Transport equation - one equation, plenty of methods

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The Fields Medal – officially known as International Medal for Outstanding Discoveries in Mathematics, is a prize awarded to two, three, or four mathematicians not over 40 years of age at each International Congress of the International Mathematical Union, a meeting that takes place every four years. The Fields Medal is often viewed as the greatest honour a mathematician can receive.

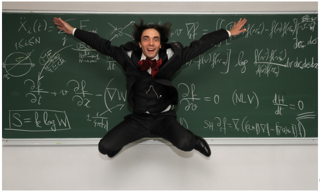
International Congress of Mathematicians

Rio de Janeiro 2018: Akshay Venkatesh, Peter Scholze, Alessio Figalli and Caucher Birkar.

Powiada robaczek: I dziadek, i babka,
I ojciec, i matka jadali wciąż jabłka

Fields heritage


Pierre-Louis Lions

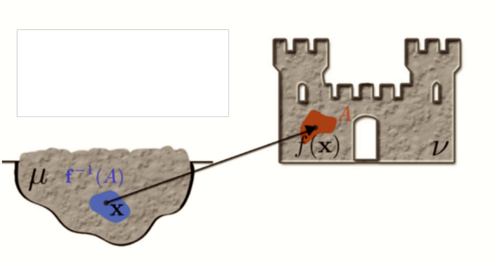

Cédric Villani

Alessio Figalli


Optimal transport

Alessio Figalli has been awarded the Fields medal for *"his contributions to the theory of optimal transport and his applications in partial differential equations, metric geometry and probability"*.

Optimal transport – the idea originated in the 18th century, when Gaspard Monge (1746-1818) - a mathematician working for Napoleon Bonaparte tried to find the most efficient way to build a network of earthen fortifications.



Alessio Figalli works in the broad areas of calculus of variations and partial differential equations, including:

- optimal transportation,
- Monge-Ampére equations,
- functional and geometric inequalities,
- elliptic PDEs of local and non-local type,
- free boundary problems,
- Hamilton-Jacobi equations,
- transport equations with rough vector-fields,
- random matrix theory.

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Transport equation

Given a vector field $b : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ find a solution $u : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ that solves

$$\partial_t u(t,x) + b(t,x)\nabla u(t,x) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n$$

$$u(0,x) = u_0(x) \quad \text{in } \mathbb{R}_+$$

If b satisfies

$$|b(t,x) - b(t,y)| \leq C|x-y|, \quad |b(t,x)| \leq c(1+|x|)$$

we can define the so-called characteristis associated with the transport equation

$$\frac{d}{dt}\phi(t,y) = b(t,\phi(t,y))$$

$$\phi(0,y) = y.$$

They are defined for all times $t \in \mathbb{R}$ and the mapping $y \mapsto \phi(t,y)$ is a diffeomorphism on $\mathbb{R}^n$.

Transport equation

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Theorem

Under the assumption that $u_0 \in C^1(\mathbb{R}^n)$ there exists a unique solution $u \in C^1(\mathbb{R}_+ \times \mathbb{R}^n)$ which is constant along the characteristics i.e.,

$$u(t,\phi(t,y)) = u_0(y)$$

for all $t \in \mathbb{R}_+, y \in \mathbb{R}^d$.

Renormalized solutions (DiPerna - Lions)

It is possible to improve the above results and optimize the assumptions on the velocity field b while keeping existence and uniqueness.

Theorem

Let b be bounded, ∇b be integrable and $\mathrm{div} b$ be bounded; then there exists a unique bounded solution to the equation

$$\partial_t u(t,x) + b(t,x)\nabla u(t,x) = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n$$

$$u(0,x) = u_0(x) \in L^\infty(\mathbb{R}^n)$$

Notion of solution

A bounded function $u(t,x)$ is called a weak solution if it satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} u(t,x)[\partial_t \varphi(t,x) + \mathrm{div}(b(t,x)\varphi(t,x))] dxdt = - \int_{\mathbb{R}^n} \varphi(0,x) u_0(x) dx$$

for all smooth test functions φ with compact support.

Renormalized solutions (DiPerna - Lions)

A solution is called a renormalized solution if not only u solves the transport equation

$$\partial_t u(t,x) + b(t,x)\nabla u(t,x) = 0$$

but also $\beta(u)$ solves the same transport equation

$$\partial_t \beta(u(t,x)) + b(t,x)\nabla \beta(u(t,x)) = 0$$

where $\beta \in C^1(\mathbb{R})$.

Transport equation in a conservative form

We can also consider the conservative form of the transport equation

$$\partial_t u(t,x) + \mathrm{div}[b(t,x)u(t,x)] = 0 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n$$

$$u(0,x) = u_0(x) \quad \text{in } \mathbb{R}_+$$

where

$$\mathrm{div}[b(t,x)u(t,x)] = \sum_{i=1}^{n} \partial_{x_i}[b_i(t,x)u(t,x)]$$

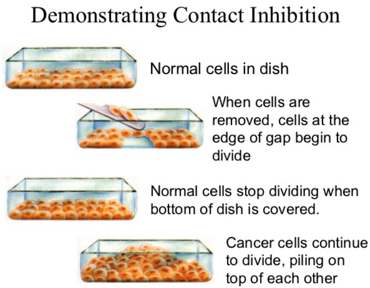
Transport equation in mathematical models

- Fluid mechanics (compressible Navier-Stokes equations, Euler equations)
- Biological models (ecology, cell division)
- Kinetic models

Let's play with various methods on our example

Our example

In the natural process of cell growth, one can observe cases where cells closely approach and come into contact with each other. This phenomenon is referred to as contact inhibition of growth between cells. It includes the effect of pushing cells away from overcrowded regions so that each cell moves in the direction of lower cell density.



Models of tissue growth

- Growth is limited by the competition for space and tissues are considered as multiphasic fluids (the phases could be interstitial water, healthy and tumor cells, extracellular matrix). The lower pressure that prevents cell multiplication by contact inhibition is called a homeostatic pressure.

The simplest case

In the simplest model the cell population density $n(x,t)$ evolves under pressure forces and cell multiplication according to the equation

$$\partial_t n - \mathrm{div}[n \nabla p] = nF(p),$$

where p is the pressure. Pressure-limited growth is described by the term $F(p)$, which typically satisfies

$$F'(p) < 0 \quad \text{and} \quad F(p_H) = 0$$

for some $p_H > 0$ (the homeostatic pressure). The pressure is assumed to be a given increasing function of the density.

A two-species model

We consider the following model with two cell populations which react similarly to the pressure in terms of their motion but undergo different growth/death rates.

$$\begin{cases} \partial_t n_1 - \mathrm{div}[n_1 \nabla p] = n_1 F_1(p) + n_2 G_1(p), & x \in \mathbb{R}^d, \ t \geq 0, \\ \partial_t n_2 - \mathrm{div}[n_2 \nabla p] = n_1 F_2(p) + n_2 G_2(p), \end{cases}$$

with

$$n := n_1 + n_2, \qquad p(n) = n^\gamma, \quad \gamma > 1.$$

P. Gwiazda, B. Perthame, A. Ś.-G. A two species hyperbolic-parabolic model of tissue growth, arXiv:1809.01867

Connections to porous medium equation

This type of equations have a very deep connection with the physical study of multiphase flows in porous media. More precisely, one considers a mixture of N immiscible (incapable of being mixed) phases. An equation for the conservation of the i-th volume phase reads:

$$\partial_t s_i + \mathrm{div}_x(\mathbf{v_i} s_i) = 0,$$

where $\mathbf{v_i}$ is the filtration speed of the i-th phase and s_i is its fraction in total volume. By Darcy's law, up to constants, one considers $\mathbf{v_i} = \nabla p_i$ where p_i is the pressure.

Different methods to attack the problem - renormalized solutions

- A tumor growth model (in multi-dimensional case) which describes the growth of two types of cell population densities with contact inhibition was considered in

M. Bertsch, D. Hilhorst, H. Izuhara, and M. Mimura. A nonlinear parabolic-hyperbolic system for contact inhibition of cell-growth. *Differ. Equ. Appl.*, 2012.

  - They used the theory of renormalized solutions of transport equation. The initial data $n_1^0 + n_2^0$ are bounded away from zero, thus vacuum does not appear.
  - Is is also shown that if the two populations are initially segregated - i.e. disjoint supports of their densities - this property remains true at all later times.This segregation property reflects the contact inhibition mechanism for the growth of the cells.

Different methods to attack the problem - renormalized solutions

- Bertsch et al. apply the results of Ambrosio, Bouchut and De Lellis on transport equations and regular Lagrangian flows – theory á la DiPerna-Lions for transport equations and ordinary differential equations, in which the usual assumption of boundedness of the divergence of the coefficients is replaced by a control on the Jacobian (or by the existence of a solution of the continuity equation which is bounded away from 0 and ∞).
- Observe that $\mathrm{div}\nabla p = \Delta p$. Is it possible that Δp is bounded?

The main observation is that, while our problem is of hyperbolic nature, we can take advantage of informations coming from the parabolic equation on n

$$\partial_t n - \mathrm{div}[n\nabla p] = n_1 F(p) + n_2 G(p) =: nR(c_1, c_2, p),$$

where we define

$$c_i = \frac{n_i}{n} \leq 1 \quad \text{and} \quad c_i(x,t) = 0 \;\; \text{when} \;\; n(x,t) = 0,$$

$$R = c_1 F(p) + c_2 G(p) \in L^\infty.$$

Next, multiplying equation for n with $p'(n)$, we compute that p satisfies porous medium equation

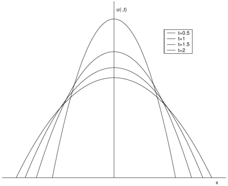
$$\partial_t p - |\nabla p|^2 - \gamma p \Delta p = \gamma p R.$$

Excursion to PME

Recall that a fundamental example of solution was obtained around 1950 by Zel'dovich, Kompaneets and Barenblatt, who found and analyzed a solution representing heat release from a point source. This solution has the explicit formula

$$U(x,t) = t^{-\alpha}\left(c - k|x|^2 t^{-2\beta}\right)_+^{\frac{1}{m-1}}$$

with $\alpha = \frac{d}{d(m-1)+2}, \beta = \frac{\alpha}{d}, k = \frac{\alpha(m-1)}{2md}, c > 0$ - arbitrary constant.



Excursion to PME

The name source-type solution comes from the fact that it takes as initial data a Dirac mass: as $t \to 0$ we have $U(x,t) \to M\delta_x$, where M depends just on constants.
The source solution has compact support in space for every fixed time.
Since this framework includes the Barenblatt solutions (in case without production term), we know that Δp may be a singular measure.

Different methods to attack the problem - optimal transport

An existence of weak solutions for a wide class of initial data without restriction about their supports or their positivity in one-dimensional case was shown in

J. A. Carrillo, S. Fagioli, F. Santambrogio, and M. Schmidtchen. Splitting schemes & segregation in reaction-(cross-)diffusion systems, to appear in *SIAM J. Math. Anal.*

The authors we proposed a variational splitting scheme combining ODEs with methods from optimal transport.

Different methods to attack the problem - optimal transport

In the absence of reactions, the system leads to the nonlinear diffusion equation for n

$$\partial_t n = \partial_x (n \partial_x p(n))$$

This equation can be understood as a gradient flow of a certain energy functional in the space of probability measures equipped with the $2-$Wasserstein metric. The idea of such a scheme is to recursively construct a sequence by solving a minimisation problem in a certain metric space (X, d).

p-Wasserstein metric

Definition

For probability measures ν, μ with finite pth moment we define

$$\mathbf{W}(\nu,\mu)^p := \inf\{ \int\limits_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p d\gamma(x,y)\},$$

where the infimum is taken over all transport plans γ, which are probability measures on the product space $\mathbb{R}^n \times \mathbb{R}^n$

p-Wasserstein metric and optimal transport problem

- For a distribution of mass $\mu(x)$ on a space X, we wish to transport the mass in such a way that it is transformed into the distribution $\nu(x)$ on the same space; transforming the 'pile of earth' μ to the pile ν. This problem only makes sense if the pile to be created has the same mass as the pile to be moved.
- Assume also that there is given some cost function $c(x, y)$ that gives the cost of transporting a unit mass from the point x to the point y.
- A transport plan to move μ into ν can be described by a function $\gamma(x, y)$, which gives the amount of mass to move from x to y.
- The plan γ is not unique; the optimal transport plan is the plan with the minimal cost out of all possible transport plans.
- If the cost of a move is simply the distance between the two points, then the optimal cost is identical to the definition of the W_1 distance.

Optimal transport approach

In this recent study of Carrillo et al., the authors exploit this structure. However a difficulty follows from the fact that the considered system contains a production term, and is thus not mass conservative. They can apply the bounded Lipschitz distance (also known as Fortet-Mourier norm)

$$p_F(\mu,\nu) = \sup_{\phi \in C^1 : \|\phi\|_{W^{1,\infty}} \leq 1} \int_{\mathbb{R}^+} \phi \, d(\mu - \nu)$$

and existence of weak solutions is shown with help of careful estimates, which were possible only in 1-dimensional case.

We consider the following compressible two cell population model

$$\begin{cases} \partial_t n_1 - \operatorname{div}[n_1 \nabla p] = n_1 F_1(p) + n_2 G_1(p), & x \in \mathbb{R}^d,\ t \geq 0, \\ \partial_t n_2 - \operatorname{div}[n_2 \nabla p] = n_1 F_2(p) + n_2 G_2(p), \end{cases}$$

with

$$n := n_1 + n_2, \qquad p(n) = n^\gamma, \quad \gamma > 1.$$

$F_i,\ G_i$ describe the division/death rates of cells.

Weak solutions

We call $n_1, n_2, p \in L^\infty((0,T)\times\mathbb{R}^d)$ a weak solution to the system if for $i = 1, 2$

$$\int_0^T \int_{\mathbb{R}^d} \left[-n_i \partial_t \psi + n_i \nabla p . \nabla \psi - \big(n_1 F_i(p) + n_2 G_i(p)\big)\psi\right] \, dxdt$$
$$= \int_{\mathbb{R}^d} n_i^0 \psi(0)\, dx$$

holds for all $\psi \in C_c^1(\mathbb{R}\times\mathbb{R}^d)$

Existence of weak solutions

- Our strategy is to ignore compactness on the cell densities and to prove strong compactness of the pressure gradient.
- It relies on the regularizing effect for porous medium equation which provides estimates of the Laplacian of the pressure.
- We improve known results in two directions; we treat higher dimension than one and we deal with vacuum.

General plan

- Construct approximate problem, which you know how to solve
- Show the uniform bounds for approximate sequences
- Pass to the limit
- Show that the limit satisfies the original problem

Where can we expect difficulties?

Consider a sequence which is not convergent

$$f_n(x) = \sin nx$$

in a usual sense (strong). But

$$\int_0^{2\pi} \sin nx \; dx = 0.$$

We can conclude that for all φ

$$\lim_{n\to\infty} \int_0^{2\pi} f_n(x)\varphi(x) \; dx = 0$$

and write

$$f_n \rightharpoonup f$$

with $f \equiv 0$. Does $f_n \cdot f_n \rightharpoonup f \cdot f$? Observe that

$$\int_0^{2\pi} \sin^2 nx \; dx = \frac{1}{2} \int_0^{2\pi} (1 - \cos 2nx) \; dx.$$

Stability of weak solutions

We denote approximate sequences $\{n_i^\varepsilon\}$, $\{p^\varepsilon\}$, $\{n^\varepsilon\}$. Let us collect all the terms where the problems can appear

$$n_i^\varepsilon \nabla p^\varepsilon, \qquad n_i^\varepsilon F_j(p^\varepsilon), \qquad n_i^\varepsilon G_j(p^\varepsilon),$$

Our strategy is to ignore compactness on the cell densities and to prove strong compactness of the pressure gradient.

Conclusion

There exists a solution

Thank you for your attention