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Topological invariants for discontinuous mappings

Differential topology in Sobolev spaces

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Why do we need

- non-differentiable
- discontinuous

mappings in **real-life applications**?

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- non-differentiable
- discontinuous

mappings in **real-life applications**?

Two reasons:

- natural: folding, breaking, changes of state;
- technical: necessary because of the available mathematical tools.

Variational principles

- Fermat's principle (Hero of Alexandria, Pierre de Fermat): Light travels through media along paths of shortest time.
- Extremal action principle (Euler, Maupertuis): Bodies travel along paths locally minimizing the *reduced action* (integral of the momentum).

Other examples: Dirichlet's principle for harmonic maps, Einstein-Hilbert action in general relativity etc.

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Other examples: Dirichlet's principle for harmonic maps, Einstein-Hilbert action in general relativity etc.

Minima of accumulative, i.e, integral quantities $\Rightarrow$ we measure distance to minimizer with integral expressions.

Harmonic maps and $W^{1,2}$

Canonical example: **harmonic maps**

A map $f : \mathbb{B}^n \to \mathbb{R}^m$ is *minimizing harmonic* if it is a minimum of the Dirichlet energy

$$\mathcal{E}(f) = \int_{\mathbb{B}^n} |Df|^2$$

among all maps with the same boundary values as f.

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What *all maps*?

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$C^1(\mathbb{B}^n)$, with distance $d(f,g) = |\mathcal{E}(f) - \mathcal{E}(g)|$?

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$C^1(\mathbb{B}^n)$, with distance $d(f, g) = |\mathcal{E}(f) - \mathcal{E}(g)|$?
Problems:

- d is not a metric (remedy: add $\|f - g\|_{L^2}$ to $d(f, g)$)
- space is not complete $\Rightarrow$ existence of minima is not guaranteed.

Harmonic functions and $W^{1,2}$

Right choice: Sobolev space $W^{1,2}(\mathbb{B}^n)$ – the **completion** of C^1 (or C^∞) maps, in the norm

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But if $n \geq 2$, then this space contains *discontinuous* functions:

$$\log(\log(e + \frac{1}{|x|})) \in W^{1,2}(\mathbb{B}^n).$$

Weak derivative and Sobolev spaces

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Definition

A function $g \in L^1_{loc}$ is the *weak derivative* in the i-th direction of $f \in L^1_{loc}(U)$, if

$$\int_U f \frac{\partial}{\partial x^i}\phi = -\int_U g\,\phi \quad \text{for any } \phi \in C^\infty_o(U).$$

We write $g = D^i f$. If $f \in C^1(U)$, then $D^i f = \frac{\partial}{\partial x^i} f$.

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Definition (Sobolev space)

$W^{1,p}(U) = \{f \in L^p(U) \ : \ D^i f \in L^p(U) \text{ for } i = 1, \dots, n\}$

If $p \leq n = \dim U$ and $n \geq 2$, then $W^{1,p}(U)$ contains discontinuous functions.

Sobolev and Orlicz-Sobolev spaces

Advantages of Sobolev spaces:

- smooth functions are dense,

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$*$ means that sufficiently nice functionals attain their minima (minimizing sequences have *weakly* convergent subsequences).

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Of interest are also Orlicz-Sobolev functions

$$W^{1,P}(U) = \{f \in L^1(U) \ : \ P(|D^i f|) \in L^1(U) \text{ for } i = 1, \dots, n\},$$

e.g., for $P(t) = t^n / \log(e+t)$ and $p < n$, $W^{1,p} \supset W^{1,P} \supset W^{1,n}$.

Topology encoded in the derivative

invertibility:
Assume X, Y are Banach, $F\colon X \to Y$ is C^1. If

$$DF(x)\colon X \to Y \text{ is a linear isomorphism} \tag{1}$$

then F is a local homeomorphism at x.
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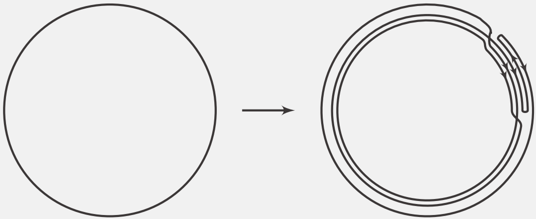
local orientation:
If $F\colon \Omega \subset \mathbb{R}^n \to F(\Omega) \subset \mathbb{R}^n$ is C^1 and

$J_F(x) = \det DF(x) > 0$, then F preserves local orientation at x.

Topology encoded in the derivative

Brouwer degree: Let M, N be closed manifolds of the same dimension.



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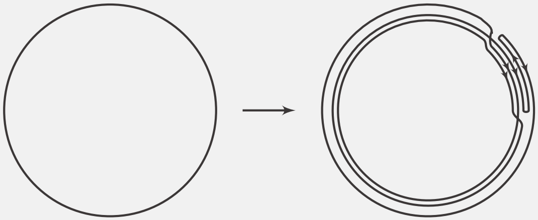

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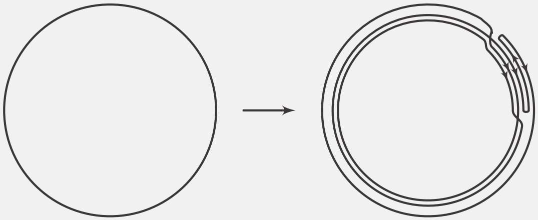

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Other topological invariants (Hopf index, characteristic classes) of a mapping can be determined if the derivative is known.

For weakly differentiable mappings, we can calculate the values of ‘invariants’, substituting the weak derivative where the strong one should go.

Two (closely related) questions:

- If two weakly differentiable mappings are suff. close in some Sobolev norm, do their ‘invariants’ match?
- If a **continuous**, non-differentiable mapping is weakly differentiable, does this value of the ‘invariant’ match the true topological invariant?

And are these ‘invariants’ of any use?

Degree for Sobolev maps

The formula for the Brouwer degree:

$$\deg F = \frac{1}{\operatorname{vol}(N^n)} \int_{M^n} \det DF(x)\, dx$$

defines a map $\deg \colon C^1(M, N) \to \mathbb{Z}$, continuous in C^1 norm.

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It is also continuous in $C^1(M,N)$ with $W^{1,n}$ norm.
We extend deg by density to arbitrary $W^{1,n}(M,N)$ maps, possibly discontinuous.

Theorem

Assume $f_m\colon M \to N$, $m = 1, 2, \ldots$, is a sequence of C^1 maps convergent in $W^{1,n}(M,N)$. Then the sequence $\deg f_m$ is constant for suff. large m.

Other invariants

Take any other invariant of a map f, e.g., homotopy class $[f]$.

Question

Assume $f_m\colon M \to N$ is a sequence of smooth / C^1 / continuous maps convergent in $W^{1,p}(M,N)$. Does the value of the invariant ($[f_m]$) become constant for large m?

If so, then the invariant (e.g., homotopy classes) can be extended, by density, to all $W^{1,p}(M,N)$.

Motivation

Problem

Find a non-trivial (non-constant) harmonic function $h : \mathbb{B} \to \mathbb{R}$.

Solution: minimize Dirichlet's energy $\mathcal{E}$ among all $W^{1,2}(\mathbb{B})$ functions with prescribed, non-trivial values on $\partial\mathbb{B}$.

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Problem

Find a non-trivial harmonic function $h : \mathbb{S}^2 \to \mathbb{S}^2$.

$\mathbb{S}^n$ has no boundary!
Solution: minimize Dirichlet's energy $\mathcal{E}$ among all $W^{1,2}(\mathbb{S}^2)$ functions with prescribed, non-zero homotopy class.
(White, Eells&Ferreira)

Function spaces

Definition

Assume N is isometrically embedded in $\mathbb{R}^k$.

$$W^{1,p}(M,N) = \{f \in W^{1,p}(M,\mathbb{R}^k) \ : \ f(x) \in N \text{ for a.e. } x \in M\}$$

Smooth mappings need not be dense in $W^{1,p}(M,N)$ for $p < n$;
$H^{1,p}(M,N)$:= the closure of C^∞ in $W^{1,p}$.

Orlicz-Sobolev maps $W^{1,P}$ between manifolds are defined in the same way; we are interested in those that are only slightly larger than $W^{1,n}$ (e.g., $P(t) = t^n / \log(e+t)$).

Homotopy classes

Theorem (White, 1986)

If M, N are compact, oriented Riemannian manifolds, $\dim M = n$, then the homotopy classes are well defined in $W^{1,n}(M, N)$.

Theorem (G., Hajłasz, 2012)

The homotopy classes are well defined in $H^{1,p}(M, N)$, $n - 1 \leq p < n$, if $\pi_n(N) = 0$, and they cannot be well defined in $H^{1,p}(\mathbb{S}^n, N)$, if $\pi_n(N) = 0$. An analogous result holds for a certain class of Orlicz-Sobolev spaces, slightly larger than $W^{1,n}(M, N)$.

Degree

Theorem (G., Hajłasz, 2012)

If M, N are compact, oriented n-dim. Riemannian manifolds w/out boundary, $n-1 \leq p < n$. Then the degree is well defined in $H^{1,p}(M,N)$ (and in the aforementioned class of Orlicz-Sobolev spaces) if and only if the universal cover of N is not a rational homology sphere (RHS).

A *rational homology n-sphere*: an n-dimensional smooth manifold with the same deRham cohomology as $\mathbb{S}^n$.

Importance of the degree

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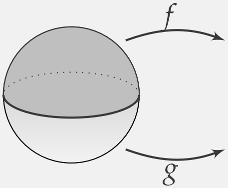
Fact

If $f, g \in C^1(\mathbb{B}^n, \mathbb{R}^n) \cap C(\overline{\mathbb{B}^n}, \mathbb{R}^n)$ and $f = g$ on $\partial\mathbb{B}$, then

$$\int_{\mathbb{B}^n} J_f = \int_{\mathbb{B}^n} J_g.$$

$F:$



$\mathbb{R}^n$ and $\deg F = \int_{\mathbb{B}^n} J_f - \int_{\mathbb{B}^n} J_g = 0.$

because $\mathbb{R}^n$ is contractible. It holds for $W^{1,n}$ maps, as well.
What if the target is not contractible?

Importance of the degree

Lemma

If M is a smooth, compact, connected, oriented, n-dim. manifold w/out boundary, $n \geq 2$, then there is a smooth mapping $f : \mathbb{S}^n \to M$ of non-zero degree if and only if the universal cover of M is a RHS.

Corollary

If $f, g \in C^1(\mathbb{B}^n, N) \cap C(\overline{\mathbb{B}^n}, N)$, $f = g$ on $\partial\mathbb{B}^n$, and the universal cover of N is not a RHS, then

$$\int_{\mathbb{B}^n} J_f = \int_{\mathbb{B}^n} J_g.$$

Continuity of mappings with positive Jacobian

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Theorem (Gol'dšteĭn&Vodop'yanov, 1976)

If $U \subset \mathbb{R}^n$ is open and $f \in W^{1,n}(U, \mathbb{R}^n)$ has positive Jacobian a.e., then f is continuous.

The same holds for our Orlicz-Sobolev maps $W^{1,P}$.
How about mappings between manifolds?

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Theorem (G., Hajłasz, 2019)

The same holds in our Orlicz-Sobolev spaces $W^{1,P}$, if the universal cover of N is not a RHS. However, if the universal cover of N is $\mathbb{S}^n$, then there exist discontinuous $W^{1,P}$ maps of positive Jacobian.

Other invariants, other directions...

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Question

Assume f is a weakly or approximately differentiable, continuous map.
How much of its topology can be read from the 'invariants' calculated using its weak/approximate derivative (which we can study and estimate by analytic means)?

- local degree/orientation $\to$ study of homeomorphisms with Jacobian changing sign (2017, 2019),
- Hopf invariant and its generalizations $\to$ study of C^1 maps with degenerate derivative, topologically non-trivial examples showing sharpness of assumptions to the Sard theorem (2018, 2019).

Thank you
for your attention