What would the rational Urysohn space and the random graph look like if they were uncountable?

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Generally, $\mathcal{K}$ will be a class of structures in some countable, relational, first-order language. We make the following assumptions on $\mathcal{K}$.

- $\mathcal{K}$ has the JEP (Joint Embedding Property),
- $\mathcal{K}$ the SP (Splitting Property Property),
- $\mathcal{K}$ is hereditary, so if $A \in \mathcal{K}$, and $B \subseteq A$, then $B \in \mathcal{K}$,
- $\mathcal{K}$ has infinitely many isomorphism types.
- $\mathcal{K}$ is closed under unions of increasing chains

(For a given countable $K \subseteq [0, \infty)$, the class of finite metric spaces with distances in K has the SP, but usually not the AP.)

Definition

Let λ be an infinite cardinal number, and S be any infinite set. Denote by $\mathrm{Fn}(S, \mathcal{K}, \lambda)$ the set

$$\{A \in \mathcal{K} | \ A \in [S]^{<\lambda}\},$$

ordered by the (reversed) inclusion.

What are properties of $\mathrm{Fn}(S, \mathcal{K}, \lambda)$?

Observation: If $|S| = \omega$, then

$$\mathrm{Fn}(S, \mathcal{K}, \omega)$$

is just the Cohen forcing.

Proposition

Forcing $\mathrm{Fn}(S, \mathcal{K}, \omega)$ *satisfies c.c.c. for any set S.*

Theorem

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"*ω with the generic structure is the Fraïssé limit of $\mathcal{F}$*".

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Theorem

(CH) Assume $\mathcal{LO}$ is the class of linear orders. $\mathrm{Fn}(\omega_2, \mathcal{LO}, \omega_1) \Vdash$ *"ω_2 with the generic structure is ccc-absolutely rigid".*

Definition (Avraham-Shelah, 1981)

An uncountable set $A \subseteq \mathbb{R}$ is *increasing*, if the following assertion holds for every $k \in \omega$:
For each sequence of pairwise disjoint k-tuples $\{(\alpha_1^\xi, \ldots, \alpha_k^\xi)\}_{\xi<\omega_1}$ of elements of A, and for every tuple of there are $\xi \neq \eta < \omega_1$, such that for $i,j = 1, \ldots, k$

$$\alpha_i^\xi < \alpha_i^\eta \iff \alpha_j^\xi < \alpha_j^\eta$$

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Proposition

If $A = (A, \leq)$ *is increasing, then* $(A, \leq)$ *is not isomorphic to* $A^* = (A, \geq)$.

Theorem (Avraham-Shelah, 1981)

Existence of an increasing set is consistent with $MA + \neg CH$.

We start with a model of *CH* and fix an increasing set A of size ω_1. We will say that a forcing $\mathbb{P}$ is *appropriate* if for each sequence of pairwise disjoint k-tuples

$$\{(p_\xi, \alpha_1^\xi, \ldots, \alpha_k^\xi)\}_{\xi<\omega_1} \subseteq \mathbb{P} \times A^k$$

there are $\xi \neq \eta < \omega_1$, such that

- for $i, j = 1, \ldots, k$

$$\alpha_i^\xi < \alpha_i^\eta \iff \alpha_j^\xi < \alpha_j^\eta,$$

- p_ξ and p_η are comparable.

Next we prove that

- Finitely supported iterations of appropriate forcings are appropriate.
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By building a suitable iteration we got the model of

$$ZFC + MA(appropriate) + 2^{\omega} = \omega_2.$$

But it turns out that $MA(appropriate) \implies MA$.

Still, there is something missing...

We didn't prove that increasing sets exist in the first place... but fortunately

$$\mathrm{Fn}(\omega_1, \mathcal{LO}, \omega) \Vdash \text{"}(\omega_1, \dot{\leq}) \text{ is increasing."}$$

(actually, *CH* implies that increasing sets exist, but it is more complicated).

The immediate consequence is

Theorem (Avraham-Shelah, 1981)

It is consistent with ZFC that there exists an uncountable set of reals A with the property that each uncountable partial function $f \subseteq A \times A$ is non-decreasing on some uncountable set.

Proof.

We work in a model of $ZFC + MA_{\omega_1} +$ "A is increasing". Given a function f we can assume that f is 1-1. We apply MA to the set

$$\{E \in [\mathrm{dom} f]^{<\omega} | f \restriction E \text{ is increasing}\}.$$

□

Can we do the same thing for other structures added by $\mathrm{Fn}(\omega_1, \mathcal{K}, \omega)$?

Definition

Let (X, d) be a metric space.

- We call (X, d) *rectangular* if it is uncountable, and for any sequence of pairwise disjoint tuples $\{(x_1^\xi, \dots, x_n^\xi) | \xi < \omega_1\} \subseteq X^n$, there are $\xi \neq \eta < \omega_1$, such that $(x_1^\xi, \dots, x_n^\xi) \circledast (x_1^\eta, \dots, x_n^\eta)$.

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- We call a pair of tuples $\bar{x} = (x_1, \ldots, x_n), \bar{y} = (y_1, \ldots, y_n) \in X^n$ *alike* if they satisfy the following axioms:

  A1 $\forall\, i, j = 1, \ldots, n\ (d(x_i, y_i) = d(x_j, y_j))$
  A2 $\forall\, i, j = 1, \ldots, n\ (d(x_i, x_j) = d(y_i, y_j))$
  A3 $\forall\, i, j = 1, \ldots, n\ (x_i \neq x_j \implies d(x_i, x_j) = d(x_i, y_j))$

  We then write $\bar{x} \circledast \bar{y}$.

Denote by *Metr* the class of rational metric spaces. $\mathrm{Fn}(\omega_1, \mathit{Metr}, \omega)$ is the partial order

$$\{(Y,d)|\ Y \in [\omega_1]^{<\omega},\ \text{and } (Y,d) \text{ is a rational metric space}\},$$

with the ordering relation being the reversed inclusion preserving the metric.

Proposition

$\mathrm{Fn}(\omega_1, Metr, \omega) \Vdash$ *"$(\omega_1, \dot{d})$ is rectangular".*

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Let $(\mathcal{M}, d)$ be the generic space we added.

Definition

A partial order $\mathbb{P}$ is *appropriate* if given any natural number $n > 0$, for each disjoint family $\{(p_\xi, x_1^\xi, \ldots, x_n^\xi) | \ \xi < \omega_1\} \subseteq \mathbb{P} \times (\mathcal{M}, d)^n$, there exist $\xi \neq \eta < \omega_1$, such that p_ξ and p_η are comparable, and $(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta)$.

Theorem

- *If $\mathbb{P}$ is appropriate then $\mathbb{P} \Vdash$ "$(\mathcal{M}, d)$ is rectangular".*
- *Finitely supported iterations of appropriate forcings are appropriate.*

It follows, that it is consistent with $ZFC + MA +$ "$2^{\omega} = \omega_2$" that $(\mathcal{M}, d)$ is rectangular.

Theorem

Let (X, d) be a rectangular rational metric space of size ω_1. MA_{ω_1} implies that any uncountable 1-1 function $f \subseteq X \times X$ is an isometry on an uncountable set.

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Proof.

We use apply Martin's Axiom to the partial order

$$\mathbb{P}_f = (\{E \in [\operatorname{dom} f]^{<\omega} | f \restriction E \text{ is an isometry}\}, \subseteq).$$

□

Definition

Let (E, d_E) be any metric space with distances in some countable set $K \subseteq [0, \infty)$. A subset $D \subseteq E$ is a *saturated subset* of E if for any finite subset $E_0 \subseteq E$, for any single-point extension $F = (E_0 \cup \{f\}, d_F)$, with distances in K, there exists $d \in D$ such that for all $e \in E_0$, we have $d_E(e, d) = d_F(e, f)$.

Definition

Let (X, d) be any metric space, with distances in a given countable set.

1. (X, d) is *separably saturated* if it has a countable saturated subset.
2. (X, d) is *hereditarily separably saturated* (HSS) if for any countable subset $A \subseteq X$, the space $X \setminus A$ is separably saturated.

How many such spaces do exist?

Theorem

Assume MA_{ω_1}. Let (X, d) be any rectangular HSS metric space of size ω_1, with distances in K. Let $Y \subseteq X$ be any HSS uncountable subspace. Then X and Y are isometric.

Corollary

Assume MA_{ω_1}. Let (X, d) be any rectangular HSS metric space of size ω_1, with distances in K. X can be decomposed into a disjoint union of λ many its isometric copies for any $\lambda \in \{2, \ldots, \omega_1\}$.

Proposition

Assume MA_{ω_1}. If (X, d) is any rectangular HSS metric space of size ω_1, with distances in K, then any finite partial isometry of (X, d) extends to a full isometry.

Theorem

Assume MA_{ω_1}. If there exists a rectangular HSS rational metric space of size ω_1, then there exist infinitely many pairwise non-isometric such spaces.

Proof.

If (X, d) is a rectangular HSS rational metric space of size ω_1, we can scale the metric:

$$d_k(x, y) = k \cdot d(x, y).$$

□

Observation: We didn't used the fact that $d : (\mathcal{M}, d) \times (\mathcal{M}, d) \to \mathbb{Q}$ is a metric! Merely a coloring of pairs. This motivates the following generalization:
We fix a class $\mathcal{K}$ is models in some language consisting of binary relations. For any $A \in \mathcal{K}$ let

$$c : A \times A \to \omega$$

assign each pair its isomorphism type.

Definition

Let $X \in \mathcal{K}$, and $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X^n$ be disjoint. We will say that they are *alike*, and write $(x_1, \ldots, x_n) \circledast (y_1, \ldots, y_n)$, if the following axioms are satisfied

A1a $\forall\, i,j = 1, \ldots, n\ c(x_i, y_i) = c(x_j, y_j)$

A1b $\forall\, i,j = 1, \ldots, n\ c(y_i, x_i) = c(y_j, x_j)$

A2a $\forall\, i,j = 1, \ldots, n\ c(x_i, x_j) = c(y_i, y_j)$

A3a $\forall\, i,j = 1, \ldots, n\ (x_i \neq x_j \implies c(x_i, x_j) = c(x_i, y_j) = c(y_i, x_j))$

Definition

$X \in \mathcal{K}$ is *rectangular* if $|X| > \omega$, and for any family of pairwise disjoint tuples $\{(x_1^\xi, \ldots, x_n^\xi) | \ \xi < \omega_1\} \subseteq X^n$, there exist $\xi \neq \eta < \omega_1$, such that $(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta)$.

In most cases $\mathrm{Fn}(\omega_1, \mathcal{K}, \omega)$ forces the generic structure to be rectangular. We fix a rectangular model $(\mathcal{X}, c)$.

Theorem

It is consistent with ZFC + MA + "$2^\omega = \omega_2$" that $(\mathcal{X}, c)$ is rectangular.

Proposition

Let $(X, c) \in \mathcal{K}$ be rectangular. MA_{ω_1} implies that any uncountable 1-1 function $f \subseteq X \times X$ is a homomorphism on an uncountable set.

If we take $K = \{0, 1, 2\}$, metric spaces with distances in K are graphs – think of two points in distance 1 as connected, and two points in distance 2 as not connected.

Definition

Let G be a graph. A subset $D \subseteq G$ is a *saturated subset* of G if for any pair of disjoint finite subsets $A, B \subseteq G$, there exists $d \in D \setminus (A \cup B)$ which is connected with each vertex in A and with no vertex in B.

Definition

If G is any graph, then

1. G is *separably saturated* if it has a countable saturated subset.
2. G is *hereditarily separably saturated* (HSS) if for any countable subset $E \subseteq G$, the graph $G \setminus E$ is separably saturated.

Proposition

$\mathrm{Fn}(\omega_1, \mathit{Graphs}, \omega) \Vdash$ "$(\omega_1, \dot{E})$ *is a rectangular HSS graph*".

Theorem

Assume MA_{ω_1}. If G is a HSS rectangular graph of size ω_1, then each uncountable HSS subgraph of G is isomorphic with G.

Theorem

Assume MA_{ω_1}. If there exists a HSS rectangular graph of size ω_1 then it is unique up to taking the complement.

Theorem

Assume MA_{ω_1}. If G is a rectangular HSS graph of size ω_1, then either G contains an uncountable clique or uncountable anticlique.

If A is a separable, dense linear order, then A is increasing iff it is rectangular.

If A is a separable, dense linear order, then A is increasing iff it is rectangular. Using this technology we can prove:

Theorem (Avraham-Shelah, 1981)

Existence of an increasing set is consistent with $MA + \neg CH$.

Theorem

Assume MA_{ω_1}. If there exists an increasing ω_1-dense linear order, then it is unique up to reversing the order. (So there are exactly two such orderings.)

More details can be found in my preprint Z. Kostana, *What would the rational Urysohn space and the random graph look like if they were uncountable?*, https://arxiv.org/abs/2102.05590