The Weak Ramsey Property

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Motivation

Motivation

Theorem (Kechris, Pestov, Todorčević 2005)

Let $\mathscr{F}$ be a relational Fraïssé class with the Fraïssé limit U and let $G = \mathrm{Aut}(U)$ be endowed with the pointwise convergence topology. TFAE:

(a) *G is extremely amenable.*

(b) *$\mathscr{F}$ has the Ramsey property and the ordering property.*

$$\underset{\text{closed}}{G \leqslant} S_\infty$$

$\mathcal{F}$ consists of finite structures with relations

- $\mathcal{F}$ is hereditary
- $\mathcal{F}$ has the joint embedding property, the amalgamation property, and ctbly many types.

Homogeneity:

$A \in \mathcal{F}$

$$\mathrm{Aut}(U) \curvearrowright \mathrm{Emb}(A, U)$$

$$\begin{array}{ccc} \mathrm{Aut}(U) & & \mathrm{Emb}(A,U) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ g & gf & f \end{array}$$

Notation

Categories

Categories will be denoted by letters $\mathfrak{S}$, $\mathfrak{C}$, $\mathfrak{L}$, etc.
Given a category $\mathfrak{C}$ and two objects $A, B \in \mathtt{Obj}(\mathfrak{C})$ the set of all $\mathfrak{C}$-arrows from A to B will be denoted by $\mathfrak{C}(A, B)$.

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The setup

We shall work with a pair $\langle \mathfrak{S}, \mathfrak{L} \rangle$, where $\mathfrak{L}$ is a category and $\mathfrak{S}$ is its full subcategory. We shall assume that $\mathrm{Obj}(\mathfrak{L})$ consists of all colimits of chains (sequences) in $\mathfrak{S}$ and all $\mathfrak{L}$-arrows are monic.
We assume $\mathfrak{S}$ is directed, i.e., for every $a, b \in \mathrm{Obj}(\mathfrak{S})$ there is $c \in \mathrm{Obj}(\mathfrak{S})$ with

$$\mathfrak{S}(a, c) \neq \emptyset \neq \mathfrak{S}(b, c).$$

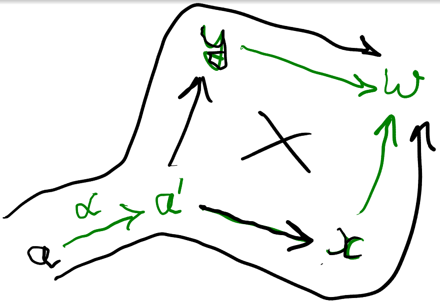
$$\begin{tikzcd} a \arrow[dr] & \\ & c \\ b \arrow[ur] & \end{tikzcd}$$

Weak Fraïssé theory

Definition

Fix $\alpha\colon a \to a'$ in $\mathfrak{S}$. We say that $\mathfrak{S}$ has the weak amalgamation property (WAP) at α if for every $\mathfrak{S}$-arrows $f\colon a' \to x$, $g\colon a' \to y$ there are $\mathfrak{S}$-arrows $f'\colon x \to w$ and $g'\colon y \to w$ satisfying

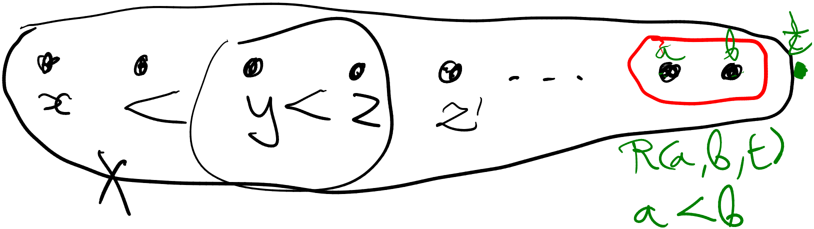
$$f' \circ f \circ \alpha = g' \circ g \circ \alpha.$$

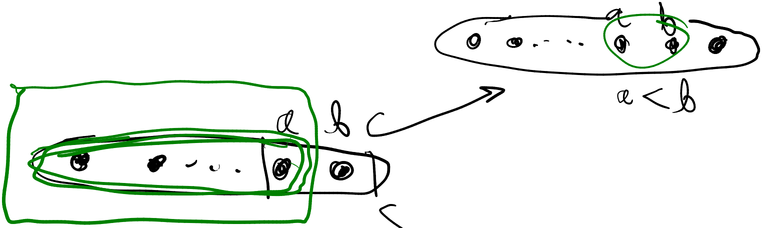


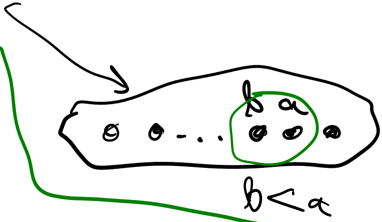
Example (Pouzet)

Let $\mathscr{F}$ be the class of all finite linearly ordered sets, where the linear ordering $<$ is replaced by the following ternary relation:

$$R(x, y, z) \iff x < y \;\&\; x < z \;\&\; y \neq z.$$







Weak Fraïssé category:

→ directed

→ with the WAP

→ countably dominated

Weak Fraïssé sequences

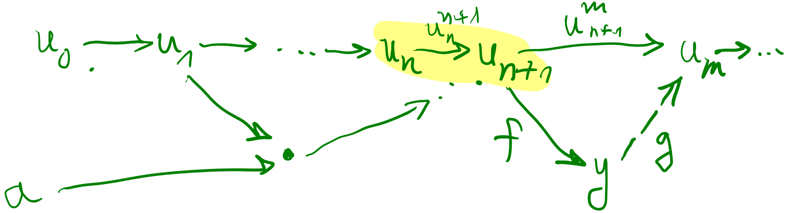
Definition

A normalized weak Fraïssé sequence in $\mathfrak{S}$ is a sequence $\vec{u}\colon \omega \to \mathfrak{S}$ satisfying the following condition.

(W) For every $n \in \omega$, for every $\mathfrak{S}$-arrow $f\colon u_{n+1} \to y$ there exist $m > n$ and an $\mathfrak{S}$-arrow $g\colon y \to u_m$ such that

$$g \circ f \circ u_n^{n+1} = u_n^m.$$

($\mathfrak{S}$ is directed)



The framework

We assume $U = \lim \vec{u}$, where $\vec{u} \colon \omega \to \mathfrak{S}$ is a normalized weak Fraïssé sequence.

Furthermore:

(F) For every $a \in \mathrm{Obj}(\mathfrak{S})$, for every $f \colon a \to \lim \vec{u}$ there are $n \in \omega$ and $f' \colon a \to u_n$ such that $f = u_n^\infty \circ f'$.

$$\begin{tikzcd} u_0 \arrow[r] & u_1 \arrow[r] & \cdots & u_n \arrow[r, "u_n^\infty"] & U \\ & & & a \arrow[u, "\Delta_n(f)"] \arrow[ur, "f"'] & \end{tikzcd}$$

$$\left.\begin{array}{l} u_n^\infty \circ \Delta_n(f) = f \\ u_n^\infty \circ \Delta'_n(f) = \end{array}\right\} \Longrightarrow \Delta'_n(f) = \Delta_n(f)$$

The topology on $G := \mathrm{Aut}(U)$

Definition

A basic neighborhood of $\mathrm{id}_U \in G$ is defined to be any set of the form

$$V_m = \{g \in G\colon g \circ u_m^\infty = u_m^\infty\},$$

where $m \in \omega$.

$$\begin{tikzcd}
U_0 \arrow[r] & U_1 \arrow[r] & \cdots \arrow[r] & U_m \arrow[r] \arrow[d, "\mathrm{id}_{U_m}"] & \cdots & U \arrow[d, "g"] \\
U_0 \arrow[r] & U_1 \arrow[r] & \cdots \arrow[r] & U_m \arrow[r] & \cdots & U
\end{tikzcd}$$

The topology on $G := \mathrm{Aut}(U)$

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where $m \in \omega$.

Claim

$\mathrm{Aut}(U)$ *is a completely metrizable non-archimedean group.*

Each V_m is an open subgroup

The weak big Ramsey property

Theorem

Assume $G = \mathrm{Aut}(U)$ is extremely amenable. Then for every $a \in \mathrm{Obj}(\mathfrak{S})$ there exists an $\mathfrak{S}$-arrow $\alpha\colon a \to a'$ satisfying:

(wB) *For every $k \in \omega$, for every finite $F \subseteq \mathfrak{L}(a', U)$, for every $\varphi\colon \mathfrak{L}(a', U) \circ \alpha \to k$ there is $g \in G$ such that φ is constant on*

$$g \circ F \circ \alpha.$$

$$a \xrightarrow{\alpha} a' \overset{f_1}{\underset{f_3}{\overset{f_2}{\rightrightarrows}}} U \qquad U \xrightarrow[\approx]{g} U$$

$G \curvearrowright \mathcal{L}(a', U)\circ\alpha$ transitively

$G \curvearrowright k^{\mathcal{L}(a', U)\circ\alpha}$ compact

If $a \xrightarrow{\alpha} a'$ is amalgamable

then

$$\begin{tikzcd}
 & & U \arrow[d, dashed, "h", "\approx"'] \\
a \arrow[r, "\alpha"] & a' \arrow[ur, "f_1"] \arrow[r, "f_2"'] & U
\end{tikzcd}$$

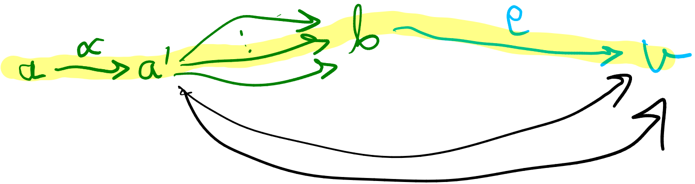
$$h \circ f_1 \circ \alpha = f_2 \circ \alpha$$

The weak Ramsey property

Definition

We say $\mathfrak{S}$ has the weak Ramsey property if for every $a \in \mathrm{Obj}(\mathfrak{S})$ there is an $\mathfrak{S}$-arrow $\alpha \colon a \to a'$ satisfying

(wR) For every $b \in \mathrm{Obj}(\mathfrak{S})$, for every $k \in \omega$, for every finite $F \subseteq \mathfrak{S}(a', b)$ there is $v \in \mathrm{Obj}(\mathfrak{S})$ such that for every $\varphi \colon \mathfrak{S}(a', v) \circ \alpha \to k$ there exists $e \colon b \to v$ such that $\varphi \restriction e \circ F \circ \alpha$ is constant.



Proposition

The weak Ramsey property implies WAP.

Proof.

Fix $a \in \mathrm{Obj}(\mathfrak{S})$, let $k = 2$, and let $\alpha \colon a \to a'$ be as in (wR). Fix $f_0, f_1 \in \mathfrak{S}$ with $\mathrm{dom}(f_0) = a' = \mathrm{dom}(f_1)$. Using directedness, choose $b \in \mathrm{Obj}(\mathfrak{S})$ and $g_0, g_1 \in \mathfrak{S}$ such that $g_i \circ f_i \in \mathfrak{S}(a', b)$ for $i = 0, 1$. Let $F = \{g_0 \circ f_0, g_1 \circ f_1\}$.

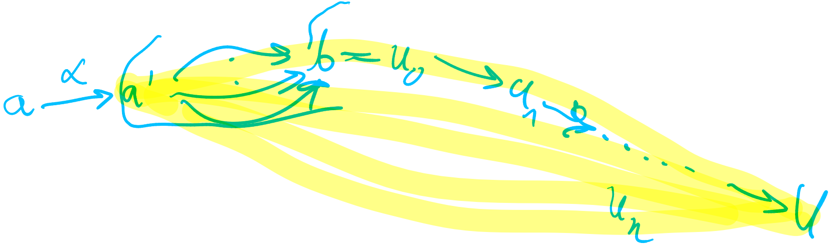
Find $v \in \mathrm{Obj}(\mathfrak{S})$ from the weak Ramsey property applied to F. Define $\varphi \colon \mathfrak{S}(a', v) \circ \alpha \to 2$ by setting $\varphi(g) = 1$ if and only if $g = g' \circ f_1 \circ \alpha$ for some $g' \in \mathfrak{S}$. The weak Ramsey property says there exists $e \colon b \to v$ such that φ is constant on $e \circ F \circ \alpha$. Note that $\varphi(e \circ (g_1 \circ f_1) \circ \alpha) = 1$, for obvious reasons. Thus also $\varphi(e \circ (g_0 \circ f_0) \circ \alpha) = 1$, which means that there exists h such that

$$e \circ g_0 \circ f_0 \circ \alpha = h \circ f_1 \circ \alpha.$$

We are done, because $e \circ g_0$ and h witness the weak amalgamation. □

Theorem

Assume $G = \mathrm{Aut}(U)$ is extremely amenable. Then $\mathfrak{S}$ has the weak Ramsey property.



Theorem

Assume $\mathfrak{S}$ has the weak Ramsey property and U is as above. Then $\mathtt{Aut}(U)$ *is extremely amenable.*

References

- A.S. Kechris, V.G. Pestov, S. Todorcevic, *Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups*, Geom. Funct. Anal. 15 (2005) 106–189
- A. Krawczyk, W. Kubiś, *Games with finitely generated structures*, preprint, `https://arxiv.org/abs/1701.05756`
- W. Kubiś, *Weak Fraïssé categories*, preprint, `https://arxiv.org/abs/1712.03300`
- D. Mašulović, *The Kechris-Pestov-Todorčević correspondence from the point of view of category theory*, preprint, `https://arxiv.org/abs/2007.14263`

Go back to Pouzet! example...

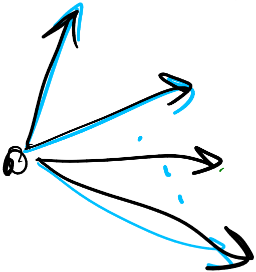
$$\mathrm{Aut}(\mathbb{U}) = \mathrm{Aut}(\mathbb{Q}, <)$$

extremely amenable

$\boxed{k\text{-AP}} \Longrightarrow \boxed{\text{WAP}}$

$\not\Longleftarrow$

4-AP



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