Topological properties in tensor products of Banach spaces

Antonio Avilés
Universidad de Murcia
with G. Martínez Cervantes, J. Rodríguez and A. Rueda Zoca

MTM2017-86182-P (AEI/FEDER, UE)
20797/PI/18 by Fundación Séneca, ACyT Región de Murcia

Tensor products

Tensor product of vector spaces

- $X \otimes Y = span\{x \otimes y\}_{x \in X, y \in Y} \subset Bilinear(X \times Y)^{\sharp}$

Tensor products

Tensor product of vector spaces

- $X \otimes Y = span\{x \otimes y\}_{x \in X, y \in Y} \subset Bilinear(X \times Y)^\sharp$

  $x \otimes y(b) = b(x, y)$

Tensor products

Tensor product of vector spaces

- $X \otimes Y = span\{x \otimes y\}_{x \in X, y \in Y} \subset Bilinear(X \times Y)^\sharp$

  $x \otimes y(b) = b(x, y)$

- $(X \otimes Y)^\sharp = Bilinear(X \times Y) = L(X, Y^\sharp) = L(Y, X^\sharp)$

Projective tensor product of Banach spaces $X \otimes_\pi Y$

Tensor products

Tensor product of vector spaces

- $X \otimes Y = span\{x \otimes y\}_{x \in X, y \in Y} \subset Bilinear(X \times Y)^\sharp$

  $x \otimes y(b) = b(x, y)$

- $(X \otimes Y)^\sharp = Bilinear(X \times Y) = L(X, Y^\sharp) = L(Y, X^\sharp)$

Projective tensor product of Banach spaces $X \otimes_\pi Y$

- $\mathscr{B}(X, Y)$, the bilinear and continuous are a Banach space

Tensor products

Tensor product of vector spaces

- $X \otimes Y = span\{x \otimes y\}_{x \in X, y \in Y} \subset Bilinear(X \times Y)^\sharp$

  $x \otimes y(b) = b(x, y)$

- $(X \otimes Y)^\sharp = Bilinear(X \times Y) = L(X, Y^\sharp) = L(Y, X^\sharp)$

Projective tensor product of Banach spaces $X \otimes_\pi Y$

- $\mathscr{B}(X, Y)$, the bilinear and continuous are a Banach space

  $\|b\| = \sup\{|b(x, y)| : \|x\| \leq 1, \|y\| \leq 1\}$

Tensor product of vector spaces

- $X \otimes Y = span\{x \otimes y\}_{x \in X, y \in Y} \subset Bilinear(X \times Y)^\sharp$

  $x \otimes y(b) = b(x, y)$

- $(X \otimes Y)^\sharp = Bilinear(X \times Y) = L(X, Y^\sharp) = L(Y, X^\sharp)$

Projective tensor product of Banach spaces $X \otimes_\pi Y$

- $\mathscr{B}(X, Y)$, the bilinear and continuous are a Banach space

  $\|b\| = \sup\{|b(x, y)| : \|x\| \leq 1, \|y\| \leq 1\}$

- $X \otimes_\pi Y = \overline{span}\{x \otimes y\} \subset \mathscr{B}(X \times Y)^*$

Tensor product of vector spaces

- $X \otimes Y = span\{x \otimes y\}_{x \in X, y \in Y} \subset Bilinear(X \times Y)^{\sharp}$

  $x \otimes y(b) = b(x, y)$

- $(X \otimes Y)^{\sharp} = Bilinear(X \times Y) = L(X, Y^{\sharp}) = L(Y, X^{\sharp})$

Projective tensor product of Banach spaces $X \otimes_{\pi} Y$

- $\mathscr{B}(X, Y)$, the bilinear and continuous are a Banach space

  $\|b\| = \sup\{|b(x, y)| : \|x\| \leq 1, \|y\| \leq 1\}$

- $X \otimes_{\pi} Y = \overline{span}\{x \otimes y\} \subset \mathscr{B}(X \times Y)^*$

- $(X \otimes_{\pi} Y)^* = \mathscr{B}(X \times Y) = \mathscr{L}(X, Y^*) = \mathscr{L}(Y, X^*)$

Preservation of topological properties

If X and Y have (P), does $X \otimes_\pi Y$ have (P)?

Preservation of topological properties

If X and Y have (P), does $X \otimes_\pi Y$ have (P)?

- X is reflexive.

Preservation of topological properties

If X and Y have (P), does $X \otimes_\pi Y$ have (P)?

- X is reflexive.
- X is (subspace of) weakly compactly generated space.
  X is SWCG $\Leftrightarrow B_{X^*}$ is Eberlein compact

Preservation of topological properties

If X and Y have (P), does $X \otimes_\pi Y$ have (P)?

- X is reflexive.
- X is (subspace of) weakly compactly generated space.
  X is SWCG $\Leftrightarrow B_{X^*}$ is Eberlein compact
- X is a (subspace of) Hilbert generated space.
  X is SHG $\Leftrightarrow B_{X^*}$ is Uniform Eberlein compact.

Preservation of topological properties

If X and Y have (P), does $X \otimes_\pi Y$ have (P)?

- X is reflexive.
- X is (subspace of) weakly compactly generated space.
  X is SWCG $\Leftrightarrow B_{X^*}$ is Eberlein compact
- X is a (subspace of) Hilbert generated space.
  X is SHG $\Leftrightarrow B_{X^*}$ is Uniform Eberlein compact.
- X is weakly Lindelöf determined space.
  X is WLD $\Leftrightarrow B_{X^*}$ is Corson compact.

Preservation can fail badly...

- A Hilbert space ℓ_2 has all good properties...

Preservation can fail badly...

- A Hilbert space ℓ_2 has all good properties... but $\{e_i \otimes e_i\}$ spans a copy of ℓ_1 inside $\ell_2 \otimes_\pi \ell_2$.

Preservation can fail badly...

- A Hilbert space ℓ_2 has all good properties... but $\{e_i \otimes e_i\}$ spans a copy of ℓ_1 inside $\ell_2 \otimes_\pi \ell_2$.

- We want $(X \otimes_\pi Y)^*$ to be small. And remember

$$(X \otimes_\pi Y)^* = \mathscr{B}(X \times Y) = \mathscr{L}(X, Y^*) = \mathscr{L}(Y, X^*).$$

Preservation can fail badly...

- A Hilbert space ℓ_2 has all good properties... but $\{e_i \otimes e_i\}$ spans a copy of ℓ_1 inside $\ell_2 \otimes_\pi \ell_2$.

- We want $(X \otimes_\pi Y)^*$ to be small. And remember

$$(X \otimes_\pi Y)^* = \mathscr{B}(X \times Y) = \mathscr{L}(X, Y^*) = \mathscr{L}(Y, X^*).$$

- So the problem is that there are too many operators $\ell_2 \longrightarrow \ell_2^*$.

When there are few bilinear maps

- If all $X \to Y^*$ are compact
  and X, Y reflexive $\Rightarrow X \otimes_\pi Y$ reflexive.

When there are few bilinear maps

- If all $X \to Y^*$ are compact and X, Y reflexive $\Rightarrow X \otimes_\pi Y$ reflexive.
- If all $X \to Y^*$ are Dunford-Pettis ($x_n \xrightarrow{w} 0 \Rightarrow Tx_n \to 0$): and X, Y WCG $\Rightarrow X \otimes_\pi Y$ WCG.

When there are few bilinear maps

- If all $X \to Y^*$ are compact and X, Y reflexive $\Rightarrow X \otimes_\pi Y$ reflexive.
- If all $X \to Y^*$ are Dunford-Pettis ($x_n \xrightarrow{w} 0 \Rightarrow Tx_n \to 0$): and X, Y WCG $\Rightarrow X \otimes_\pi Y$ WCG.
- If all $X \to Y^*$ and $Y \to X^*$ are c-Dunford-Pettis and X, Y SWCG $\Rightarrow X \otimes_\pi Y$ SWCG.

When there are few bilinear maps

- If all $X \to Y^*$ are compact
  and X, Y reflexive $\Rightarrow X \otimes_\pi Y$ reflexive.
- If all $X \to Y^*$ are Dunford-Pettis ($x_n \xrightarrow{w} 0 \Rightarrow \; Tx_n \to 0$):
  and X, Y WCG $\Rightarrow X \otimes_\pi Y$ WCG.
- If all $X \to Y^*$ and $Y \to X^*$ are c-Dunford-Pettis
  and X, Y SWCG $\Rightarrow X \otimes_\pi Y$ SWCG.
- If all $X \to Y^*$ and $Y \to X^*$ have separable range
  and X, Y WLD $\Rightarrow X \otimes_\pi Y$ WLD.

When there are few bilinear maps

- If all $X \to Y^*$ are compact
  and X, Y reflexive $\iff X \otimes_\pi Y$ reflexive. (approx. prop.)

- If all $X \to Y^*$ are Dunford-Pettis ($x_n \xrightarrow{w} 0 \Rightarrow Tx_n \to 0$):
  and X, Y WCG $\Rightarrow X \otimes_\pi Y$ WCG.

- If all $X \to Y^*$ and $Y \to X^*$ are c-Dunford-Pettis
  and X, Y SWCG $\Rightarrow X \otimes_\pi Y$ SWCG.

- If all $X \to Y^*$ and $Y \to X^*$ have separable range
  and X, Y WLD $\iff X \otimes_\pi Y$ WLD. (always)

The tensor products $\ell_p(I) \otimes_\pi \ell_q(I)$

- $1/p + 1/q \geq 1 \Rightarrow \ell_p \otimes_\pi \ell_q \supset \ell_1$.
- $1/p + 1/q < 1 \Rightarrow \ell_p \otimes_\pi \ell_q$ is reflexive.

Theorem

When $1/p + 1/q < 1$, the space $\ell_p \otimes_\pi \ell_q$ is a subspace of a Hilbert generated space.

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,
  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$
- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$
  $X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$

  $X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$

- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$

  $X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$

- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

X, Y SWCG

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$

  $$X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$$

- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

X, Y SWCG
$\Rightarrow$ B_{X^*}, B_{Y^*} Eberlein

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$

  $$X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$$

- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

X, Y SWCG
$\Rightarrow B_{X^*}, B_{Y^*}$ Eberlein
$\Rightarrow B_{X^*} \times B_{Y^*}$ Eberlein

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$

  $X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$

- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

X, Y SWCG
$\Rightarrow$ B_{X^*}, B_{Y^*} Eberlein
$\Rightarrow$ $B_{X^*} \times B_{Y^*}$ Eberlein
$\Rightarrow$ $C(B_{X^*} \times B_{Y^*})$ WCG

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$

  $X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$

- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

X, Y SWCG
$\Rightarrow$ B_{X^*}, B_{Y^*} Eberlein
$\Rightarrow$ $B_{X^*} \times B_{Y^*}$ Eberlein
$\Rightarrow$ $C(B_{X^*} \times B_{Y^*})$ WCG
$\Rightarrow$ $X \otimes_\varepsilon Y$ SWCG.

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$

  $$X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$$

- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

X, Y WLD
$\Rightarrow B_{X^*}, B_{Y^*}$ Eberlein
$\Rightarrow B_{X^*} \times B_{Y^*}$ Eberlein
$\Rightarrow C(B_{X^*} \times B_{Y^*})$ WCG
$\Rightarrow X \otimes_\varepsilon Y$ SWCG.

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,
  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$
- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$
  $X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$
- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

X, Y WLD
$\Rightarrow B_{X^*}, B_{Y^*}$ Corson
$\Rightarrow B_{X^*} \times B_{Y^*}$ Eberlein
$\Rightarrow C(B_{X^*} \times B_{Y^*})$ WCG
$\Rightarrow X \otimes_\varepsilon Y$ SWCG.

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$

  $X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$

- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

X, Y WLD
$\Rightarrow B_{X^*}, B_{Y^*}$ Corson
$\Rightarrow B_{X^*} \times B_{Y^*}$ Corson
$\Rightarrow C(B_{X^*} \times B_{Y^*})$ WCG
$\Rightarrow X \otimes_\varepsilon Y$ SWCG.

The injective tensor product $X \otimes_\varepsilon Y$

- $X \otimes_\varepsilon Y = \overline{span}\{x \otimes y\} \subset C(B_{X^*} \times B_{Y^*})$,

  $(x \otimes y)(x^*, y^*) = x^*(x) \cdot y^*(y)$

- $(X \otimes_\varepsilon Y)^* = \mathscr{B}_{int}(X \times Y) = \mathscr{L}_{int}(X, Y^*) = \mathscr{L}_{int}(Y, X^*)$

  $X \longrightarrow L_\infty(\mu) \longrightarrow L_1(\mu) \longrightarrow Y^*$

- $C(K) \otimes_\varepsilon X = C(K, X)$. $C(K) \otimes_\varepsilon C(L) = C(K \times L)$.

X, Y WLD
$\Rightarrow B_{X^*}, B_{Y^*}$ Corson
$\Rightarrow B_{X^*} \times B_{Y^*}$ Corson
$\not\Rightarrow C(B_{X^*} \times B_{Y^*})$ WLD
$\Rightarrow X \otimes_\varepsilon Y$ SWCG.

WLD in injective tensor products

$X \otimes_\varepsilon Y$ is WLD if and only if
X and Y are WLD and
all integral operators $X \to Y^*$ and $Y \to X^*$ have separable range.

WLD in injective tensor products

$X \otimes_\varepsilon Y$ is WLD if and only if
X and Y are WLD and
all integral operators $X \to Y^*$ and $Y \to X^*$ have separable range.

- A sufficient condition is that either $C(B_{X^*})$ or $C(B_{Y^*})$ are WLD.

WLD in injective tensor products

$X \otimes_\varepsilon Y$ is WLD if and only if
X and Y are WLD and
all integral operators $X \to Y^*$ and $Y \to X^*$ have separable range.

- A sufficient condition is that either $C(B_{X^*})$ or $C(B_{Y^*})$ are WLD.
- When $X = Y$, it is necessary and sufficient that $C(B_{X^*})$ is WLD.

Property (C)

X has (C) iff every x^* in the w^*-closure of a bounded dual set is in the closure of a sequence of convex combinations.

- If X has the λ-BSAP property,and $X \otimes_\varepsilon X$ has property (C), then all measures on B_{X^*} are of countable type.
- (Plebanek, Sobota) If $C(K \times K)$ has property (C) then all measures on K have countable type.