Varieties Generated by Plactic-like Monoids: Identities and Representations

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Overview

This talk addresses the following themes:

- Investigate identities and varieties generated by plactic-like monoids
- Briefly recall foundational results on the plactic monoid
- Focus on the hypoplactic and sylvester monoids: algebraic and combinatorial perspectives
- Use embedding theorems and faithful matrix representations to characterize their equational theories
- Highlight structural differences and varietal properties across monoid families

Background: Identities and Varieties

Identity

- A (monoid) identity is a formal equation $u = v$ where u, v are words over an alphabet X of variables. It is *non-trivial* if $u \neq v$.
- A monoid M *satisfies* $u = v$ if substituting any element of M for each symbol in X gives an equality that holds in M.

For example,

- Any commutative monoid satisfies $xy = yx$.
- Any nilpotent group of class 2 satisfies $xyzyx = yxzxy$ [Neumann & Taylor 1963].

Variety

A variety is a class of monoids closed under taking homomorphic images, submonoids, and direct products.

Equational Basis

A set Σ of identities is a basis for a variety $\mathcal{V}$ if every identity satisfied by $\mathcal{V}$ is a consequence of identities in Σ.

- A variety is *finitely based* if it admits a finite basis.
- The *axiomatic rank* is the least natural number such that the variety admits a basis where the number of distinct variables occurring in each identity of the basis does not exceed said number.

Matrix Representations over Semirings

Definition

Let S be a semiring. A representation of a monoid M over S is a homomorphism $\rho : M \to M_n(S)$, where $M_n(S)$ is the monoid of $n \times n$ matrices over S.

- We seek **faithful** representations: ρ is injective.
- Relevant semirings: tropical semiring and fields of characteristic 0.

Example

Tropical semiring: $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$.
Its addition and multiplication operations $\oplus$ and $\otimes$ are defined by $x \oplus y = \max\{x, y\}$ and $x \otimes y = x + y$.

- The set of $n \times n$ upper-triangular tropical matrices is denoted $\mathrm{UT}_n(\mathbb{T})$. (An upper-triangular tropical matrix has all entries below the main diagonal equal to $-\infty$.)

The Plactic Monoid

- $\mathcal{A} = \{1 < 2 < \cdots\}$ be the set of natural numbers viewed as an infinite ordered alphabet;
- For $n \in \mathbb{N}$, let $\mathcal{A}_n = \{1 < 2 < \cdots < n\}$;
- For any $u \in \mathcal{A}^*$, let $\mathrm{P}_{\text{plac}}(u)$ be the Young tableau computed from u using Schensted's algorithm;

$u = 542152153123$

$\mathrm{P}_{\text{plac}}(u) =$

5				
4	5	5		
2	2	3		
1	1	1	2	3

- The **plactic congruence** $\equiv_{\text{plac}}$ relates words in $\mathcal{A}^*$ that have the same image under $u \mapsto \mathrm{P}_{\text{plac}}(u)$.

Definition - Plactic monoid

The **plactic monoid** plac is the factor monoid $\mathcal{A}^*/{\equiv_{\text{plac}}}$.
The **plactic monoid of rank** n is the factor monoid $\mathcal{A}_n^*/{\equiv_{\text{plac}}}$ (where $\equiv_{\text{plac}}$ is naturally restricted to $\mathcal{A}_n^*$).

The plactic monoid of rank 2

Theorem (Jaszuńska and Okniński (2011))

The Chinese monoid embeds into a direct product of copies of the bicyclic monoid and the infinite cyclic group.

- $plac_2$ is isomorphic to the Chinese monoid of rank 2.

Theorem (Adjan (1966))

The bicyclic monoid $B = \langle p, q \mid qp = 1 \rangle$ satisfies the (Adjan's) identity

$$xy\, yx\, xy\, xy\, yx = xy\, yx\, yx\, xy\, yx.$$

Corollary

$plac_2$ satisfies Adian's identity.

Representations and varieties of plac_2

Let $A = \begin{bmatrix} 1 & 1 \\ -\infty & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -\infty & 1 \end{bmatrix}$.

The mapping $\rho : \mathcal{A}_2^* \to \mathrm{UT}_2(\mathbb{T})$ given by $1 \mapsto A$ and $2 \mapsto B$ factors to give a homomorphism from plac_2 into a submonoid of $\mathrm{UT}_2(\mathbb{T})$.

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The monoids B, plac_2 and $\mathrm{UT}_2(\mathbb{T})$ generate the same variety.

Theorem (Chen et al. (2016))

The variety generated by the monoid $\mathrm{UT}_2(\mathbb{T})$ of two-by-two upper triangular tropical matrices is nonfinitely based.

The plactic monoid of rank 3

Theorem (Kubat and Okniński (2015))

The plactic monoid $plac_3$ of rank 3 satisfies the identity $wv\,vw\,vw = wv\,wv\,vw$, where v and w are the left and right hand-side of Adjan's identity.
$plac_3$ does not satisfy Adian's identity.

Idea of the proof:

Exhibit a simple proof of Adjan's result, showing a faithful representation $\phi : B \to End(V)$;
Find a faithfull representation $\phi : N_1/(cba = 1) \to End(V)$ where N_1 is a homomorphic image of $plac_3$, for which $plac_3$ and $N_1/(cba = 1)$ satisfy the same "kind" of identities.

The plactic monoid of rank 3

For any $w \in \mathcal{A}^+$ and any $p, q \in \mathcal{A}$, define:

$$w_{pq} = \begin{cases} \text{the maximal length of a nondecreasing subsequence in } w \text{ with entries in the interval } [p, q] & \text{if } p \leq q, \\ -\infty & \text{if } p > q. \end{cases}$$

Define a map $\phi_n \colon \mathcal{A}^* \to \mathrm{UT}_n(\mathbb{T})$ by $\phi_n(w) = \left[w_{pq}\right]$.

Lemma (Cain, Klein, et al. (2017))

The map ϕ_n is a homomorphism that factors to give a homomorphism $\phi_n \colon \mathrm{plac}_n \to \mathrm{UT}_n(\mathbb{T})$.

The Plactic Monoid of Rank 3

- Define $f_n \colon \mathcal{A}_n^* \to \mathcal{A}_n^*$ by:

$$f_n(i) = n(n-1)\cdots(i+1)(i-1)\cdots 1$$

and for $w = a_1 \cdots a_k \in \mathcal{A}_n^*$ set $f_n(w) = f_n(a_k)\cdots f_n(a_1)$.

- f_n factors to give an antihomomorphism $f_n \colon \mathrm{plac}_n \to \mathrm{plac}_n$.
- Let $F = \left[F_{ij}\right]$ be the $n \times n$ tropical matrix where:

$$F_{i,n+1-i} = 0 \quad \text{for } i = 1, 2, \ldots, n$$

and all other entries are $-\infty$.

- Define the involution $\pi_n \colon \mathrm{UT}_n(\mathbb{T}) \to \mathrm{UT}_n(\mathbb{T})$ by:

$$x \mapsto (FxF)^T$$

(i.e., conjugation by F followed by transposition).

- π_n is an antiautomorphism.

The Plactic Monoid of Rank 3

- Define:

$$\sigma_n = \pi_n \circ \phi_n \circ f_n$$

Since f_n and π_n are antihomomorphisms, σ_n is a homomorphism.

Theorem (Cain, Klein, et al. (2017))

The map $\Psi_3 \colon \text{plac}_3 \to \mathrm{UT}_3(\mathbb{T}) \times \mathrm{UT}_3(\mathbb{T})$ defined by:

$$\Psi_3(u) = (\phi_3(u), \sigma_3(u))$$

is an embedding.

- Izhakian (2019) also showed a faithful representation of plac_3.

Infinite rank plactic monoid

Theorem (Cain, Klein, et al. (2017))

The plactic monoid plac_n does not satisfy any non-trivial identity of length less than or equal to n.

Theorem (Cain, Klein, et al. (2017))

The infinite-rank plactic monoid plac does not satisfy any non-trivial identity.

Identities in upper triangular tropical matrix

Define

$$u_0(p,q) = p, \qquad v_0(p,q) = q,$$
$$u_1(p,q) = pq\textcolor{red}{p}pq, \qquad v_1(p,q) = pq\textcolor{red}{q}pq,$$

and inductively define

$$u_n(p,q) = u_1\big(u_{n-1}(p,q), v_{n-1}(p,q)\big), \quad v_n(p,q) = v_1\big(u_{n-1}(p,q), v_{n-1}(p,q)\big)$$

for $n \geq 2$.

- $u_1(xy, yx) = v_1(xy, yx)$ is Adian's identity.

Theorem (Cain, Klein, et al. (2017))

The monoid of $n \times n$ upper-triangular tropical matrices $\mathrm{UT}_n(\mathbb{T})$ satisfies the identity $u_{n-1}(xy, yx) = v_{n-1}(xy, yx)$.

The plactic monoid of rank n

Define a map $\rho_n : [n]^* \to M_{2^{[n]}}(\mathbb{T})$ by letting, for $P, Q \in 2^{[n]}$,

$$\rho_n(x)_{P,Q} = \begin{cases} -\infty & \text{if } |P| \neq |Q| \text{ or } P \nleq Q; \\ 1 & \text{if } |P| = |Q| \text{ and } x \in \cup[P,Q]; \\ 0 & \text{otherwise (that is, if } |P| = |Q|,\ P \leq Q,\ x \notin \cup[P,Q]). \end{cases}$$

for each generator $x \in [n]$.

Theorem (Johnson and Kambites (2021))

The map $\rho_n : \mathrm{plac}_n \to M_{2^{[n]}}(\mathbb{T})$ induces a faithful representation of plac_n.

Corollary

The plactic monoid of each finite rank satisfies a non-trivial semigroup identity.

The plactic monoid of rank n

Theorem (Johnson and Kambites (2021))

For each n, plac_n satisfies all semigroup identities satisfied by $\mathrm{UT}_d(\mathbb{T})$ where d is the integer part of $\frac{n^2}{4} + 1$.

Theorem (Johnson and Kambites (2021))

The variety generated by $\mathrm{UT}_n(\mathbb{T})$ is contained in the variety generated by plac_n.

Corollary (Johnson and Kambites (2021))

The variety generated by plac_3 coincides with the variety generated by $\mathrm{UT}_3(\mathbb{T})$.

The Hypoplactic Monoid: Combinatorics

- Combinatorial objects are quasi-ribbon tableaux;
- For any word $u \in \mathcal{A}^*$, let $P_{hypo}(u)$ be the quasi-ribbon tableau computed from u using Krob–Thibon algorithm;
- Words u, v are congruent ($u \equiv_{hypo} v$) if they produce the same quasi-ribbon tableau $P_{hypo}(u) = P_{hypo}(v)$.

$$u = 6135461254 \mapsto P_{hypo}(u) =$$

1	1	2			
		3	4	4	
				5	
				6	6

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Definition

The *hypoplactic monoid* is the quotient $\mathcal{A}^* / \equiv_{hypo}$.
The *hypoplactic monoid of rank n* is the factor monoid $\mathcal{A}_n^*/{\equiv_{\mathrm{hypo}}}$ (where $\equiv_{\mathrm{hypo}}$ is naturally restricted to $\mathcal{A}_n^*$).

Hypoplactic Congruence and Inversions

Inversion Condition

Two words are equivalent in the hypoplactic monoid if and only if they share:

- The same content (number of occurrences of each letter)
- The same set of adjacent inversions (e.g., 2–1 inversion)

Example

$u = 31214$ has 3–2 and 2–1 inversions.
$v = 21341$ has a 2–1 inversion.

This characterization underpins the embedding results.

Embedding into Products of Rank-2 Monoids

Theorem (Cain, Malheiro, and Ribeiro (2022))

For $n \geq 2$, the hypoplactic monoid of rank n embeds into a direct product of copies of the rank-2 hypoplactic monoid.

- Let u_{ij} be homomorphisms capturing inversions between pairs $i < j$.
- The embedding $\phi_n : hypo_n \to \prod_{i<j} hypo_2$ is injective.
- Monoids of rank ≥ 2 satisfy the same identities.

Example

A word over $\{1, 2, 3\}$ is mapped into projections via u_{12}, u_{13}, u_{23}.

Characterization of Identities in *hypo*

Theorem (Cain, Malheiro, and Ribeiro (2022))

An identity $u = v$ is satisfied by *hypo* if and only if:

- It is balanced (i.e., same content);
- For all x, y, xy appears as a subsequence in u iff it appears in v

Example

$xyxy = xyyx$ is satisfied.
$xyyx = yxxy$ is satisfied.

Consequence

All hypoplactic monoids of rank ≥ 2 generate the same variety $\mathcal{V}_{hypo}$.

A Finite Basis for $\mathcal{V}_{hypo}$

Theorem (Cain, Malheiro, and Ribeiro (2022))

The variety $\mathcal{V}_{hypo}$ is finitely based. A basis consists of the identities:

$$\begin{aligned} (L) \quad & xyzxty = yxzxty \\ (M) \quad & xzxytx = xzyxtx \\ (R) \quad & xzytxy = xzytyx \end{aligned}$$

- These identities are minimal (cannot be omitted).
- The axiomatic rank of $\mathcal{V}_{hypo}$ is 4.

Representations: A Key Submonoid

- Define matrices over a semiring S:

$$I = \begin{bmatrix} 1_S & 1_S \\ 0_S & 0_S \end{bmatrix}, \quad J = \begin{bmatrix} 0_S & 0_S \\ 0_S & 1_S \end{bmatrix}$$

- Let: $K = JI, \quad L = IJ, \quad E =$ identity matrix
- Observations:
  - K is the 2×2 zero matrix.
  - $\mathcal{H} = \{E, I, J, K, L\}$ forms a submonoid of $\mathrm{UT}_2(S)$.
- Presentation of $\mathcal{H}$:

$$\mathcal{H} \cong \langle I, J \mid I^2 = I,\ J^2 = J,\ IJI = JI = JIJ \rangle$$

- $\mathcal{H}$ is isomorphic to $\mathcal{C}_3$ the monoid of order-preserving, extensive transformations of the 3-element chain.

Homomorphisms into $\mathcal{H}$

- For $i < j \in \mathbb{N}$, define a monoid homomorphism:

$$h_{ij} : \mathbb{N}^* \to \mathrm{UT}_2(S)$$

  with:
  - $i \mapsto I, \quad j \mapsto J$
  - $k \in (i,j) \mapsto K$
  - all other letters $\mapsto E$
- Image of h_{ij} is $\mathcal{H} = \{E, I, J, K, L\}$
- Computation for a word $w \in \mathbb{N}^*$:

$$h_{ij}(w) = \begin{cases} E & \text{if } w \text{ contains no symbols in the interval } [i,j]; \\ I & \text{if } w \text{ contains } i \text{ and no other symbols from } [i,j]; \\ J & \text{if } w \text{ contains } j \text{ and no other symbols from } [i,j]; \\ L & \text{if } w \text{ contains } i \text{ and } j \text{, no other symbols from } [i,j] \text{, and no scattered subword } ji \text{; and} \\ K & \text{otherwise.} \end{cases}$$

Faithful Representation of hypo

- For $n \geq 2$ and $1 \leq i < j \leq n$, the map h_{ij} factors through the hypoplactic monoid of rank n:

$$h_{ij} : \mathrm{hypo}_n \to \mathcal{H}$$

Theorem (Cain, Johnson, et al. (2022))

The hypoplactic monoid monoid hypo_n (respectively, hypo) embeds into a direct product of n copies (respectively, countably infinite copies) of $(\mathbb{N}, +)$ with $\binom{n}{2}$ copies (respectively, countably infinite copies) of the finite monoid $\mathcal{H}$.

Theorem (Cain, Johnson, et al. (2022))

Let S be a commutative unital semiring with zero containing an element of infinite multiplicative order. The hypoplactic monoid of rank n admits a faithful representation by upper triangular matrices of size n^2 over S having block-diagonal structure with largest block of size 2 (or size 1 if $n = 1$).

Variety Generated by the Hypoplactic Monoid

- Let J_k be the set of identities $u = v$ such that:

  u and v have the same scattered subwords of length $\leq k$.

- Notation:
  - $\mathcal{V}_{\mathrm{Comm}}$: variety of commutative monoids (balanced identities)
  - $\mathcal{V}_{\mathrm{B}}$: variety generated by the bicyclic monoid
  - $\mathcal{J}_k$: variety defined by J_k
- **Result of Volkov (2004):** $\mathcal{J}_k$ is generated by any of the following monoids:
  - Boolean unitriangular matrices of size $k + 1$
  - Reflexive binary relations on a $k + 1$ element set
  - Order-preserving and extensive transformations of a $k + 1$-element chain
- In particular: the monoid $\mathcal{H}$ (with 5 elements) generates $\mathcal{J}_2$

Variety Generated by the Hypoplactic Monoid

- Tishchenko (1980): Any monoid with 5 elements generates a finitely based variety $\Rightarrow \mathcal{J}_2$ is finitely based.
- Blanchet-Sadri (1993, 1994): $\mathcal{J}_k$ is finitely based if and only if $k \leq 3$.
- Basis for $\mathcal{J}_2$ includes:

$$xyxzx = xyzx, \quad xyxy = yxyx$$

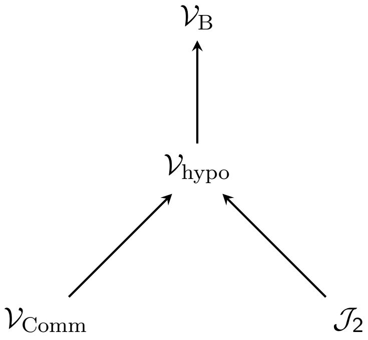
Theorem (Cain, Johnson, et al. (2022))

Let $n \geq 2$. The variety generated by the hypoplactic monoid of rank n is:

1. a proper subvariety of $\mathcal{V}_{\mathrm{B}}$;
2. the join of $\mathcal{V}_{\mathrm{Comm}}$ and $\mathcal{J}_2$;
3. equal to the variety generated by the infinite-rank hypoplactic monoid.

Varietal Inclusions

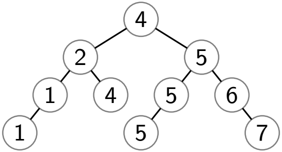
- The variety generated by the hypoplactic monoid sits strictly between two classical varieties:



$$\mathcal{V}_{\mathrm{hypo}} = \mathcal{V}_{\mathrm{Comm}} \vee \mathcal{J}_2$$

The Sylvester Monoid: Combinatorics

- Combinatorial objects are binary search trees (BST);
- For any word $u \in \mathcal{A}^*$, let $P_{sylv}(u)$ be the BST computed from u using the right strict insertion;
- Words u, v are congruent $(u \equiv_{sylv} v)$ if they produce the BST $P_{sylv}(u) = P_{sylv}(v)$.

$u = 5451761524 \mapsto P_{sylv}(u) =$ 

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Definition

The *sylvester monoid* is the quotient $\mathcal{A}^*/\equiv_{sylv}$.
The *sylvester monoid of rank n* is the factor monoid $\mathcal{A}_n^*/\equiv_{\mathrm{sylv}}$ (where $\equiv_{\mathrm{sylv}}$ is naturally restricted to $\mathcal{A}_n^*$).

Sylvester Congruence and precedences

- Let $a < b$ in a word $u \in \mathcal{A}$;
- We say u has a $b - a$ *right precedence* if:
  - when reading u from right to left, b occurs before the first occurrence of a; and
  - for any letter c in u such that $a < c < b$, the symbol c does not occur before the first occurrence of a.
- The number of occurrences of b before the first occurrence of a is the index of the right precedence.

Example

- The word 3123 has a $2 - 1$ and a $3 - 2$ right precedence, both of index 1, however, it does not have a $3 - 1$ right precedence, since 2 occurs before the first occurrence of 1;
- the word 2313 has a $3 - 1$ right precedence of index 1 and a $3 - 2$ right precedence of index 2;
- the word 3132 has a $2 - 1$ right precedence of index 1, and does not have a $3 - 1$ right precedence.

Sylvester Congruence and precedences

Precedence Condition

Two words are equivalent in the sylvester monoid if and only if they share:

- The same content (number of occurrences of each letter)
- The same right precedences.

This characterization underpins the embedding results.

Embedding into Products of Rank-2 Monoids

Theorem (Cain, Malheiro, and Ribeiro (2023))

For $n \geq 2$, the sylvester monoid of rank n embeds into a direct product of copies of the rank-2 sylvester monoid.

- The embedding strategy is similar to the hypoplactic case, via homomorphisms between 2-letter alphabets.
- This implies that the identities of the rank-2 case characterize the entire family.
- All sylvester monoids of rank ≥ 2 satisfy the same identities.

Characterization of Identities in *sylv*

Theorem (Cain, Malheiro, and Ribeiro (2023))

An identity $u = v$ is satisfied by the sylvester monoid if and only if:

- $u = v$ is balanced;
- For any variables x and y, the number of occurrences of y before the first occurrence of x in u, is equal to the number of occurrences of y before the first occurrence of x in v.

Example

- $xyxy = yxxy$ is the shortest up to to equivalence.

A Finite Basis for $\mathcal{V}_{sylv}$

Theorem (Cain, Malheiro, and Ribeiro (2023))

The variety $\mathcal{V}_{sylv}$ is finitely based. A basis consists of the identity:

$$(L) \quad xyzxty = yxzxty$$

- This identity reflects the structural invariance under BST insertion.
- Axiomatic rank is 4.

Representations: a key submonoid

- Define upper triangular matrices over a semiring S:

$$I = \begin{bmatrix} 1_S & 1_S \\ 0_S & 0_S \end{bmatrix}, \quad J = \begin{bmatrix} 1_S & 0_S \\ 0_S & \alpha \end{bmatrix},$$

  where α is an element of infinite multiplicative order in the commutative semiring S.
- We have I idempotent, $JI = I$, and for any $k \in \mathbb{N}_0$,

$$J^k = \begin{bmatrix} 1_S & 0_S \\ 0_S & \alpha^k \end{bmatrix} \text{ and } IJ^k = \begin{bmatrix} 1_S & \alpha^k \\ 0_S & 0_S \end{bmatrix}.$$

- Let $\mathcal{M}$ be the submonoid of $\mathrm{UT}_2(S)$ generated by $\{I, J\}$.
- Presentation of $\mathcal{M}$:

$$\langle I, J \mid JI = I = I^2 \rangle.$$

Homomorphisms h_{ij} for Sylvester

- Consider the monoid homomorphism $s : [2]^* \to \mathrm{UT}_2(S)$ defined by $1 \mapsto I$ and $2 \mapsto J$. Note that the image of s is $\mathcal{M}$, specifically:

$$s(w) = \begin{cases} J^k & \text{if } w = 2^k \\ IJ^k & \text{if } w = w'12^k. \end{cases}$$

- The homomorphism $s : [2]^* \to \mathrm{UT}_2(S)$ factors to give a homomorphism from sylv_2 to the monoid $\mathcal{M}$.

Theorem (Cain, Johnson, et al. (2022))

The sylvester monoid sylv_n (respectively, sylv) embeds into a direct product of n copies (respectively, countably infinite copies) of $(\mathbb{N}_0, +)$ and $\binom{n}{2}$ copies of the (infinite) monoid $\mathcal{M}$.

Faithful Matrix Representation of sylv_n

Theorem (Cain, Johnson, et al. (2022))

Let S be a commutative unital semiring with zero containing an element of infinite multiplicative order. The sylvester monoid of rank n admits a faithful representation by upper triangular matrices of size n^2 over S having block-diagonal structure with largest block of size 2 (or size 1 if $n = 1$).

- Each block corresponds to a 2×2 representation from $\mathcal{M}$.
- The representation reflects BST insertion structure.

Variety Generated by the Sylvester Monoid

Theorem (Cain, Johnson, et al. (2022))

Let $n \geq 2$. Then the variety of monoids generated by the sylvester monoid of rank n is:

1. a proper subvariety of **B**;
2. the variety generated by the (infinite) monoid $\mathcal{M}$;
3. not contained in the join of **Comm** and any variety generated by a finite monoid;
4. equal to the variety generated by the (infinite-rank) sylvester monoid.

Faithful Representations of Plactic-like Monoids

Theorem (Cain, Johnson, et al. (2022))

The hypoplactic, stalactic, taiga, sylvester, Baxter, and right patience sorting monoids of rank n admit faithful representations as monoids of upper triangular matrices over:

- The tropical semiring
- Any field of characteristic 0

- The algebraic and combinatorial structures underpin these representations.
- Matrices are upper triangular with entries reflecting relative orderings and multiplicities.

Structure of the Image of the Representation and Varietal Consequences

- The image of the representation forms a submonoid of $M_n(S)$.
- Dimension and structure determined by the alphabet size and insertion behavior.
- Representations allow reduction of identity questions to matrix identities.
- The varieties generated by hypoplactic, stalactic, taiga monoids are proper subvarieties of the plactic variety.

✓THANK YOU FOR YOUR ATTENTION

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