The Curse of Dimensionality for Continuous Problems

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Curse of Dimensionality

ε	**error demand**
d	**the (large) number of variables**
$n(\varepsilon, d)$	**the minimal cost, to be defined**

Many problems suffer from the *curse of dimensionality*

$$n(\varepsilon, d) \;\geq c\;(1 + C)^{\,d}$$

for infinitely many d with $c, C \;>\; 0$.

Multivariate Approximation

F_d the space of d-variate real infinitely differentiable functions $f:[0,1]^d \to \mathbb{R}$ with the norm

$$\|f\|_{F_d} = \sup_{\alpha} \|D^\alpha f\|_{L_\infty(0,1]^d}$$

Here, $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_d]$ with $\alpha_j = 0, 1, \dots$ and

$$D^\alpha \, f = \frac{\partial^{\alpha_1 + \cdots + \alpha_d}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \, f$$

Is the unit ball of F_d large ?

We want to approximate

$$\mathrm{APP}_d : \, F_d \to \, L_\infty([0,1]^d), \quad \mathrm{APP}_d f = f, \quad \|\mathrm{APP}_d\| = 1$$

Algorithms

$$\mathrm{APP}_d f = f \;\approx\; A_{n,d}(f) := \phi_{n,d}(L_1(f), L_2(f), \dots, L_n(f)), \qquad L_j \in F_d^*$$

Worst Case Setting

algorithm error $\quad e(A_{n,d}) = \sup\limits_{\|f\|_{F_d} \le 1} \|f - A_{n,d}(f)\|_{L_\infty([0,1]^d)}$

nth minimal error $\quad e(n,d) = \inf\limits_{A_{n,d}} e(A_{n,d})$

information complexity $\quad n(\varepsilon, d) = \min\{n : \; e(n,d) \le \varepsilon\}$

Theorem

Rate of Convergence – Excellent !!!

For all r we have

$$\begin{aligned} e(n,d) &= \mathcal{O}(n^{-r}) \\ n(\varepsilon,d) &= \mathcal{O}(\varepsilon^{-1/r}) \end{aligned}$$

But how long do we have to wait for this excellent rates ?

Remember:

the factors in the big $\mathcal{O}$ notation may depend on d and r.

Curse is present !!!

$$\begin{aligned} e(n,d) &= 1 && \textbf{for all } n = 0,1,\dots,2^{\lfloor d/2 \rfloor}-1 \\ n(\varepsilon,d) &\geq 2^{\lfloor d/2 \rfloor} && \textbf{for all } \varepsilon \in (0,1) \textbf{ and } d = 1,2,\dots \end{aligned}$$

So we have to wait exponentially long
to enjoy excellent rates !!!!

Remarks

- **holds for** $L_p([0,1]^d)$
- **holds even if** F_d **is the space of** d**-variate polynomials of first degree in each variable**
- **proof based on identifying two functions** f **and** $-f$ **for which** $L_j(f) = 0$ **for** $j = 1, 2, \ldots, n$ **and** $\|f\|_{F_d} = 1$
- **Novak and W [2009], Weimar [2012], Werschulz and W [2009]**
- **but if**

$$\|f\|_{F_d} := \sup_{\alpha} \|D^{\alpha} f\| \le 1 \quad \textbf{is replaced by} \quad \sum_{|\alpha| \ge 0} [\alpha!]^{-1} \|D^{\alpha} f\| \le 1$$

then the curse is *not* present, Vybiral [2014].

Multivariate Integration

For $f \in F_d$ we want to approximate

$$I_d(f) := \int_{[0,1]^d} f(t)\,\mathrm{d}t \quad \approx \quad A_{n,d}(f)$$

- **Algorithms:**

$$A_{n,d}(f) = \phi_{n,d}(f(x_1), f(x_2), \dots, f(x_n)) \quad \text{with} \quad x_j \in [0,1]^d$$

- **Minimal Worst Case Error:**

$$e(n,d) = \inf_{A_{n,d}} \sup_{\|f\|_{F_d} \le 1} |I_d(f) - A_{n,d}(f)|$$

- **Worst Case Information Complexity:**

$$n(\varepsilon, d) \ = \ \min\{\, n \mid \ e(n,d) \le \varepsilon \,\}$$

Multivariate Integration for Smooth Functions

$$K = \{K_d\} \qquad K_d > 0$$

$$F_d = C_d^r(K) := \{f : [0,1]^d \to \mathbb{R} \, : \, \|f\|_{\max} \le 1, \ \|D^\alpha f\|_{\max} \le K_d \, \forall \, |\alpha| \in [1,r]\}$$

Bakhvalov [1959]

$$n(\varepsilon, d) = \Theta(\varepsilon^{-d/r})$$

but factors in the Θ-notation depend on d and r. Curse?

Sukharev [1979]: The curse holds for $r = 1$ and $K_d \equiv 1$.

Otherwise, curse?

Multivariate Integration for Smooth Functions

$$C_d^r(K) := \{f : [0,1]^d \to \mathbb{R} : \ |f(x)| \le 1, \ \ |D^\alpha f(x)| \le K_d \ \forall \ |\alpha| \in [1,r]\}$$

What are necessary and sufficient conditions for $\{K_d\}$ to have the curse of dimensionality for multivariate integration?

Theorem (Hinrichs, Novak, Ullrich, W [2012])

The curse holds for $C_d^r(K)$ iff $\liminf_{d\to\infty} K_d \sqrt{d} > 0$

Multivariate Integration for Korobov Spaces

$$r = \{r_j\} \quad \textbf{with} \quad 1 \le r_1 \le r_2 \le \cdots$$

H_{r_j}**:** 1**-periodic** $f : [0,1] \to \mathbb{C}$**,** $f^{(r_j-1)}$ **abs. cont,** $f^{(r_j)} \in L_2$

$$\|f\|_{H_{r_j}}^2 = \left| \int_0^1 f(t)\,\mathrm{d}t \right|^2 + \int_0^1 \left| f^{(r_j)}(t) \right|^2 \mathrm{d}t$$

For $d \ge 1$**,**

$$F_d = H_{d,r} = H_{r_1} \otimes H_{r_2} \otimes \cdots \otimes H_{r_d}$$

Usually, it is assumed that $r_j \equiv r$

Theorem

Let $r_j \equiv r$. Then there exists $c_r, C_r > 0$ such that

$$n(\varepsilon, d) \; > \; c_r \, (1 + C_r)^{\,d}$$

Based on Hickernell+W [2001] and Novak+W[2001], see also Sloan+W[2001]

Multivariate integration for Korobov space
with arbitrarily smooth functions
suffers from the curse of dimensionality

How to cope with the curse of dimensionality

- **switch to spaces of increased smoothness with respect to successive variables**

- **switch to weighted spaces, i.e., groups of variables are of varying importance**

- **switch to a more lenient setting, i.e, from the worst case setting to the randomized or average case setting**

Increasing Smoothness

Multivariate integration for Korobov spaces in the worst case setting with $r_1 \le r_2 \le \cdots$.

But we now allow to increase r_j

Let

$$R := \limsup_{k\to\infty} \frac{\ln\, k}{r_k}$$

Theorem

If $R < 2\,\ln\, 2\pi$ then

- **no curse**
- $n(\varepsilon, d) \le C\,\varepsilon^{-p(1+p/2)}$ **with** $p := \max(r_1^{-1}, R/\ln\, 2\pi) < 2,$ **i.e., strong polynomial tractability**

Based on Papageorgiou+W [09], Kuo, Wasilkowski+W[09]

Weighted Spaces

Major research activities in last 20 years...

In particular, for $r_j \equiv r$ and $\gamma = \{\gamma_j\}$, redefine H_{r_j,γ_j} with

$$\|f\|^2_{H_{r_j,\gamma_j}} \;=\; \left|\int_0^1 f(t)\,\mathrm{d}t\right|^2 \;+\; \frac{1}{\gamma_j}\int_0^1 \left|f^{(r_j)}(t)\right|^2 \mathrm{d}t$$

For $d \geq 1$,

$$H_{d,r} = H_{r_1,\gamma_1} \otimes H_{r_2,\gamma_2} \otimes \cdots \otimes H_{r_d,\gamma_d}$$

Theorem

- **Gnewuch+W[08]**

  $\lim_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_j}{d} = 0$ **iff no curse,**

- **Hickernell+W[01]**

  $\limsup_{d\to\infty} \frac{\sum_{j=1}^{d} \gamma_j}{\ln\, d} < \infty$ **iff polynomial tractability,**

  i.e., $n(\varepsilon, d) \le C\, d^q\, \varepsilon^{-p}$

- **Hickernell+W[01]**

  $\sum_{j=1}^{\infty} \gamma_j < \infty$ **iff strong polynomial tractability,**

  i.e., $n(\varepsilon, d) \le C\, \varepsilon^{-p}$

More Lenient Settings

From Worst Case Setting to

- **Randomized Setting**
- **Average Case Setting**

Average Case Setting $\leq$ Randomized Setting

Randomized Setting

- **Algorithms:**

$$A^{\omega}_{n,d}(f) = \phi^{\omega}_{n,d}(f(x_{1,\omega}), f(x_{2,\omega}), \dots, f(x_{n(\omega),\omega})) \quad \text{for a random } \omega$$

- **Minimal Randomized Error:**

$$e(n,d) = \inf_{A^{\omega}_{n,d}} \sup_{\|f\|_{H_{d,r}} \le 1} \left[\mathbb{E}\, |I_d(f) - A^{\omega}_{n,d}(f)|^2\right]^{1/2}$$

- **Randomized Information Complexity:**

$$n(\varepsilon, d) \;=\; \min\{\, n \,|\;\; e(n,d) \,\le\, \varepsilon \,\}$$

Monte Carlo Algorithm

$$A_{n,\omega}(f) = \frac{1}{n} \sum_{j=1}^{n} f(x_{j,\omega})$$

with

$x_{j,\omega}$ **iid with uniform distribution over** $[0,1]^d$

Sloan+W[01] for Korobov spaces, obvious for $C_d^r(K)$ spaces

- $n(\varepsilon, d) \le \varepsilon^{-2}$
- **no curse and strong polynomial tractability**

Conclusions

- **Many multivariate problems suffer from the curse of dimensionality in the worst case setting**
- **We may sometimes break the curse of dimensionality by**
  - **switching to spaces of increased smoothness with respect to successive variables**
  - **switching to weighted spaces, i.e., groups of variables are of varying importance**
  - **switching to a more lenient setting, i.e., from the worst case setting to the randomized or average case setting**

Book

More can be found in

Tractability of Multivariate Problems

Erich Novak and Henryk Woźniakowski

- **Volume I: Linear Information (2008)**
- **Volume II: Standard Information for Functionals (2010)**
- **Volume III: Standard Information for Operators (2012)**

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