Using generalized decision ensembles to solve multi-class decision problems

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January 2021

Generalized decision function (1)

- Let $(U, A \cup D)$ be a data table with distinguished decision attribute(s) in D.
- For $B \subseteq A$ we define function $\partial_{D/B}: U \to 2^{V_D}$ such that
$$\partial_{D/B}(u) = \{D(u'): B(u') = B(u)\}$$
whereby V_D denotes the set of all (vectors of) values of D which occur in U and $B(u)$ denotes the vector of values, which $u \in U$ takes on B.
- We say that $B \subseteq A$ is a ∂-superreduct, if and only if
$$\forall_{u \in U} \left(\partial_{D/A}(u) = \partial_{D/B}(u) \right)$$

Generalized decision function (2)

- We do not need to assume a fixed set of decisions D. For $X, Y \subseteq A$ we can consider function $\partial_{X/Y}: U \rightarrow 2^{V_X}$. For $X, Y, Z \subseteq A$ we can consider condition

$$\forall_{u \in U} \left(\partial_{X/Y}(u) = \partial_{X/Y \cup Z}(u) \right) \qquad (*)$$

- We can equivalently consider $\partial_{X/Y}: V_Y \rightarrow 2^{V_X}$ such that

$$\partial_{X/Y}(y) = \{x \in V_X : x \wedge y\}$$

whereby $x \wedge y$ means that x and y occur together in U

- We can then equivalently rewrite (*) as follows:

$$\forall_{y \in V_Y} \forall_{z \in V_Z} \left(y \wedge z \Rightarrow \partial_{X/Y}(y) = \partial_{X/Y \cup Z}(yz) \right)$$

Multivalued dependency (MVD)

- For $(U, A \cup D)$ and $B \subseteq A$, the MVD $B \twoheadrightarrow D$ holds, if and only if:

  If two tuples of $(U, A \cup D)$ agree on all attributes of B, then their components in D may be swapped, and the result will be two tuples that are also in $(U, A \cup D)$.

- **Proposition** $B \subseteq A$ is a ∂-superreduct, if and only if $B \twoheadrightarrow D$ holds.

- For (U, A) and $X, Y, Z \subseteq A$, $X \cup Y \cup Z \neq A$, we can have the embedded multivalued dependency $Y \twoheadrightarrow_Z X$ which is equivalent to $\partial_{X/Y} = \partial_{X/Y \cup Z}$

Discernibility property of ∂

- **Proposition** $B \subseteq A$ is a ∂-superreduct in $(U, A \cup D)$, if and only if

$$\forall_{u,u' \in U} \left(\partial_{D/A}(u) \neq \partial_{D/A}(u') \Rightarrow B(u) \neq B(u')\right)$$

- In the nomenclature of relational databases this means that $B \twoheadrightarrow D$, if and only if $B \rightarrow \partial_{D/A}$ whereby $\rightarrow$ denotes the functional dependency.

- Interestingly, I couldn’t find such a fact in the literature on databases.

- By the way, is the name „discernibility property” the best choice here?

Relational semi-graphoids

- Let us define conditional independence of X from Z subject to Y as follows:

  $$\forall_{x \in V_X} \forall_{y \in V_Y} \forall_{z \in V_Z} (P(x, y) > 0 \wedge P(y, z) > 0 \Rightarrow P(x, y, z) > 0)$$

  which means that

  the range of values permitted for X is not restricted by the choice of Z, once Y is fixed.

- **Proposition** The above statement holds, if and only if there is $\partial_{X/Y} = \partial_{X/YZ}$ Therefore, let's denote it as $I_\partial(X|Y|Z)$.

- By the way, if $X \cup Y \cup Z = A$, then we talk about saturated independences.

Symmetry of generalized decisions

- **Proposition** The following statements are equivalent to each other:

$$\forall_{u\in U}\left(\partial_{X/Y}(u) = \partial_{X/YZ}(u)\right) \qquad \forall_{u\in U}\left(\partial_{Z/Y}(u) = \partial_{Z/XY}(u)\right)$$

$$\forall_{u\in U}\left(\partial_{XZ/Y}(u) = \partial_{X/Y}(u) \times \partial_{Z/Y}(u)\right)$$

- The following forms are useful to think about the above statements:

$$\forall_{y\in V_Y}\forall_{z\in V_Z}\left(y \wedge z \Rightarrow \partial_{X/Y}(y) = \partial_{X/YZ}(y,z)\right)$$

$$\forall_{x\in V_X}\forall_{y\in V_Y}\forall_{z\in V_Z}(y \wedge z \Rightarrow (x \wedge y \Rightarrow x \wedge y \wedge z))$$

- Given the symmetry, one may write $I_\partial(X;Z|Y)$ instead of $I_\partial(X|Y|Z)$.

Generalized decision ensembles

- We want to use collections of the smallest subsets $B_1 \dots B_m \subseteq A$ such that

$$\forall_{u \in U}\left(\partial_{D/A}(u) = \bigcap_{i=1}^{m} \partial_{D/B_i}(u)\right)$$

- Consider

$$B_1 = \{a_1, a_2, a_3\}$$
$$B_2 = \{a_3, a_4, a_5\}$$

a_1	a_2	a_3	a_4	a_5	D
No	No	No	No	No	green
No	No	Yes	No	Yes	green
No	No	Yes	No	No	red
No	Yes	No	Yes	No	red
No	Yes	No	No	No	blue
Yes	No	Yes	No	Yes	blue

$$(a_1 = No \wedge a_2 = Yes \wedge a_3 = No) \Rightarrow (D = blue \vee D = red)$$
$$(a_3 = No \wedge a_4 = No \wedge a_5 = No) \Rightarrow (D = blue \vee D = green)$$

Generalized decision decomposition (1)

- Consider $B, C \subseteq A$, $B \cup C = A$, such that $\partial_{D/A} = \partial_{D/B} \cap \partial_{D/C}$
  Could such condition have something in common with $I_\partial(B; C|D)$?

- **Proposition** If $I_\partial(X; Y|Z)$ then $\forall_{u \in U} \left(\partial_{Z/XY}(u) = \partial_{Z/X}(u) \cap \partial_{Z/Y}(u)\right)$
  But not conversely.

- **Proof** Recall that $I_\partial(X; Y|Z)$ can be rewritten as
  $$\forall_{x \in V_X} \forall_{y \in V_Y} \forall_{z \in V_Z} \big((x \wedge z) \wedge (y \wedge z) \Rightarrow (x \wedge y \wedge z)\big) \qquad (*)$$
  On the other hand, our decomposition condition is equivalent to
  $$\forall_{x \in V_X} \forall_{y \in V_Y} \forall_{z \in V_Z} \big((x \wedge y) \wedge (x \wedge z) \wedge (y \wedge z) \Rightarrow (x \wedge y \wedge z)\big) \qquad (**)$$

Generalized decision decomposition (2)

- **Proposition** The following statements are equivalent to each other:

$$\forall_{u\in U}\left(\partial_{Z/XY}(u) = \partial_{Z/X}(u) \cap \partial_{Z/Y}(u)\right)$$
$$\forall_{u\in U}\left(\partial_{Y/XZ}(u) = \partial_{Y/X}(u) \cap \partial_{Y/Z}(u)\right)$$
$$\forall_{u\in U}\left(\partial_{X/YZ}(u) = \partial_{X/Y}(u) \cap \partial_{X/Z}(u)\right)$$

Given this kind of „3-symmetry", we denote the above as $I_\partial(X;Y;Z)$.

- $I_\partial(X;Y;Z) \nRightarrow I_\partial(X;Y|Z)$
  $\nRightarrow I_\partial(X;Z|Y)$
  $\nRightarrow I_\partial(Y;Z|X)$

X	**Y**	**Z**
No	No	No
No	No	Yes
No	Yes	No
Yes	No	No

Stronger decomposition/synthesis

- Consider the following constraint:

$$\forall_{x\in V_X}\forall_{y\in V_Y}\begin{cases}(x\wedge y)\Rightarrow\left(\partial_{Z/XY}(xy)=\partial_{Z/X}(x)\cap\partial_{Z/Y}(y)\right)\\ \neg(x\wedge y)\Rightarrow\left(\partial_{Z/X}(x)\cap\partial_{Z/Y}(y)=\emptyset\right)\end{cases}$$

- **Proposition** The above is equivalent to $I_\partial(X;Y|Z)$.

- **Proof** Let us rewrite the second above component as

$$\forall_{x\in V_X}\forall_{y\in V_Y}\forall_{z\in V_Z}\left(\neg(x\wedge y)\Rightarrow\left(\neg(x\wedge z)\vee\neg(y\wedge z)\right)\right)$$

Together with (*), this becomes to be equivalent to (**).

Thank You!

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January 2021