”The tragedy of the commons” in the dynamic context

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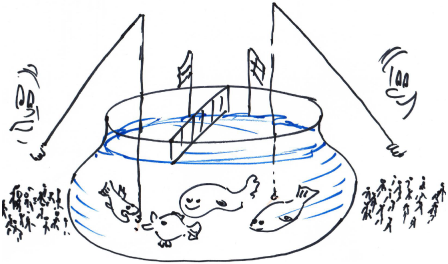
The motivating example



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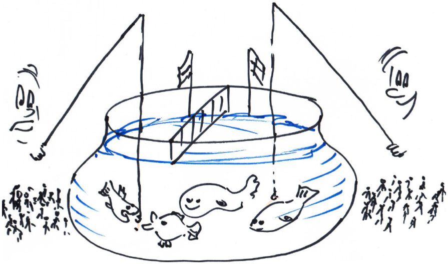

Extraction of a common fishery

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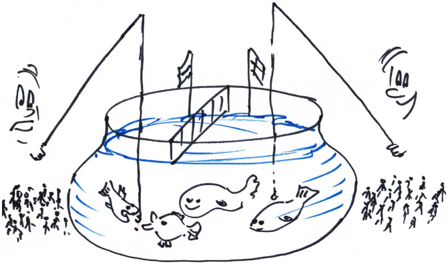

Extraction of a common fishery with possibility of extinction, *d*ivision into Excusive Economic Zones

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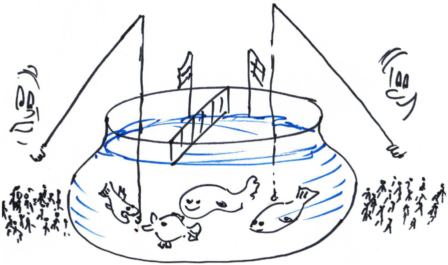

Extraction of a common fishery with possibility of extinction, *d*ivision into Excusive Economic Zones and inherent constraints

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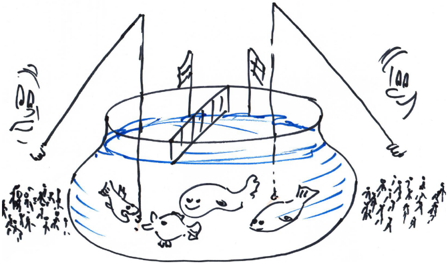

Extraction of a common fishery with possibility of extinction, *d*ivision into Excusive Economic Zones and inherent constraints with possibility to model many fishermen

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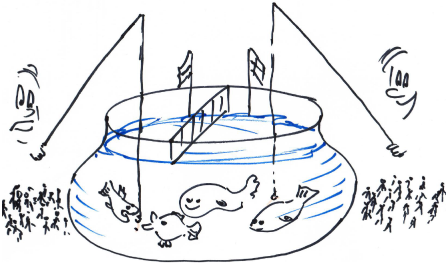

Extraction of a common fishery with possibility of extinction, *d*ivision into Excusive Economic Zones and inherent constraints with possibility to model many fishermen by the simplest possible model.

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- At least 2 agents + sets to choose from + aim + interaction.

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- or best responds to the strategies of the others.

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- So, calculation of a Nash equilibrium requires solving a set of optimization problems in players' strategy spaces coupled by finding a fixed point of the resulting best response correspondence in the space of strategy profiles.

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- What if players either do choose their strategies sequentially, there is a hierarchy or one of them has informational advantage (i.e. s/he can calculate the best response function of the other player or players). Then, instead of Nash, we consider a Stackelberg equilibrium.

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- For more than two players there may be different level of hierarchy or some players at the same level: e.g. a leader and many followers playing Nash between them given the leader's strategy (and the leader knows it and takes into its calculation). So, an optimization nested with a set of optimizations coupled by a fixed point

A short introduction to game theory cont. 3

- We are also interested in Pareto optimal profiles, i.e. profiles $\bar{S}$ such that there exists no profile S with
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Example: a static fishery in an unconstrained word

- A fishery with two identical fishing firms, $\mathbb{S}_i = \mathbb{R}_+$, linear costs of fishing $10S_i$, price dependent on the amount of fish on the market $100 - (S_1 + S_2)$.

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- The best response correspondences $B_1(S_2) = \operatorname*{Argmax}_{S_1 \geqslant 0}(100 - (S_1 + S_2))S_1 - 10S_1 = \left\{ \frac{90 - S_2}{2} \right\}$, $B_2(S_1)$ analogously.

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- The social optimum $\operatorname*{Argmax}_{S_1, S_2 \geqslant 0}(100 - (S_1 + S_2))S_1 - 10S_1 + (100 - (S_1 + S_2))S_2 - 10S_2 = \{(S_1, S_2) : S_1 + S_2 = 45\}$, price 55.

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- The social optimum $\operatorname*{Argmax}_{S_1, S_2 \geqslant 0}(100 - (S_1 + S_2))S_1 - 10S_1 + (100 - (S_1 + S_2))S_2 - 10S_2 = \{(S_1, S_2) : S_1 + S_2 = 45\}$, price 55. So, if additionally, $S_1 = S_2$, then higher profits for both.

Example: a static fishery cont.

- The Stackelberg equilibrium: After calculating $B_2(S_1)$, the leader optimizes $$S_1 \in \operatorname*{Argmax}_{S_1 \geqslant 0} (100 - (S_1 + \tfrac{90 - S_1}{2}))S_1 - 10S_1 = 45.$$

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- The leader extracts more than at a Nash equilibrium and gets more payoff that at the symmetric cooperative solution and it makes the follower extract as in the symmetric cooperative solution and get less payoff than at Nash.
- That may be only the matter of informational advantage and kindly informing the follower about the resulting choice!

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- In games related to extraction of common (or interrelated) resources: the fact that the social optimum is not a Nash equilibrium

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- Usually solved by enforcement: changing a game by adding a benevolent social planner – a Stackelberg leader modifying payoffs of the rest of the players by e.g. a tax in order that the previous social optimum is a Nash equilibrium given his strategy.

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- But they all are dynamic problems!

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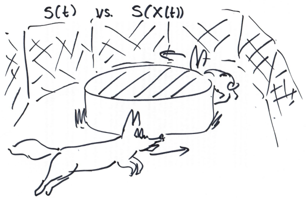

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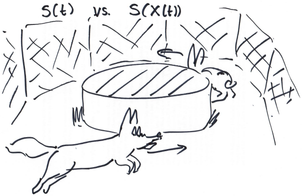

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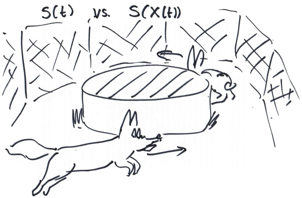

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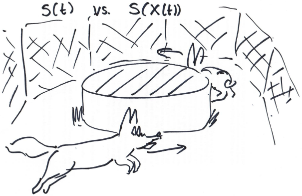

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- For some problems with a continuum of players, also a decomposition method (introduced and developed in A. Wiszniewska-Matyszkiel: Positivity 2002, C& C 2003, IGTR 2002, 2003, JOTA 2014) can be used and the results for open loop and feedback are equivalent in a wider class of problems (JOTA 2014).

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- So, let's add the inherent constraints to LQDG and we will have a nice model, with quite standard and nice results.

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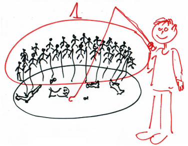

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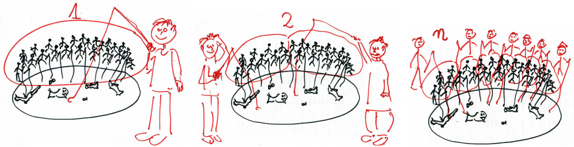

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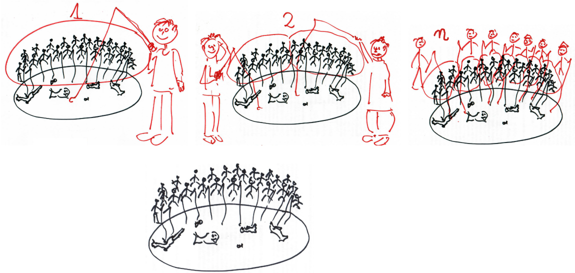

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- To model this, the set of players $\mathbb{I}$ is $\{1,\ldots,n\}$ or $[0,1]$
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- Consequently, the aggregate extraction $u=\sum_{i=1}^{n}\frac{s_i}{n}$ in the case of n players and $u=\int_{[0,1]}s_i\mathrm{d}\lambda(i)$ (λ means the Lebesgue measure) in the case of continuum of players.
- Rule of growth with fishing $X_i(t+1)=(1+\xi)X(t)-U(t)$ (generalized later).
- To make depletion possible, we set $c=(1+\xi)$.
- Discounting by a discount factor $\beta=\frac{1}{1+\xi}$ (the golden rule).
- Current payoff of player i: $P_i(s)=\text{price}(s)s_i-\text{cost}(s_i)$ (auxiliary notation $P(s_i,u)$ or $P(s_i,s_{\sim i})$).
- So we have a linear-quadratic dynamic game with linear state-dependent constraints on controls.

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Standard infinite horizon Bellman sufficient condition

- For a dynamic optimization problem
  - maximize $J(\bar{t}, \bar{x}, U) = \sum_{t=\bar{t}}^{\infty} g(X(t), U(X(t), t), t)\delta^{t-\bar{t}}$,

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  [3] A Wiszniewska-Matyszkiel, R. Singh, 2020, *Necessity of the Terminal Condition in the Infinite Horizon Dynamic Optimization Problems with Unbounded Payoff*, Automatica, https://doi.org/10.1016/j.automatica.2020.109332.

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- But $V(x) = hx^2 + gx$ with negative h which solves **(BE)** (and it is the only quadratic solution of it) is not the value function.

Counterexample!!!

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- There is also a solution with the only piecewise quadratic V that fulfils both **(BE)** and **(TC)**.

Social optimum cont.

Theorem 1

(a) The value function per player is

$$\bar{V}(x) = \begin{cases} \hat{g} \cdot x + \frac{\hat{h}}{2} \cdot x^2 & \text{if } x \in (0, \frac{\hat{s}}{\xi}), \\ \tilde{k} & \text{otherwise,} \end{cases}$$

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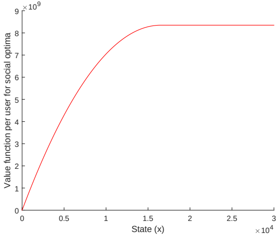

Figure: Value function per player for social optimum

Social optimum cont. 2

Theorem 1 cont. **(b)** A profile defined by

$$\hat{S}_i^{SO}(x) = \begin{cases} \xi x, & x \in (0, \frac{\hat{s}}{\xi}), \\ \hat{s} & \text{otherwise}, \end{cases}$$

is the unique social optimum both for *n* players and the continuum of players.

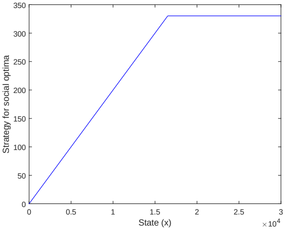

Figure: Strategy of each player at social optimum

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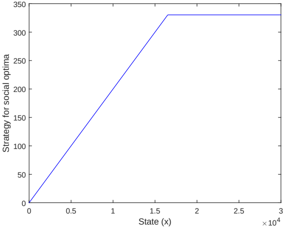

Figure: Strategy of each player at social optimum

For piecewise defined $\bar{V}$ and **s**, the Bellman equation has to be checked again!

Nash equilibrium for continuum of players

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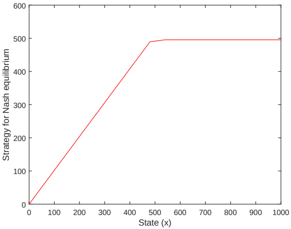

Figure: Strategy of each player at the Nash equilibria

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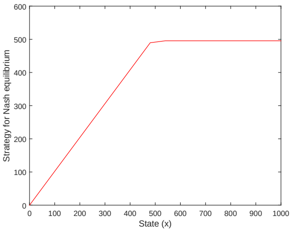

Figure: Strategy of each player at the Nash equilibria

- Exploitation many times larger than at the social optimum.

Nash equilibrium for continuum of players

Theorem 2 **(a)** The profile defined by

$$\hat{S}_i^{\mathrm{NE}}(x) = \begin{cases} (1+\xi)x & \text{for } x \leqslant \hat{x}_1, \\ \frac{A-f}{2} & \textit{otherwise}, \end{cases}$$

for $\hat{x}_1 = \frac{A-f}{2(1+\xi)}$, is the only feedback Nash equilibrium profile (up to measure equivalence).

Nash equilibrium for continuum of players

Theorem 2 cont. **(b)** The function defined by

$$\bar{V}_i^{\mathrm{NE}}(x) = \begin{cases} P_{\mathrm{depl}}(x), & \text{for } x \leqslant \hat{x}_1 \\ \frac{\sum_{k=1}^{N}(A-f)^2\beta^{k-1}}{8} + \beta^N P_{\mathrm{depl}}\left((1+\xi)^N x - \frac{(A-f)\sum_{k=1}^{N}\beta^{k-1}}{2}\right) & \text{for } x \in (\hat{x}_N, \hat{x}_{N+1}], \\ \frac{(A-f)^2}{8}\cdot\frac{(1+\xi)}{\xi} & \text{otherwise,} \end{cases}$$

for $P_{\mathrm{depl}}(x) = P((1+\xi)x, (1+\xi)x)$ (payoff resulting from immediate depletion of the resource) and $\hat{x}_N = \frac{A-f}{2}\sum_{k=1}^{N}\beta^k$ for $N \geqslant 1$, is the value function for optimization problem for the continuum of players game.

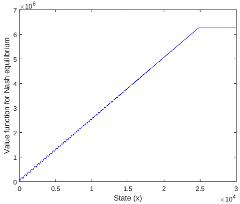

Figure: Value function at Nash equilibrium for continuum of players

Value function through a magnifying glass

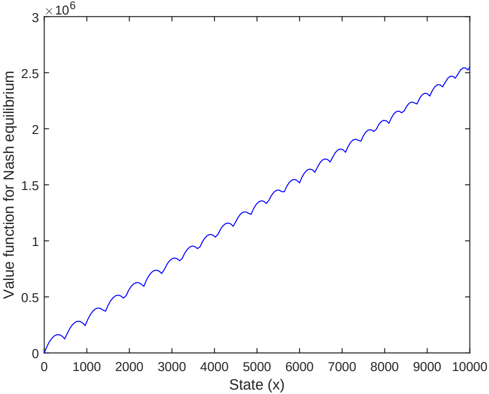

Figure: Value function at Nash equilibrium for continuum of players – zoomed view

Nash equilibrium for continuum of players cont2.

Theorem 2 cont. **(c)** For $x \in (\hat{x}_N, \hat{x}_{N+1}]$ with $\hat{x}_0 = 0$, the resource will be depleted/extracted in $N + 1$ stages, while for $x \geqslant \hat{x}_\infty = \lim_{N\to\infty} \hat{x}_N$, the resource will never be depleted.

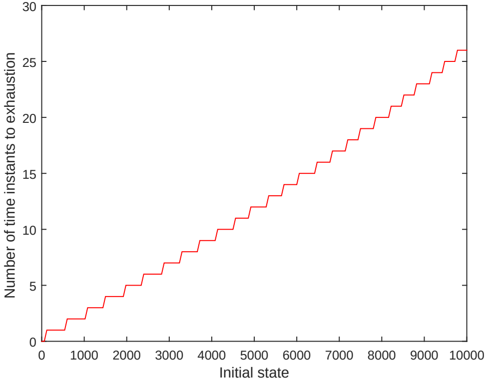

Figure: Number of time moments to resource exhaustion at Nash equilibrium for continuum of players

Nash equilibria for *n* players

- For *n* players, a similar value function to that for continuum of players, with number of stages to depletion nonstrictly increasing as *x* increases, can be expected.

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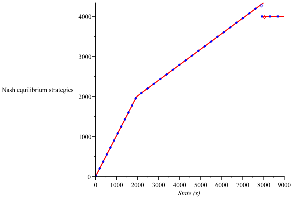

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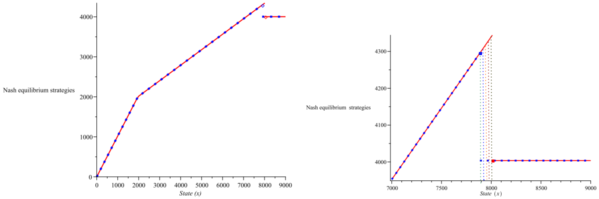

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(b) two symmetric Nash equilibria—zoomed view

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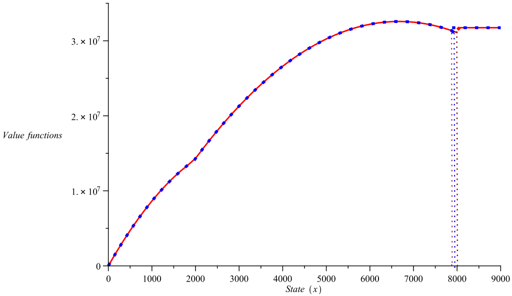

Figure: Two stage truncation of the game—the value functions at two symmetric Nash equilibria

Enforcing social optimality by a tax or tax-subsidy system

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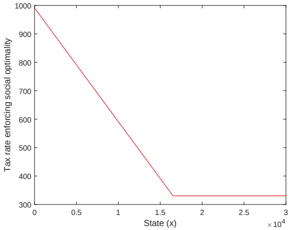

Figure: Rate of tax enforcing social optimality for continuum of

Enforcing social optimality by a tax – cont.

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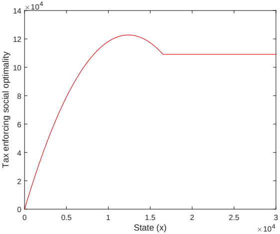

Figure: Tax enforcing social optimality for continuum of players

Enforcing social optimality by a tax – cont.

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Extensions of the model and introducing carrying capacity

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- All the above results remain valid if we appropriately modify the dynamics of the state above $\frac{\hat{s}}{\xi}$ in order to take into account the carrying capacity of the environment.

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- A class of optimal control problems (i.e. $n = 1$), analogous to our social optimality but for a whole interval of possible discount factors (a slightly more impatient decision makers): for a candidate V^f for the value function calculated analogously as in Theorem 1 and a control S^f from the rhs of the (BE) with V^f, for every $\epsilon > 0$, there is a discount factor close to the golden rule such that the Bellman equation is fulfilled everywhere besides an ϵ-neighbourhood of 0, while S^f is far from the optimal control while V^f from the value function on the set of all reasonable states (i.e. below $\frac{\hat{s}}{\xi}$). [5] R. Singh, A. Wiszniewska-Matyszkiel, 2020, *A class of linear quadratic dynamic optimization problems with state dependent constraints*, MMOR, 91, 325–355.
- Nested induction (backward and forward) plus concave analysis needed to derive the optimal control analytically – piecewise linear with infinitely many pieces.

Continuous time

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- General theory for such problems still not developed (viscosity solutions for infinite horizon, sufficiency and necessity, etc.)

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- Adding even very inherent constraints can change the solutions drastically, with several surpsises.

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Thank you for your attention!