Sharp weighted martingale inequalities

Extended Abstract of the PhD Thesis

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1 Introduction and statement of the problem

The thesis studies several weighted extensions of the celebrated inequalities for martingale transforms established by Burkholder more than fifty years ago. The topic has its roots in the following properties of the classical Haar system $(h_n)_{n\geqslant 0}$. Recall that this collection consists of functions on $[0,1)$ given by

$$\begin{aligned}
&h_0 = [0,1), && h_1 = [0,1/2) - [1/2,1),\\
&h_2 = [0,1/4) - [1/4,1/2), && h_3 = [1/2,3/4) - [3/4,1),\\
&h_4 = [0,1/8) - [1/8,1/4), && h_5 = [1/4,3/8) - [3/8,1/2),\\
&h_6 = [1/2,5/8) - [5/8,3/4), && h_7 = [3/4,7/8) - [7/8,1)
\end{aligned}$$

and so on (here we have identified a set with its indicator function). A classical inequality due to Marcinkiewicz [19], based on the earlier work of Paley [25], can be stated as follows. If $1 < p < \infty$, then there is a finite constant c_p depending only on p such that for all $n \geqslant 0$ and any numbers $a_0, a_1, \ldots, a_n \in \mathbb{R}$, $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}$ we have

$$\left\| \sum_{k=0}^{n} \varepsilon_k a_k h_k \right\|_{L^p(0,1)} \leqslant c_p \left\| \sum_{k=0}^{n} a_k h_k \right\|_{L^p(0,1)}. \tag{1.1}$$

This remarkable property is called unconditionality and plays a significant role in harmonic analysis, approximation theory and geometry of Banach spaces: see e.g. the monographs [15, 26, 28] and consult the references therein.

In the sixties Burkholder showed that the inequality (1.1) can be reformulated and generalized to the probabilistic setting. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space endowed with a discrete-time filtration, that is, a non-decreasing sequence $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ of sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geqslant 0}$, $g = (g_n)_{n\geqslant 0}$ be adapted real-valued martingales, i.e., sequences of integrable random variables satisfying $\mathbb{E}(f_{n+1}|\mathcal{F}_n) = f_n$ and $\mathbb{E}(g_{n+1}|\mathcal{F}_n) = g_n$ for each $n \geqslant 0$. Define the associated difference sequences $df = (df_n)_{n\geqslant 0}$, $dg = (dg_n)_{n\geqslant 0}$ by

$df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n = 1, 2, \ldots$, and analogously for dg. Equivalently, we have the identities

$$f_n = \sum_{k=0}^{n} df_k, \qquad g_n = \sum_{k=0}^{n} dg_k, \qquad n = 0,\ 1,\ 2,\ \ldots.$$

We say that g is a martingale transform of f, if there exists a predictable sequence $v = (v_n)_{n \geqslant 0}$ such that $dg_n = v_n df_n$ for each $n \geqslant 0$. Here by predictability of v we mean that for each $n \geqslant 0$, the random variable v_n is measurable with respect to $\mathcal{F}_{(n-1)\vee 0}$. Moreover, if the values of v are in the interval $[-1, 1]$, then we will say that g is a ± 1-transform of f. There is a very natural general problem: to compare magnitudes of a martingale f and its ± 1-transform g, where the size is measured in terms of norms in various function spaces (e.g., L^p-norms, weak-L^p norms, etc.). The aforementioned result of Burkholder [10] is the following.

Theorem 1.1. *For any $1 < p < \infty$, there is a constant C_p depending only on p such that if f is a martingale and g is its ± 1-transform, then*

$$||g_n||_{L^p} \leqslant C_p ||f_n||_{L^p}, \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$

This does extend (1.1). Indeed, if the probability space is the interval $[0, 1)$ with its Borel subsets and Lebesgue's measure, then the Haar system $(h_n)_{n \geqslant 0}$, and hence also any collection of the form $(a_n h_n)_{n \geqslant 0}$, are martingale differences sequences (relative to the natural filtration). It turns out that the above theorem has many profound applications in harmonic analysis, for example it yields inequalities for the Hilbert transform, Riesz transforms and a wider class of singular operators and Fourier multipliers. Because of these applications, it is of interest and importance to identify the optimal (i.e., the least possible) value of the constant C_p. The first „optimal" result of this type was obtained by Burkholder, who in the late seventies established weak-type $(1, 1)$ estimate with the best constant 2:

$$||g_n||_{L^{1,\infty}} \leqslant 2||f_n||_{L^1}, \qquad n = 0, 1, 2, \ldots,$$

using the „good-lambda" technique (where $||\xi||_{L^{p,\infty}} = \sup(\lambda^p \mathbb{P}(|\xi| > \lambda))^{\frac{1}{p}}$ denotes a weak-L^p norm of ξ). In the eighties Burkholder developed a unified approach to estimates of such type and applied it to obtain the following statement.

Theorem 1.2. *Let $1 < p < \infty$ and $p^* = \max\left\{p, \frac{p}{p-1}\right\}$. For every martingale f and its ± 1-transform g we have*

$$||g_n||_{L^p} \leqslant (p^* - 1)||f_n||_{L^p}, \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

The constant $p^* - 1$ is optimal, even in the special case (1.1) *(in other words, the unconditional L^p-constant of the Haar system is equal to $p^* - 1$).*

In the same paper Burkholder established a weak-type (p, p) estimate for $1 \leqslant p \leqslant 2$. A case for $p > 2$ was solved by Suh [27].

Theorem 1.3. *Fix* $1 \leqslant p < \infty$. *For every martingale* f *and its* ± 1*-transforma* g *we have the following*

$$||g_n||_{L^{p,\infty}} \leqslant C_p ||f_n||_{L^p}, \qquad n = 0, 1, 2, \ldots, \tag{1.4}$$

where

$$C_p = \begin{cases} \left(\frac{2}{\Gamma(p+1)}\right)^{1/p} & \textit{jeśli } 1 \leqslant p \leqslant 2, \\ \left(\frac{p^{p-1}}{2}\right)^{1/p} & \textit{jeśli } 2 \leqslant p < \infty. \end{cases}$$

The above constant is optimal.

We turn our attention to the version of (1.4) for $p = \infty$. Let ξ be a random variable on a non-atomic probability space. We define ξ^*, the decreasing rearrangement of ξ, by

$$\xi^*(t) = \inf\{\lambda \geqslant 0 : \mathbb{P}(|\xi| > \lambda) \leqslant t\}.$$

Then $\xi^{**} : (0, 1] \to [0, \infty)$, the maximal function of ξ, is given by the formula

$$\xi^{**}(t) = \frac{1}{t} \int_0^t \xi^*(s) \mathrm{d}s, \qquad t \in (0, 1],$$

or equivalently,

$$\xi^{**}(t) = \frac{1}{t} \sup \left\{ \int_E |\xi| \mathrm{d}\mathbb{P} : E \in \mathcal{F}, \mathbb{P}(E) = t \right\}.$$

These objects enable the introduction of the weak-L^∞ space. Following Bennett, DeVore and Sharpley [5], we set

$$||\xi||_{weak(\mathbb{P})} = \sup_{t \in (0,1]} (\xi^{**}(t) - \xi^*(t))$$

and define $weak(\mathbb{P}) = \{\xi : ||\xi||_{weak(\mathbb{P})} < \infty\}$. Coming back to martingale context, we have the following statement proved by Osękowski in [23].

Theorem 1.4. *For every martingale* f *and its* ± 1*-transform* g *we have the inequality*

$$||g_n||_{weak(\mathbb{P})} \leqslant 2||f_n||_{L^\infty}, \qquad n = 0, 1, 2, \ldots, \tag{1.5}$$

and the constant 2 *is optimal.*

There exist also a substitute for (1.3) in the endpoint case $p = 1$. The idea is to enlarge the right-hand side of this estimate, by replacing the first norm of f_n by the first norm of the maximal function $|f|^*_n = \sup_{0 \leqslant k \leqslant n} |f_k|$, $n = 0,\ 1,\ 2,\ \ldots$. Then we have the following fact, established by Burkholder in [12].

Theorem 1.5. *If* f *is a martingale and* g *is its* ± 1*-transform, then*

$$||g_n||_{L^1} \leqslant \eta || \, |f|^*_n ||_{L^1}, \qquad n = 0,\ 1,\ 2,\ \ldots, \tag{1.6}$$

where $\eta = 2.536\ldots$ *is the unique solution of the equation* $\eta - 3 = -\exp\left(\frac{1-\eta}{2}\right)$. *The constant is the best possible.*

Differential subordination. All the results formulated above can be studied in the less restrictive setting in which the assumption of g being a martingale transform of f is relaxed. Suppose that f and g are adapted martingales. Following Burkholder [11], we say that g is differentially subordinate to f, if for any $n \geqslant 0$ we have $|dg_n| \leqslant |df_n|$ almost surely. It is immediate that if g is a martingale transform of f with respect to some predictable sequence with values in $[-1, 1]$, then g is differentially subordinate to f. On the other hand, one easily constructs a pair (f, g) satisfying the differential subordination such that g is not a transform of f. It turns out that Theorems 1.2, 1.3, 1.4 and 1.5 remain valid in this more general context, with unchanged constants (which, of course, remain best possible).

Inequalities for continuous-time martingales. The notions and results discussed above can be also considered in the context of continuous-time processes. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with a right-continuous filtration $(\mathcal{F}_t)_{t\geqslant 0}$, such that $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)_{t\geqslant 0}$, $Y = (Y_t)_{t\geqslant 0}$ be adapted martingales: for each $s \geqslant 0$ the variables X_s, Y_s are integrable and we have $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$, $\mathbb{E}(Y_t|\mathcal{F}_s) = Y_s$ whenever $t > s$. As usual, we assume that both processes are càdlàg: their trajectories are right-continuous and have limits from the left.

Suppose that $Y = (Y_t)_{t\geqslant 0}$ is of special form: it is the stochastic integral, with respect to $X = (X_t)_{t\geqslant 0}$, of some predictable process $H = (H_t)_{t\geqslant 0}$ which takes values in $[-1, 1]$:

$$Y_t = H_0 X_0 + \int_{0+}^{t} H_s \mathrm{d}X_s, \qquad t \geqslant 0.$$

We will write $Y = H \cdot X$ in such a case. This is the obvious analogue of a martingale transform from the discrete-time setting. One can also extend the notion of the differential subordination to the case of continuous processes. We say that a martingale Y is differentially subordinate by X, if the process $([X]_t - [Y]_t)_{t\geqslant 0}$ is nondecreasing and nonnegative. Here $[X]$ denotes a square bracket (see [13]).

It can be shown that Theorems 1.2, 1.3, 1.4 and 1.5 remain valid in the context of continuous-time differentially subordinate martingales.

All four results mentioned above are only a small part of a large class of inequalities for semimartingales and their transforms, studied by Burkholder and his students in the eighties/nineties and Osękowski in the last ten years. All the sharp estimates discussed above can be obtained by using a very efficient and flexible method, which reduces the whole problem to that of finding a special function satisfying appropriate majorization and concavity-type requirements. The technique has its origins in the theory of stochastic optimal control developed by Bellman in the fifties and has become a powerful tool widely used in probability theory and harmonic analysis. There is a natural question, which is the main motivation behind the dissertation: Can this topic, which so well understood for the classical (nonweigthed) theory be generalized for the weighted context? What kind of problems can we encounter? We will start with necessary definitions. Assume that $(\Omega,\ \mathcal{F},\ \mathbb{P})$ is a fixed probability space. Every nonnegative integrable random variable will be called a weight. Every weight w gives rise to weighted spaces L^p, $L^{p,\infty}$ given by

$$L^p(w) = \left\{ f : \Omega \longrightarrow \mathbb{R} : ||f||_{L^p(w)} = \left(\int_\Omega |f|^p w \mathrm{d}\mathbb{P} \right)^{\frac{1}{p}} < \infty \right\},$$

$$L^{p,\infty}(w) = \left\{ f : \Omega \longrightarrow \mathbb{R} : ||f||_{L^{p,\infty}(w)} = \sup_{\lambda>0} (\lambda^p w(|f| > \lambda))^{\frac{1}{p}} < \infty \right\},$$

where $w(A) = \int_A w \mathrm{d}\mathbb{P}$. Moreover any weight w generates a martingale $(w_n)_{n \geqslant 0} = (\mathbb{E}(w|\mathcal{F}_n))_{n \geqslant 0}$, which will also be denoted by w. The first weighted estimates appeared in the work of Muckenhoupt [20] about boundedness in L^p of Hardy-Littlewood maximal operator. This result also has the following martingale interpretation. We want to characterize, for a fixed $1 < p < \infty$, those w, for which there is a finite constant $C_{p,w}$ such that

$$|||f|_n^*||_{L^p(w)} \leqslant C_{p,w} ||f_n||_{L^p(w)}, \qquad n = 0, 1, 2, \dots, \tag{1.7}$$

for all martingales f and its maximal functions $|f|_n^* = \sup_{0 \leqslant k \leqslant n} |f_k|$. This is a weighted extension of a classical Doob's maximal inequality. It turns out that the necessary and sufficient condition for validity of (1.7) is the so called Muckenhoupt's condition A_p defined as follows:

Definition 1. (A_1) A weight w is called a Muckenhoupt A_1 weight, if there is a finite deterministic constant C, such that

$$w_n^* \leqslant C w_n$$

almost surely for $n = 0, 1, 2, \dots$. The smallest admissible C is called A_1 characteristic of weight w and is denoted by $[w]_{A_1}$.

(A_p) Let $1 < p < \infty$. A weight w is called Muckenhoupt weight A_p, if a random variable $v = w^{1/(1-p)}$ is integrable and there is a constant C, such that

$$w_n v_n^{p-1} \leqslant C$$

almost surely for each $n = 0, 1, 2, \dots$. Here $(v_n)_{n \geqslant 0}$ denotes a martingale defined as $v_n = \mathbb{E}(w^{1/(1-p)}|\mathcal{F}_n)$. The smallest admissible C is called A_p characteristic of weight w and is denoted as $[w]_{A_p}$.

(A_∞) A weight w is called Muckenhoupt A_∞, if a random variable $v = \log(w)$ is integrable and there is a constant C, such that

$$w_n \exp(-v_n) \leqslant C$$

almost surely for all $n = 0, 1, 2, \dots$. Here $(v_n)_{n \geqslant 0}$ denotes a martingale defined as $v_n = \mathbb{E}(\log(w)|\mathcal{F}_n)$. The smallest admissible C is called A_∞ characteristic of weight w and is denoted as $[w]_{A_\infty}$.

The above definitions are valid also for continuous-time filtrations.

The important problem is to establish quantitative version of the estimate (1.7) with the optimal dependence of a constant $C_{p,w}$ on the weight characteristic $[w]_{A_p}$. More precisely, we have the following question. For a fixed p, what is the optimal exponent α_p such that $C_{p,w} \leqslant c_p[w]_{A_p}^{\alpha(p)}$, where c_p depends only on p. This kind of problem has appeared in the work of Buckley [9] concerning the estimate (1.7) (it turns out that the optimal exponent is $\frac{1}{p-1}$).

The main goal of this dissertation is to establish weighted versions of the theorems 1.2, 1.3, 1.4 and 1.5 with the optimal dependence on the weight characteristic.

2 Results

2.1 Weighted strong-type inequalities for stochastic integrals

Let f be a discrete-time martingale and let g be its ± 1-transform. A celebrated result of Burkholder [11] asserts that for $1 < p < \infty$ we have the sharp bound

$$\|g\|_{L^p} \leqslant (p^* - 1)\|f\|_{L^p}, \tag{2.1}$$

where $p^* = \max\{p, p/(p-1)\}$. Here we have used the notation $\|f\|_{L^p} = \sup_{n\geqslant 0} \|f_n\|_{L^p}$ (and similarly for $\|g\|_{L^p}$)

The inequality (2.1) was proved using the Bellman function technique: as we have described in Chapter 1, it suffices to find $B : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

1° $B(x, y) \leqslant 0$ if $|y| \leqslant |x|$;

2° $B(x, y) \geqslant |y|^p - (p^* - 1)|x|^p$

3° for any x, y and d, e with $|e| \leqslant |d|$, the function $t \mapsto B(x + td, y + te)$ is concave on $\mathbb{R}$.

Burkholder proved that the function

$$B(x, y) = \alpha_p(|y| - (p^* - 1)|x|)(|x| + |y|)^{p-1}$$

(where α_p is a certain constant depending only on p) enjoys all the properties. It turns out that this function can be applied in seemingly unrelated areas of mathematics. Namely, there is a deep and unexpected connection of B with the geometric function theory, particularly with the theory of quasiconformal mappings, rank-one convex functionals and the properties of Beurling-Ahlfors operator: see [1, 2, 3, 16, 17] and consult the references therein. In other words, although the function B originates in the probabilistic estimate (2.1), its explicit formula is of independent interest and importance in contexts far and beyond martingale theory.

The above observation was one of the motivations for our research. There is a natural and interesting question about the following weighted version of (2.1):

$$\|g\|_{L^p(w)} \leqslant C_{p,w}\|f\|_{L^p(w)}, \qquad w \in A_p,$$

to be valid for all uniformly integrable martingales f, g such that g is a ± 1-transform of f. Here $\|f\|_{L^p(w)}$, $\|g\|_{L^p(w)}$ are the weighted L^p-norms of the limit random variables $\lim_{n\to\infty} f_n$, $\lim_{n\to\infty} g_n$, and the multiplicative constant $C_{p,w}$ depends on p and the weight w. Moreover we will be interested in the extraction of the sharp dependence of $C_{p,w}$ on the A_p characteristic of w. Namely, the problem is to identify the least exponent κ_p for which

$$\|g\|_{L^p(w)} \leqslant C_p[w]_{A_p}^{\kappa_p}\|f\|_{L^p(w)},$$

where C_p depends on p only.

For martingales on one-dimensional probability space (the interval $[0,1]$ with Lebesgue's measure and the dyadic filtration), this problem was answered by Wittwer [31]: the optimal κ_p is equal to $\max\{1, 1/(p-1)\}$. The proof was based on several reductions and depended heavily on the dyadic structure. The general case for arbitrary filtrations was established independently by Thiele, Treil and Volberg [29] and Lacey [18]. The extension of the inequality to continuous-time, differentially subordinate and uniformly integrable martingales was obtained in the recent work of Domelevo and Petermichl [14]:

$$\|Y\|_{L^p(W)} \leqslant c_p[W]_{A_p}^{\max\{1,1/(p-1)\}}\|X\|_{L^p(W)},$$

for every $1 < p < \infty$. Here, as before, $\|X\|_{L^p(W)}$ and $\|Y\|_{L^p(W)}$ are the usual weighted L^p-norms of the limiting random variables $\lim_{t\to\infty} X_t$ and $\lim_{t\to\infty} Y_t$. It can be shown, using standard extrapolation techniques, that it is enough to study the above inequality for the case $p = 2$ only:

$$||Y||_{L^2(W)} \leqslant c_2[W]_{A_2}||X||_{L^2(W)}. \tag{2.2}$$

The proof of this special bound, presented in [14], rests on duality and a number of complicated Bellman functions (involving six variables). There is a natural question whether the estimate (2.2) can be established by the direct application of the Bellman function method. The appearance of A_2 weights forces the introduction of two additional arguments into the function and hence the problem reduces to the construction of an explicit function of four variables, enjoying the appropriate concavity and size conditions similar to 1°-3° above. Here is the result for continuous-time, continuous-path martingales, obtained by Bañuelos, Osękowski and the author [4]. Interestingly, as a by-product, our approach will allow us to obtain the following stronger maximal estimate.

Theorem 2.1. *Suppose that W is an A_p weight and X, Y are continuous-path martingales such that Y is stochastic integral, with respect to X, of some predictable process H taking values in [-1,1]. Then for any $1 < p < \infty$ there is finite constant C_p depending only on p such that*

$$|||Y|^*||_{L^p(W)} \leqslant C_p[W]_{A_p}^{\max\{1,1/(p-1)\}}||X||_{L^p(W)}. \tag{2.3}$$

The exponent $\max\{1, 1/(p-1)\}$ *is the best possible.*

2.2 Weighted weak-type inequalities for stochastic integrals

The next result concerns weak-type (p, p) inequalities for stochastic integrals, where $1 < p < \infty$. Let us recall first the unweighted weak-type estimate established in the works of Burkholder [11], Suh [27] and Wang [30].

Theorem 2.2. *Suppose that X and Y are martingales such that Y is a stochastic integral, with respect to X, of some predictable process H with values in $[-1,1]$. Then for any $1 < p < \infty$ we have*

$$||Y||_{L^{p,\infty}} \leqslant C_p||X||_{L^p}, \tag{2.4}$$

where the optimal constant satisfies $C_p^p = 2/\Gamma(p+1)$ for $1 < p \leqslant 2$ and $C_p^p = p^{p-1}/2$ for remaining p.

We will study the following weighted extension of (2.4). Again we assume that all adapted martingales have continuous paths. This is a joint work with A. Osękowski (in preparation).

Theorem 2.3. *Fix $1 < p < \infty$. Let X, Y be martingales such that Y is a stochastic integral, with respect to X, of some predictable process H with values in $[-1,1]$. Then for any A_p weight W we have*

$$\|Y\|_{L^{p,\infty}(W)} \leqslant C_p[W]_{A_p}\|X\|_{L^p(W)}, \tag{2.5}$$

for some constant C_p depending only on p. The linear dependence on the A_p characteristic of W is optimal: for any $\kappa < 1$ and any $K > 0$, there is a weight W, a real-valued martingale X and a predictable process H with values in $\{-1,1\}$ such that the stochastic integral $Y = H \cdot X$ satisfies $\|Y\|_{L^{p,\infty}(W)} > K[W]_{A_p}^{\kappa}\|X\|_{L^p(W)}$.

2.3 Weighted weak type (∞,∞) inequality for differentially subordinate martingales

In the next part of the dissertation we study weighted weak-type inequalities in the endpoint case $p = \infty$. We will need the notions of nondecreasing rearrangement ξ^* and its maximal function ξ^{**} presented in the introduction. Following Bennett, DeVore and Sharpley [5], we set

$$||\xi||_{weak(\mathbb{P})} = \sup_{t\in(0,1]} (\xi^{**}(t) - \xi^*(t))$$

and define $weak(\mathbb{P}) = \{\xi : ||\xi||_{weak(\mathbb{P})} < \infty\}$. The main motivation behind introducing these spaces comes from the interpolation theory. First, note that this space contains $L^\infty(\Omega)$. Next, one can show that if a linear operator T is bounded from L^1 to $L^{1,\infty}$ and from L^∞ to $weak(\mathbb{P})$, then it can be extended to a bounded operator on all L^p spaces, $1 < p < \infty$. In other words, in the context of the above weak-L^∞ spaces, we have a substitute of Marcinkiewicz interpolation theorem for operators which are unbounded on L^∞ (more on this topic can be found in [5]).

Let us recall the main result of Osękowski [23].

Theorem 2.4. *Let X, Y be martingales such that Y is differentially subordinate to X. Then we have the inequality*

$$||Y||_{weak(\mathbb{P})} \leqslant 2||X||_{L^\infty} \tag{2.6}$$

and the constant 2 is the best possible.

The following weighted extension of this inequality comes from the joint work with A. Osękowski [7].

Theorem 2.5. *Let X, Y be continuous-path martingales such that X is bounded and Y is differentially subordinate to X. Then for any A_∞ weight W with $\|W\|_1 = 1$, we have*

$$||Y||_{weak(W)} \leqslant 97[W]_{A_\infty}||X||_{L^\infty}. \tag{2.7}$$

The linear dependence on the A_∞ characteristics is optimal.

2.4 Weighted maximal inequality for martingale transforms

The last result of the dissertation is the weighted extension of the following Burkholder maximal inequality:

Theorem 2.6. *If f, g are martingales satisfying $dg_n = \varepsilon_n df_n$, $n = 0, 1, 2, \ldots$ for some predictable sequence $\varepsilon = (\varepsilon_n)_{n\geqslant 0}$ with values in $[-1, 1]$, then*

$$||g||_{L^1} \leqslant \eta|| \, |f|^* ||_{L^1}, \tag{2.8}$$

where $\eta = 2.536\ldots$ is the unique solution of the equation $\eta - 3 = -\exp\left(\frac{1-\eta}{2}\right)$. The constant is the best possible.

We will be interested in discrete-time martingales. It turns out, that for the weighted extension of the above estimate, we need to impose some additional assumption concerning regularity of a filtration. We will need the following definition.

Definition 2. Let $\theta \in (0, 1/2]$ be a fixed parameter. A filtration $(\mathcal{F}_n)_{n\geqslant 0}$ is said to be θ-regular, if $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and for any $n \geqslant 0$, every atom A of $\mathcal{F}_n$ splits into a finite number $A_1, A_2, \ldots, A_k$ of atoms of $\mathcal{F}_{n+1}$ satisfying $\mathbb{P}(A_j) \geqslant \theta\mathbb{P}(A)$, $j = 1, 2, \ldots, k$.

Regular filtrations are natural extensions of dyadic filtrations used widely in harmonic analysis. For a fixed dimension d, the dyadic filtration of the space $\left([0, 1]^d, \mathcal{B}([0, 1]^d), |\cdot|\right)$ is 2^{-d}-regular in the above sense.

We are interested in a weighted version of (2.8):

$$|| \, |g|^* ||_{L^1(w)} \leqslant C_{\theta,w}|| \, |f|^* ||_{L^1(w)}. \tag{2.9}$$

Observe that we have maximal functions on both sides of the equation. The following result was obtained in a joint work with A. Osękowski [8].

Theorem 2.7. *Fix $\theta \in (0, 1/2]$. Let f, g be martingales adapted to a θ-regular filtration such that g is a transform of f by a predictable sequence with values in $[-1, 1]$. Then for any A_∞ weight w we have*

$$|| \, |g|^* ||_{L^1(w)} \leqslant 769 \, \theta^{-2} [w]_{A_\infty} || \, |f|^* ||_{L^1(w)}. \tag{2.10}$$

The dependence on the A_∞ characteristics of the weight is optimal.

A weaker result for Haar multipliers and A_p weights was obtained in [24]:

$$\left|\left| \max_{0 \leqslant n \leqslant N} \left| \sum_{k=0}^{n} \varepsilon_k a_k h_k \right| \right|\right|_{L^1(w)} \leqslant C_p [w]_{A_p} \left|\left| \max_{0 \leqslant n \leqslant N} \left| \sum_{k=0}^{n} a_k h_k \right| \right|\right|_{L^1(w)}, \tag{2.11}$$

where $1 < p < \infty$, w is a dyadic A_p weight, N is a nonnegative integer, $a_0, a_1, \ldots, a_N$ are natural numbers, $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N$ is a sequence of signs and $(h_k)_{k=0}^{\infty}$ is a Haar system on the interval $[0, 1)$. Observe that (2.10) extends this result in two directions. Firstly, we consider a more general case of θ-regular filtrations. Secondly, since $[w]_{A_\infty} \leqslant [w]_{A_p}$, the inequality (2.10) is stronger.

A constant in (2.10) depends on θ and tends to infinity, when θ goes to 0. The last result in the dissertation is showing that this dependence is necessary, even under a weaker condition A_p for a fixed $p > 1$. We have the following theorem:

Theorem 2.8. *Let $p > 1$ and let K be an arbitrary positive constant. Then there is a positive integer d, a martingale f on d-dimensional dyadic probability space, an A_p weight w satisfying $[w]_{A_p} \leqslant 2$ and a predictable sequence v with values in $\{-1, 1\}$ such that the associated martingale transform g satisfies*

$$||g||_{L^1(w)} > K || |f|^* ||_{L^1(w)}.$$

Literatura

[1] K. Astala, T. Iwaniec, I. Prause and E. Saksman, *Burkholder integrals, Morrey's problem an quasiconformal mappings*, J. Amer. Math. Soc. **25** (2012), 507-531.

[2] K. Astala, T. Iwaniec, I. Prause and E. Saksman, *A hunt for sharp Lp-estimates and rank-one convex variational integrals*, Filomat **29** (2015), 245-261.

[3] A. Baernstein II and S. J. Montgomery-Smith, *Some conjectures about integral means of ∂f and $\bar{\partial} f$*, Complex Analysis and Differential Equations, Acta Univ. Upsaliensis Skr. Uppsala Univ. C. Organ. Hist. **64**, Uppsala Univ., Uppsala, (1999), 92-109.

[4] R. Bañuelos, M. Brzozowski and A. Osękowski, *Burkholder's function and a weighted L^2 bound for stochastic integrals*, to appear in Proceedings of the American Mathematical Society.

[5] C. Bennett, R. A. DeVore and R. Sharpley, *Weak-L^∞ and BMO*, Ann. of Math. **113** (1981), 601-611.

[6] R. Bellman, *Dynamic programming*, Reprint of the 1957 edition. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, (2010).

[7] M. Brzozowski and A. Osękowski, *Sharp weighted weak type (∞,∞) inequality for differentially subordinate martingales*. Statistics and Probability Letters **155** (2019), 108561.

[8] M. Brzozowski and A. Osękowski, *Weighted maximal inequalities for martingale transforms*, to appear in Probability Math. Statist.

[9] S. M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340** (1993), 253–272.

[10] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. **37** (1966), 1494–1504.

[11] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), 647–702.

[12] D. L. Burkholder, *Sharp norm comparison of martingale maximal functions and stochastic integrals*, Proceedings of the Norbert Wiener Centenary Congress, 1994 (East Lansing, MI, 1994), 343-358, Proc. Sympos. Appl. Math., **52**, Amer. Math. Soc., Providence, RI, (1997).

[13] C. Dellacherie and P. A. Meyer, *Probabilities and potential* B, North-Holland, Amsterdam, (1982).

[14] K. Domelevo and S. Petermichl, *Differential subordination under change of law*, Ann. Probab. **47** (2019), 896-925.

[15] T. Hytönen, J.M.A.M. van Neerven, M.C. Veraar, and L. Weis, *Analysis in Banach Spaces. Volume I: Martingales and Littlewood-Paley Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, (2016).

[16] T. Iwaniec, *Extremal inequalities in Sobolev spaces and quasiconformal mappings*, Z. Anal. Anwendungen **1** (1982), 1-16.

[17] T. Iwaniec, *Nonlinear Cauchy-Riemann operators in* $\mathbb{R}^n$, Trans. Amer. Math. Soc.**354**(2002), 1961-1995.

[18] M. T. Lacey, *An elementary proof of the A_2 bound*, Israel Journal of Mathematics, **217**: 181-195, (2017).

[19] J. Marcinkiewicz, *Quelques théorèmes sur les séries orthogonales*, Ann. Soc. Polon. Math. **16** (1937), 84-96.

[20] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc **165** (1972), 207–226.

[21] A. Osękowski, *Sharp martingale and semimartingale inequalities*, Monografie Matematyczne **72**, Birkhäuser, 2012.

[22] A. Osękowski, *Sharp LlogL inequality for differentially subordinated martingales*, Illinois J. Math. **52**, Vol. **3** (2008), 745–756.

[23] A. Osękowski, *Sharp Weak-Type Inequality for the Haar System, Harmonic Functions and Martingales*, Bulletin of the Polish Academy of Sciences Mathematics **62** (2014), pp. 187-196.

[24] A. Osękowski, *Weighted maximal inequalities for the Haar system*, Monatshefte für Mathematik **186** (2018), 321-336.

[25] R. E. A. C. Paley, *A remarkable series of orthogonal functions*, Proc. London Math. Soc. **34** (1932), 241-264.

[26] I. Singer, *Bases in Banach Spaces, Grundlehren der mathematischen Wissenschaften* **154**, Springer Verlag, 2010.

[27] Y. Suh, *A sharp weak type* (p,p) *inequality* $(p > 2)$ *for martingale transforms and other subordinate martingales*, Trans. Amer. Math. Soc. **357** (2005), 1545-1564.

[28] V. Temlyakov, *Sparse Approximation with Bases*, Springer Basel, (2015).

[29] C. Thiele, S. Treil, and A. Volberg, *Weighted martingale multipliers in the non-homogeneous setting and outer measure spaces*, Advances in Mathematics, **285**:1155-1188, (2015).

[30] G.Wang, *Differential subordination and strong differential subordination for continuous time martingales and related sharp inequalities*, Ann. Probab. **23** (1995), 522-551.

[31] J. Wittwer, *A sharp estimate on the norm of the martingale transform*, Math. Res. Lett. **7(1)**, 1-12 (2000).