |\ g \in \operatorname{dom} p\},$$

$$E_h = \{p \in \mathbb{P} |\ h \in \operatorname{rg} p\}.$$

We verify the c.c.c. property. Fix any uncountable subset $\{p_\gamma |\ \gamma < \omega_1\} \subseteq \mathbb{P}$. Using Δ-system Lemma we can write

$$\operatorname{dom} p_\gamma = (x_1, \dots, x_k, x^\gamma_{k+1}, \dots, x^\gamma_m),$$

$$\operatorname{rg} p_\gamma = (y_1, \dots, y_k, y^\gamma_{k+1}, \dots, y^\gamma_m),$$

where tuples $(x^\gamma_{k+1}, \dots, x^\gamma_m)$ are pairwise disjoint, and moreover $p_\gamma(x_i) = y_i$, and $p_\gamma(x^\gamma_i) = y^\gamma_i$ for each $\gamma < \omega_1$. By rectangularity and Proposition 5.2.7 we can assume that tuples

$$\{(x^\gamma_{k+1}, \dots, x^\gamma_m) |\ \gamma < \omega_1\},$$

as well as

$$\{(y^\gamma_{k+1}, \dots, y^\gamma_m) |\ \gamma < \omega_1\}$$

are pairwise alike.

For all $k < i \neq j \leq m$, and $\xi \neq \eta < \omega_1$ we have

$$x_i^\eta \; E \; x_j^\xi \iff x_i^\eta \; E \; x_j^\eta \iff$$
$$y_i^\eta \; E \; y_j^\eta \iff y_i^\eta \; E \; y_j^\xi$$

What remains is the case $i = j$. But look at the function given by $x_{k+1}^\eta \mapsto y_{k+1}^\eta$, for $\eta < \omega_1$. This is a bijection between an uncountable subgraph of G and an uncountable subgraph of H, or equivalently, H^c. By our assumption, it cannot be a homomorphism into H^c. We conclude that there are $\eta \neq \xi < \omega_1$ such that

$$x_{k+1}^\eta \; E \; x_{k+1}^\xi \iff y_{k+1}^\eta \; E \; y_{k+1}^\xi,$$

and by rectangularity

$$x_i^\eta \; E \; x_i^\xi \iff y_i^\eta \; E \; y_i^\xi,$$

for all $i = k+1, \ldots, m$. Clearly $p_\xi \cup p_\eta \in \mathbb{P}$. □

Corollary 5.3.10. *Assume MA_{ω_1}. If there exists a HSS rectangular graph of size ω_1 then it is unique up to taking the complement.*

Proof. Fix two HSS rectangular graphs G and H, both of size ω_1. It is sufficient to prove that either no uncountable graph F can be embedded both into G and H, or no uncountable graph F can be embedded both into G and H^c. Suppose towards contradiction that $F_0, F_1 \subseteq G$ are uncountable subgraphs, and there exist embeddings $i_0 : F_0 \hookrightarrow H$, $i_1 : F_1 \hookrightarrow H^c$. The function given by $i_0(f) \mapsto i_1(f)$ is a bijection between uncountable subsets of H. By Proposition 5.2.9, on some two points it must be a homomorphism. But this contradicts the choice of i_0 and i_1. □

Theorem 5.3.11. *Assume MA_{ω_1}. If G is a rectangular graph of size ω_1, then either G contains an uncountable clique or an uncountable anticlique.*

Proof. In the light of Proposition 5.2.9, G clearly can't contain both. Assume that G doesn't contain an uncountable anticlique. We can represent G as (ω_1, E), and consider the partial order

$$\mathbb{P} = \{F \subseteq \omega_1 | \; F \text{ is a finite clique in } G\},$$

ordered by reversed inclusion. If we can show that $\mathbb{P}$ is c.c.c, Lemma 5.1.11 will provide us with an uncountable clique in G. Suppose that $\{F_\xi | \; \xi < \omega_1\}$ is an uncountable subset of $\mathbb{P}$. We can write

$$F_\xi = (f_1, \ldots, f_k, f_{k+1}^\xi, \ldots, f_m^\xi),$$

where tuples $(f^{\xi}_{k+1}, \dots, f^{\xi}_{m})$ are pairwise disjoint. By the virtue of Martin's Axiom, and Proposition 5.2.7, we can also assume that they are pairwise alike. By our assumption the set $\{f^{\xi}_{k+1} | \ \xi < \omega_1\}$ is not an anticlique, so we will find $\xi \neq \eta < \omega_1$, such that $f^{\eta}_{k+1} \ E \ f^{\xi}_{k+1}$. It is now standard to check that $F_\eta \cup F_\xi \in \mathbb{P}$. □

5.3.3 Separable ω_1-dense Linear Orders

What do rectangular linear orders look like? After unwinding the axioms for the $\circledast$ relation, we see that $(x_1, \dots, x_n) \circledast (y_1, \dots, y_n)$ translates to the following three axioms:

A1c $\forall \, i, j = 1, \dots, n \ (x_i < y_i \iff x_j < y_j)$

A2c $\forall \, i, j = 1, \dots, n \ (x_i < x_j \iff y_i < y_j)$

A3c $\forall \, i, j = 1, \dots, n \ ((x_i \neq x_j) \implies (x_i < x_j \iff x_i < y_j \iff y_i < x_j))$

If we omitted A3c, we would obtain what the authors of [5] call an *increasing order* (actually, one can show that in the class of separable, dense linear orders these two notions coincide). An order added by $\mathrm{Fn}(\omega_1, \mathcal{LO}, \omega)$ is a rectangular, separable, ω_1-dense linear order, which under MA_{ω_1} is also homogeneous. MA_{ω_1} imposes a great deal of regularity on the class of separable, homogeneous ω_1-dense linear orders – for example two such orderings are isomorphic precisely when they are bi-embeddable. Implications of MA_{ω_1} for this class, as well as other axioms like OCA_{ARS}, have been extensively studied in [5]. Not surprisingly, many properties of rectangular linear orders resemble those of graphs.

Theorem 5.3.12. *Assume MA_{ω_1}, and suppose that L is a separable, ω_1-dense, rectangular linear order. If K is any other separable, ω_1-dense, rectangular linear order, then $K \simeq L$ if and only if K and L^* do not contain a common uncountable suborder (L^* denotes L with the reversed ordering).*

Proof.

"⇒." We must show that there is no uncountable strictly decreasing function $f \subseteq L \times L$. But this follows directly from Proposition 5.2.9.

"⇐." Assume that there is no uncountable linear order which embeds both into L and L^*. We proceed like in the proofs of Theorems 5.3.1 and 5.3.9. We can decompose L and K into disjoint unions of countable dense subsets $\{L_\alpha\}_{\alpha<\omega_1}$, and $\{K_\alpha\}_{\alpha<\omega_1}$ respectively. Let $\mathbb{P}$ consist of finite partial isomorphisms between

L and K which map elements from K_α to L_α, for all $\alpha < \omega_1$. The following sets are dense for $l \in L$, $k \in K$:

$$D_k = \{p \in \mathbb{P} |\ k \in \operatorname{dom} p\},$$
$$E_l = \{p \in \mathbb{P} |\ l \in \operatorname{rg} p\}.$$

We verify the c.c.c. property. Fix any uncountable subset $\{p_\gamma |\ \gamma < \omega_1\} \subseteq \mathbb{P}$. Using Δ-system Lemma, we can write

$$\operatorname{dom} p_\gamma = (x_1, \ldots, x_k, x^\gamma_{k+1}, \ldots, x^\gamma_m),$$
$$\operatorname{rg} p_\gamma = (y_1, \ldots, y_k, y^\gamma_{k+1}, \ldots, y^\gamma_m),$$

where tuples $(x^\gamma_{k+1}, \ldots, x^\gamma_m)$ are pairwise disjoint, and moreover $p_\gamma(x_i) = y_i$, and $p_\gamma(x^\gamma_i) = y^\gamma_i$, for each $\gamma < \omega_1$. By rectangularity and Proposition 5.2.7 we can assume, that tuples

$$\{(x^\gamma_{k+1}, \ldots, x^\gamma_m) |\ \gamma < \omega_1\},$$

as well as

$$\{(y^\gamma_{k+1}, \ldots, y^\gamma_m) |\ \gamma < \omega_1\},$$

are pairwise alike.

For all $k < i \neq j \leq m$, and $\xi \neq \eta < \omega_1$ we have

$$x^\eta_i < x^\xi_j \iff x^\eta_i < x^\eta_j \iff$$
$$y^\eta_i < y^\eta_j \iff y^\eta_i < y^\xi_j$$

What remains is the case $i = j$. Look at the function given by $x^\eta_{k+1} \mapsto y^\eta_{k+1}$, for $\eta < \omega_1$. This is a bijection between an uncountable subset of L and an uncountable subset of K. By our assumption it cannot be decreasing, so there are $\eta \neq \xi < \omega_1$ such that

$$x^\eta_{k+1} < x^\xi_{k+1} \iff y^\eta_{k+1} < y^\xi_{k+1},$$

and by rectangularity

$$x^\eta_i < x^\xi_i \iff y^\eta_i < y^\xi_i,$$

for all $i = k+1, \ldots, m$. Clearly $p_\xi \cup p_\eta \in \mathbb{P}$. □

Just like in the case of graphs, one can easily prove

Corollary 5.3.13. *Assume MA_{ω_1}. If there exists a rectangular, separable, ω_1-dense linear order then it is unique up to reversing the order.*

These orders are also mimimal in the same sense as metric spaces from Theorem 5.3.1. The reader will have no difficulty with adjusting its proof to obtain

Theorem 5.3.14. *Assume MA_{ω_1} and let K, L be a pair of separable, ω_1-dense linear orders. If K embeds into L, and L is rectangular, then L and K are isomorphic.*

5.4 Concluding Remarks

Is the theory we develop entitled to be called "the uncountable Fraïssé theory with finite supports"? We left it to the reader's opinion. Obviously, similarity of forcing notions

$$\mathrm{Fn}\,(\omega, \mathcal{K}, \omega)$$

and

$$\mathrm{Fn}\,(\omega_1, \mathcal{K}, \omega)$$

shows that some "genericity" is common to Fraïssé limits and rectangular models. On the other hand, properties of models added by the latter forcing seem heavily dependent on the set-theoretic background, unlike those of countable homogeneous models. Moreover, it looks like Proposition 5.2.9 exhibits some strange asymmetry of rectangular models in the presence of MA_{ω_1}, which distinguishes them from their countable counterparts.

The most attractive direction of further research is perhaps to represent a separable, ω_1-dense linear order as a Fraïssé-style structure, in case it is unique. By Theorem 5.3.12, in such situation no increasing (rectangular) set of reals can exist.

It is not hard to see, that many of our results can be restated in much broader generality. For instance:

- In all models of ZFC built in Chapter 5 one can have $2^\omega > \omega_2$. This is because MA_{ω_1} ensures that each c.c.c. forcing satisfies the Knaster condition, and so is appropriate.
- We introduced hereditarily separably saturated metric spaces, but one could of course define hereditarily separably saturated models for arbitrary first order theory, given it is phrased in a language consisting only of binary relational symbols.
- There is really nothing particular about the class of linear orders in Theorem 4.2.3, and ω_2 clearly can be replaced by any bigger cardinal.

However, our goal was to present some ideas and look for possible generalizations of known theories, rather than providing an exhaustive exposition in the most general manner. There are many natural questions about rectangular models, which weren't considered in the dissertation.

Question 5.4.1. Is there any reasonable analog of rectangularity for models added by

$$\mathrm{Fn}\,(\omega_2, \mathcal{K}, \omega_1)?$$

Question 5.4.2. Which small (for instance countable) groups can appear as the automorphism groups of ω_1-dense, separable linear orders?

In particular, is it consistent that

$$\mathrm{Aut}\,(A, \leq) \simeq (\mathbb{Q}, +)$$

for $A \subseteq \mathbb{R}$?

Question 5.4.3. Are there any natural conditions for a class $\mathcal{K}$, ensuring that

$$\mathrm{Fn}\,(S, \mathcal{K}, \omega)$$

is always equivalent to an iteration of the Cohen forcings?

Question 5.4.4. Is there anything interesting to be said about forcings

$$\mathrm{Fn}\,(\kappa, \mathcal{K}, \lambda)$$

if κ and λ are very big rather than very small, say λ is weakly compact?

Question 5.4.5. What can be said about the automorphism groups of rectangular models?

The last one seems quite ambitious.

Question 5.4.6. Can we have an analog of the Baumgartner's Theorem, from the beginning of Chapter 5, for any structures different from linear orders?

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