The Curse of Dimensionality for Continuous Problems

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Curse of Dimensionality

${\mathcal E}$	error demand
d	the (large) number of variables
n(arepsilon,d)	the minimal cost, to be defined

Many problems suffer from the *curse of dimensionality*

 $n(\varepsilon, d) \ge c (1+C)^d$

for infinitely many d with c, C > 0.

Multivariate Approximation

 F_d the space of *d*-variate real infinitely differentiable functions $f: [0,1]^d \to \mathbb{R}$ with the norm

$$\|f\|_{F_d} = \sup_{\alpha} \|D^{\alpha}f\|_{L_{\infty}(0,1]^d}$$

Here, $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_d]$ with $\alpha_j = 0, 1, \dots$ and

$$D^{\alpha} f = \frac{\partial^{\alpha_1 + \dots + \alpha_d}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f$$

Is the unit ball of F_d large ?

We want to approximate

 $\operatorname{APP}_d: F_d \to L_{\infty}([0,1]^d), \quad \operatorname{APP}_d f = f, \quad ||\operatorname{APP}_d|| = 1$

Algorithms

$$\operatorname{APP}_d f = f \approx A_{n,d}(f) := \phi_{n,d}(L_1(f), L_2(f), \dots, L_n(f)), \quad L_j \in F_d^*$$

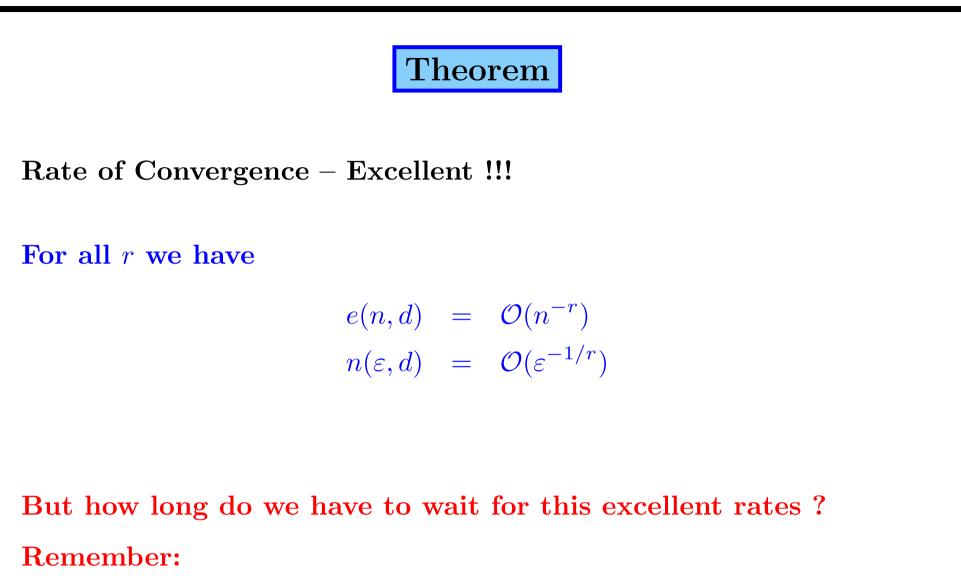
Worst Case Setting

algorithm error

nth minimal error

information complexity

$$e(A_{n,d}) = \sup_{\|f\|_{F_d} \le 1} \|f - A_{n,d}(f)\|_{L_{\infty}([0,1]^d)}$$
$$e(n,d) = \inf_{A_{n,d}} e(A_{n,d})$$
$$n(\varepsilon,d) = \min\{n : e(n,d) \le \varepsilon\}$$



the factors in the big \mathcal{O} notation may depend on d and r.

Curse is present !!!

$$\begin{array}{lll} e(n,d) &=& 1 & \quad \mbox{for all } n = 0, 1, \dots, 2^{\lfloor d/2 \rfloor} - 1 \\ n(\varepsilon,d) &\geq& 2^{\lfloor d/2 \rfloor} & \quad \mbox{for all } \varepsilon \in (0,1) \mbox{ and } d = 1, 2, \dots \end{array}$$

So we have to wait exponentially long to enjoy excellent rates !!!!

Remarks

- holds for $L_p([0,1]^d)$
- holds even if F_d is the space of *d*-variate polynomials of first degree in each variable
- proof based on identifying two functions f and -f for which $L_j(f) = 0$ for j = 1, 2, ..., n and $||f||_{F_d} = 1$
- Novak and W [2009], Weimar [2012], Werschulz and W [2009]

• but if

$$||f||_{F_d} := \sup_{\alpha} ||D^{\alpha}f|| \le 1$$
 is replaced by $\sum_{|\alpha|\ge 0} [\alpha!]^{-1} ||D^{\alpha}f|| \le 1$

then the curse is not present, Vybiral [2014].

Multivariate Integration

For $f \in F_d$ we want to approximate

$$I_d(f) := \int_{[0,1]^d} f(t) \,\mathrm{d}t \quad \approx \quad A_{n,d}(f)$$

• Algorithms:

$$A_{n,d}(f) = \phi_{n,d}(f(x_1), f(x_2), \dots, f(x_n))$$
 with $x_j \in [0, 1]^d$

• Minimal Worst Case Error:

$$e(n,d) = \inf_{A_{n,d}} \sup_{\|f\|_{F_d} \le 1} |I_d(f) - A_{n,d}(f)|$$

• Worst Case Information Complexity:

$$n(\varepsilon, d) = \min\{n \mid e(n, d) \le \varepsilon\}$$

Multivariate Integration for Smooth Functions

 $K = \{K_d\} \qquad K_d > 0$

$$F_d = C_d^r(K) := \{ f : [0,1]^d \to \mathbb{R} : \|f\|_{\max} \le 1, \|D^{\alpha}f\|_{\max} \le K_d \,\forall \, |\alpha| \in [1,r] \}$$

Bakhvalov [1959]

$$n(\varepsilon, d) = \Theta(\varepsilon^{-d/r})$$

but factors in the Θ -notation depend on d and r. Curse?

Sukharev [1979]: The curse holds for r = 1 and $K_d \equiv 1$.

Otherwise, curse?

Multivariate Integration for Smooth Functions

 $C_d^r(K) := \{ f : [0,1]^d \to \mathbb{R} : |f(x)| \le 1, |D^{\alpha}f(x)| \le K_d \ \forall \ |\alpha| \in [1,r] \}$

What are necessary and sufficient conditions for $\{K_d\}$ to have the curse of dimensionality for multivariate integration?

Theorem (Hinrichs, Novak, Ullrich, W [2012])

The curse holds for $C_d^r(K)$ iff $\liminf_{d\to\infty} K_d \sqrt{d} > 0$

Multivariate Integration for Korobov Spaces $r = \{r_j\} \quad \text{with} \quad 1 \le r_1 \le r_2 \le \cdots$ $H_{r_j}: \quad 1\text{-periodic } f: [0,1] \to \mathbb{C}, \ f^{(r_j-1)} \text{ abs. cont}, \ f^{(r_j)} \in L_2$ $\|f\|_{H_{r_j}}^2 = \left|\int_0^1 f(t) \, \mathrm{d}t\right|^2 + \int_0^1 \left|f^{(r_j)}(t)\right|^2 \, \mathrm{d}t$

For $d \geq 1$,

$$F_d = H_{d,r} = H_{r_1} \otimes H_{r_2} \otimes \cdots \otimes H_{r_d}$$

Usually, it is assumed that $r_j \equiv r$

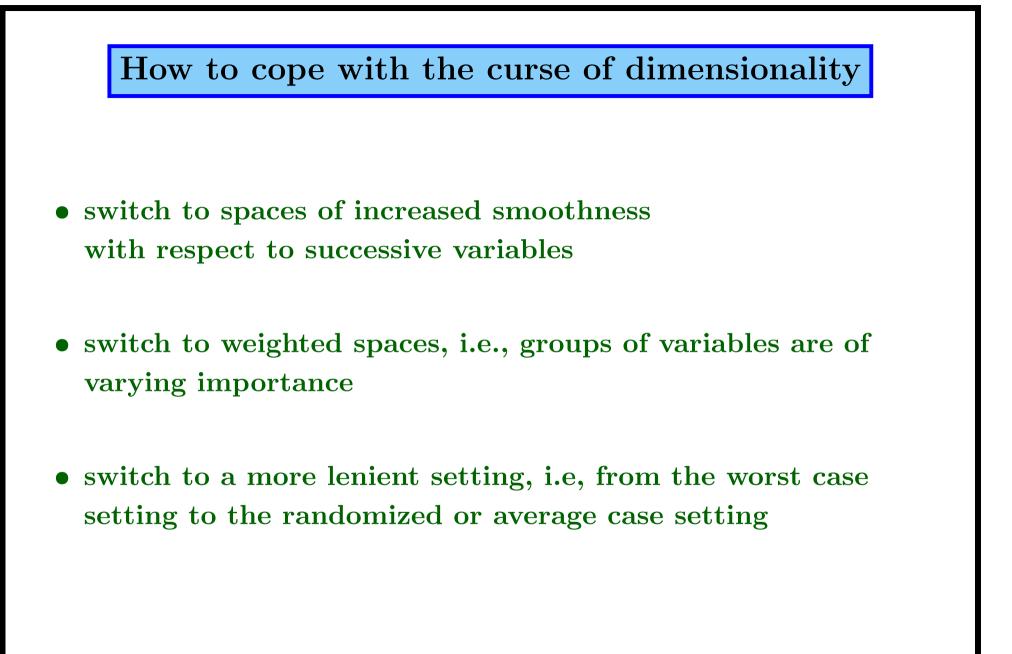
Theorem

Let $r_j \equiv r$. Then there exists $c_r, C_r > 0$ such that

 $n(\varepsilon, d) > c_r \left(1 + C_r\right)^d$

Based on Hickernell+W [2001] and Novak+W[2001], see also Sloan+W[2001]

Multivariate integration for Korobov space with arbitrarily smooth functions suffers from the curse of dimensionality



Increasing Smoothness

Multivariate integration for Korobov spaces in the worst case setting with $r_1 \leq r_2 \leq \cdots$.

But we now allow to increase r_i

\mathbf{Let}

$$R := \limsup_{k \to \infty} \frac{\ln k}{r_k}$$

Theorem

If $R < 2 \ln 2\pi$ then

- no curse
- $n(\varepsilon, d) \leq C \varepsilon^{-p(1+p/2)}$ with $p := \max(r_1^{-1}, R/\ln 2\pi) < 2$, i.e., strong polynomial tractability

Based on Papageorgiou+W [09], Kuo, Wasilkowski+W[09]

Weighted Spaces

Major research activities in last 20 years...

In particular, for $r_j \equiv r$ and $\gamma = \{\gamma_j\}$, redefine H_{r_j,γ_j} with

$$\|f\|_{H_{r_j,\gamma_j}}^2 = \left\|\int_0^1 f(t) \,\mathrm{d}t\right\|^2 + \frac{1}{\gamma_j} \int_0^1 \left|f^{(r_j)}(t)\right|^2 \,\mathrm{d}t$$

For $d \geq 1$,

$$H_{d,r} = H_{r_1,\gamma_1} \otimes H_{r_2,\gamma_2} \otimes \cdots \otimes H_{r_d,\gamma_d}$$

Theorem

• Gnewuch+W[08]

 $\lim_{d\to\infty} \frac{\sum_{j=1}^d \gamma_j}{d} = 0$ iff no curse,

• Hickernell+W[01]

 $\limsup_{d\to\infty} \frac{\sum_{j=1}^d \gamma_j}{\ln d} < \infty \quad \text{iff} \quad \text{polynomial tractability,}$ i.e., $n(\varepsilon, d) \leq C d^q \varepsilon^{-p}$

• Hickernell+W[01]

 $\sum_{j=1}^{\infty} \gamma_j < \infty \quad \text{iff} \quad \text{strong polynomial tractability,}$ i.e., $n(\varepsilon, d) \leq C \varepsilon^{-p}$

More Lenient Settings

From Worst Case Setting to

- Randomized Setting
- Average Case Setting

Average Case Setting \leq Randomized Setting

Randomized Setting

• Algorithms:

 $A_{n,d}^{\omega}(f) = \phi_{n,d}^{\omega}(f(x_{1,\omega}), f(x_{2,\omega}), \dots, f(x_{n(\omega),\omega}))$ for a random ω

• Minimal Randomized Error:

$$e(n,d) = \inf_{\substack{A_{n,d}^{\omega} \ \|f\|_{H_{d,r}} \le 1}} \sup_{\|E\|_{H_{d,r}} \le 1} \left[\mathbb{E} |I_d(f) - A_{n,d}^{\omega}(f)|^2 \right]^{1/2}$$

• Randomized Information Complexity:

 $n(\varepsilon, d) = \min\{n \mid e(n, d) \le \varepsilon\}$

Monte Carlo Algorithm

$$A_{n,\omega}(f) = \frac{1}{n} \sum_{j=1}^{n} f(x_{j,\omega})$$

with

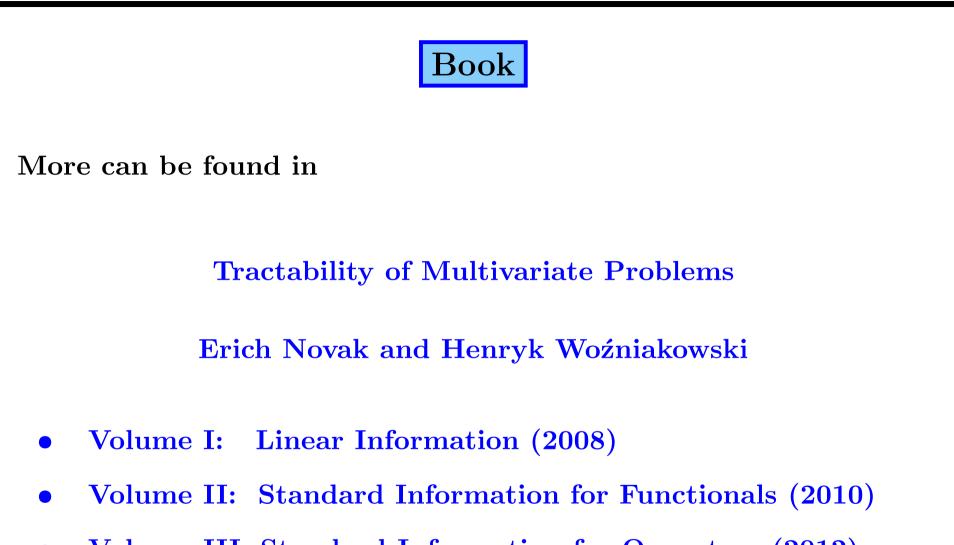
 $x_{j,\omega}$ iid with uniform distribution over $[0,1]^d$

Sloan+W[01] for Korobov spaces, obvious for $C^r_d(K)$ spaces

- $n(\varepsilon, d) \leq \varepsilon^{-2}$
- no curse and strong polynomial tractability

Conclusions

- Many multivariate problems suffer from the curse of dimensionality in the worst case setting
- We may sometimes break the curse of dimensionality by
 - switching to spaces of increased smoothness
 with respect to successive variables
 - switching to weighted spaces, i.e., groups of variables are of varying importance
 - switching to a more lenient setting, i.e., from the worst case setting to the randomized or average case setting



• Volume III: Standard Information for Operators (2012)

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