Universal autohomeomorphisms of $\mathbb{N}^*$

Jan van Mill

University of Amsterdam

Warsaw
November 17, 2021

- *All spaces under discussion here are Tychonoff and all results are joint work with Klaas Pieter Hart.*
- There are at least two notions of universality of autohomeomorphisms of topological spaces.
- Let h be an autohomeomorphism of the space X. We say that h is *universal* for a class of pairs (Y, g), where Y is a space and g is an autohomeomorphism of Y, there exists a continuous surjection $s\colon X \to Y$ such that the diagram

$$\begin{array}{ccc} X & \xleftarrow{\ h\,\approx\ } & X \\ {\scriptstyle \exists\, s}\big\downarrow & & \big\downarrow{\scriptstyle \exists\, s} \\ \forall\, Y & \xleftarrow[\ \forall\, g\,\approx\ ]{} & \forall\, Y \end{array}$$

commutes.

- Of course, this definition only makes sense if for the pairs (Y, g) that we consider it is true that Y is a continuous image of X.
- Our basic space of interest is $\mathbb{N}^*$, the Čech-Stone remainder $\beta\mathbb{N} \setminus \mathbb{N}$ of the natural numbers $\mathbb{N}$ endowed with the discrete topology.
- The underlying set of $\beta\mathbb{N}$ is the set of all ultrafilters p in $\mathscr{P}(\mathbb{N})$. Its topology is generated by the collection $\{A^+ : A \in \mathscr{P}(\mathbb{N})\}$, where $A^+ = \{p \in \beta\mathbb{N} : A \in p\}$.
- The space $\mathbb{N}$ is identified with the set of all *fixed* ultrafilters in $\mathscr{P}(\mathbb{N})$.
- The spaces $\beta\mathbb{N}$ and $\mathbb{N}^*$ are among the best studied spaces in set theory and topology.
- The above type of universality for $\mathbb{N}^*$ was thoroughly investigated by Will Brian, *Universal flows of* $\mathscr{P}(\omega)/\mathrm{fin}$, Israel J. Math. 233 (2019), 453-500.

- We are interested in the following *dual* notion of universality:
- An autohomeomorphism h on a space X is *universal* for a class of pairs (Y, g), where Y is a space and g is an autohomeomorphism of Y, if for every such pair there is an embedding $e\colon Y \to X$ such that the diagram

$$\begin{array}{ccc} X & \xleftarrow{h\,\approx} & X \\ \exists\, e \uparrow & & \uparrow \exists\, e \\ \forall\, Y & \xleftarrow[\forall\, g\,\approx]{} & \forall\, Y \end{array}$$

  commutes; that is, h extends the copy of g on $e(Y)$.
- Of course, this definition only makes sense if for the pairs (Y, g) that we consider it is true that Y can be embedded in X.

- Consider $X = K^{\mathbb{Z}}$, where K is the Cantor set 2^ω, and let $h\colon X \to X$ be the shift mapping $h(x)_n = x_{n+1}$, $x \in X, n \in \mathbb{Z}$. Then (X, h) is universal for the class of all pairs (Y, f), where Y is zero-dimensional, separable metrizable, and $f\colon Y \to Y$ is an autohomeomorphism.
- This is a folklore result, and the proof is obvious.
- We may assume that $Y \subseteq 2^\omega$. Consider the embedding $e\colon Y \to K^X$ defined by

$$e(x) = (\cdots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \cdots,).$$

  Then, clearly, $h \circ e = e \circ f$.
- The same proof works for Cantor cubes 2^τ, $\tau \geq \omega$, and zero-dimensional spaces of weight at most τ.
- And, the same proof works for Tychonoff cubes $\mathbb{I}^\tau$, $\tau \geq \omega$, and Tychonoff spaces of weight at most τ.

- So there are many universal autohomeomorphisms on various spaces. Related is the classical work on linearizations of group actions in Banach spaces.
- *Does $\mathbb{N}^*$ have a universal autohomeomorphism?*
- As we mentioned above, Brian dealt with the first case of universality that we discussed. We will deal with the second case.
- The question that we study is: is there an autohomeomorphism h of $\mathbb{N}^*$ such that for any pair (Y, g), where Y is a closed subspace of $\mathbb{N}^*$ with autohomeomorphism g, there is an embedding $e\colon Y \to \mathbb{N}^*$ such that the diagram

$$\begin{array}{ccc} \mathbb{N}^* & \xleftarrow{\;h\,\approx\;} & \mathbb{N}^* \\ {\scriptstyle \exists\, e}\uparrow & & \uparrow{\scriptstyle \exists\, e} \\ \forall\, Y & \xleftarrow[\forall\, g\,\approx]{} & \forall\, Y \end{array}$$

commutes; that is, h extends the copy of g on $e(Y)$.

- Shelah proved that it is consistent that all autohomeomorphisms of $\mathbb{N}^*$ are trivial (see his 1998 book on proper forcing).
- This means that for every autohomeomorphism f of $\mathbb{N}^*$ there are finite subsets E and F of $\mathbb{N}$, and a bijection $\pi\colon \mathbb{N}\setminus E \to \mathbb{N}\setminus F$ such that on $\mathbb{N}^*$, f coincides with the Stone extension $\beta\pi$ of π.
- In particular this means that there are only $\mathfrak{c}$ many autohomeomorphisms of $\mathbb{N}^*$.
- The fixed-point set of any trivial autohomeomorphism is *clopen*. Indeed, let f, E, F and π be as above. Then the fixed-point set of $\beta\pi \restriction \mathbb{N}^*$ coincides with $A^+ \cap \mathbb{N}^*$, where $A = \{n \in \mathbb{N} : \pi(n) = n\}$.
- In Shelah's model universal autohomeomorphisms do not exist. To prove that, all we need to show is that there is a closed subspace of $\mathbb{N}^*$ with an autohomeomorphism whose fixed-point set is not (relatively) clopen.

- This is easy. We let L be the ordinal $\omega_1 + 1$ endowed with its G_δ-topology. Thus all points other than ω_1 are isolated and the neighbourhoods of ω_1 are exactly the co-countable sets that contain it. Then L is a P-space of weight $\aleph_1$ and it is known that βL can be embedded in $\mathbb{N}^*$. We define $f\colon L \to L$ such that ω_1 is the only fixed point of βf. Split ω_1 into two disjoint uncountable sets, say E and F. Let $\pi\colon E \to F$ be a bijection. Now put

$$
\begin{aligned}
f(\omega_1) &= \omega_1, \\
f(e) &= \pi(e) \qquad\qquad (e \in E), \\
f(f) &= \pi^{-1}(f) \qquad\qquad (f \in F).
\end{aligned}
$$

  Then f is an involution, ω_1 is its only fixed-point, and it is not difficult to show that ω_1 is the only fixed-point of βf.

- Hence there are models of set theory in which $\mathbb{N}^*$ has no universal autohomeomorphism.
- The situation dramatically changes under the Continuum Hypothesis (abbreviated: CH).
- Walter Rudin showed that under CH, $\mathbb{N}^*$ has $2^{\mathfrak{c}}$ many homeomorphisms. Hence in the presence of CH there is more chance than in the Shelah model that one of them is universal.

Theorem

$\mathbb{N}^*$ *has a universal homeomorphism under* CH.

- For the proof, we need CH many, many times.

- The following well-known result is due to Parovičenko:

Theorem

Assume CH. Then $\mathbb{N}^*$ is topologically the unique space X with the following properties:

1. *X is a compact zero-dimensional F-space of weight $\mathfrak{c}$.*
2. *Nonempty G_δ's in X have infinite interior.*

- A compact space X is an *F-space* if disjoint open F_σ-subsets of X have disjoint closures.
- (So this is a (very) weak form of extremal disconnectivity: the closure of every open subset is open.)
- It is known that Parovičenko's characterization of $\mathbb{N}^*$ implies CH (van Douwen & van Mill).

- Results in the same spirit that we will use are that under CH:
  1. The closed subspaces of $\mathbb{N}^*$ are characterized as the class of all compact zero-dimensional F-spaces of weight $\mathfrak{c}$ (Louveau).
  2. Every closed subspace of $\mathbb{N}^*$ can be re-embedded as a nowhere dense closed P-set (Balcar, Frankiewicz).
  3. Every homeomorphism between closed nowhere dense P-sets in $\mathbb{N}^*$ can be extended to an autohomeomorphism of $\mathbb{N}^*$ (van Douwen & van Mill).
- Let Aut denote the group of autohomeomorphisms of $\mathbb{N}^*$.
- Let $\sigma\colon \mathrm{Aut} \times \mathbb{N}^* \to \mathbb{N}^*$ be the natural action. That is,

$$\sigma(f,p) = f(p).$$

- This action is continuous when Aut carries the compact-open topology.
- So it is also continuous when Aut carries the stronger G_δ-topology.

- Define an autohomeomorphism $h\colon \mathrm{Aut} \times \mathbb{N}^* \to \mathrm{Aut} \times \mathbb{N}^*$ by

$$h(f, p) = (f, f(p)).$$

- Now if X is a closed subset of $\mathbb{N}^*$ and $g\colon X \to X$ is an autohomeomorphism then we can re-embed X as a nowhere dense closed P-set and we can then find an $f \in \mathrm{Aut}$ such that $f \restriction X = g$. We transfer this embedded copy of X to $\{f\} \times \mathbb{N}^*$ in $\mathrm{Aut} \times \mathbb{N}^*$; for this copy of X we then have $h \restriction X = g$. It follows that h satisfies the universality condition. (We used CH already twice.)
- So $\mathrm{Aut} \times \mathbb{N}^*$ contains witnesses of all autohomeomorphism's of compact subspaces of $\mathbb{N}^*$.
- But $\mathrm{Aut} \times \mathbb{N}^*$ is not compact, it is not $\mathbb{N}^*$!
- By a result of Negrepontis, $\mathrm{Aut} \times \mathbb{N}^*$ is an F-space, being the product of a P-space and a compact F-space.
- (This explains why we used the G_δ-topology on Aut.)

- The weight of Aut $\times\ \mathbb{N}^*$ is obviously $\mathfrak{c}$.
- Under CH, Aut $\times\ \mathbb{N}^*$ is *ultraparacompact* ($=$ every open cover has a disjoint (cl)open refinement).
- Using this, it is not difficult to construct a Boolean subalgebra $\mathbb{B}$ of the algebra of clopen subsets of Aut $\times\ \mathbb{N}^*$ that is closed under h and h^{-1}, of cardinality $\mathfrak{c}$, and that has the property that for every pair of countable subsets A and B of $\mathbb{B}$ such that $a \cap b = \emptyset$ whenever $a \in A$ and $b \in B$ there is a $c \in \mathbb{B}$ such that $a \subseteq c$ and $b \cap c = \emptyset$ for all $a \in A$ and $b \in B$.
- The Stone space $\mathrm{st}(\mathbb{B})$ of $\mathbb{B}$ is then a compactification of Aut $\times\ \mathbb{N}^*$ that is a compact zero-dimensional F-space of weight $\mathfrak{c}$, with an autohomeomorphism $\bar{h}$ that extends h. (β(Aut $\times\ \mathbb{N}^*$) is too big.)
- We embed $\mathrm{st}(\mathbb{B})$ into $\mathbb{N}^*$ as a nowhere dense P-set and extend $\bar{h}$ to an autohomeomorphism H of $\mathbb{N}^*$.
- Then H is the desired universal homeomorphism of $\mathbb{N}^*$.

Biography

- J. van Mill, *An introduction to $\beta\omega$*, Handbook of Set-Theoretic Topology (K. Kunen and J.E. Vaughan, eds.), North-Holland, Amsterdam, (1984), pp. 503-567.
- E. K. van Douwen, J. van Mill, The homeomorphism extension theorem for $\beta\omega \setminus \omega$, Ann. New York Acad. Sci., 704 (1993), 345-350.
- K. P. Hart and J. van Mill, *Universal autohomeomorphisms of $\mathbb{N}^*$*, 2021, to appear in Proc. Amer. Math. Soc.

- https://staff.fnwi.uva.nl/j.vanmill/
- Thank you for listening! ☺