# The automorphism group of the random poset

Aleksandra Kwiatkowska

joint work with Aristotelis Panagiotopoulos

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Let $G$ be a Polish group.

**Proposition**

*The following conditions are equivalent:*

1. $G$ *is a closed subgroup of* $S_\infty = \mathrm{Sym}(X)$ *– topological group of all bijections of a countable set* $X$*, equipped with the pointwise convergence topology;*
2. $G$ *has a neighbourhood basis of the identity that consists of open subgroups;*
3. $G$ *is an automorphism group of a countable first-order structure;*
4. $G$ *is an automorphism group of a countable ultrahomogeneous relational first-order structure.*

# Ultrahomogeneous structures

**Definition**

A countable first-order structure $M$ is ultrahomogeneous if every automorphism between finitely generated substructures of $M$ can be extended to an automorphism of the whole $M$.

**Example**

- rationals with the ordering
- the random graph
- the random poset
- the rational Urysohn metric space

# How do we obtain countable ultrahomogeneous structures?

A countable family $\mathcal{F}$ of finitely generated structures is a Fraïssé family if:

1. (F1) (hereditary property: HP) if $A \in \mathcal{F}$ and $B \subseteq A$ is finitely generated then $B \in \mathcal{F}$;
2. (F2) (joint embedding property: JEP) for any $A, B \in \mathcal{F}$ there is $C \in \mathcal{F}$ and embeddings from $A$ to $C$ and from $B$ to $C$;
3. (F3) (amalgamation property: AP) for $A, B_1, B_2 \in \mathcal{F}$ and embeddings $\varphi_1 \colon A \to B_1$ and $\varphi_2 \colon A \to B_2$, there exist $C$, and embeddings $\psi_1 \colon B_1 \to C$ and $\psi_2 \colon B_2 \to C$ such that $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$.

# Fraïssé's Theorem

**Theorem (Fraïssé)**

*For every Fraïssé family $\mathcal{F}$ there is a unique countable ultrahomogeneous structure $M$ (called Fraïssé limit), such that the set of finitely generated substructures of $M$ is equal to $\mathcal{F}$.*

# Examples

## Example

- $\mathcal{F}$ = the family of finite linear orders
  Fraïssé limit = rationals with the ordering
- $\mathcal{F}$ = the family of finite graphs
  Fraïssé limit = the random graph
- $\mathcal{F}$ = the family of finite partially ordered sets (posets)
  Fraïssé limit = the random poset
- $\mathcal{F}$ = the family of finite metric spaces with rational distances
  Fraïssé limit = the rational Urysohn metric space

# Automatic continuity, small index property

**Definition**

A Polish group $G$ is automatically continuous if every abstract homomorphism from $G$ to a separable topological group is continuous.

**Definition**

A Polish group $G$ has the small index property if any subgroup of index $< 2^{\aleph_0}$ is open.

# Ample generics 1

## Definition

A topological group $G$ has ample generics if for every $n$ the diagonal conjugacy action of $G$ on $G^n$ given by $(g,(h_1,\ldots,h_n)) \mapsto (gh_1g^{-1},\ldots,gh_ng^{-1})$ has a comeager orbit.

## Example

- (Hrushovski, 1992) automorphism group of the random graph
- (Solecki, 2005) automorphism group of the rational Urysohn space
- (Kwiatkowska, 2012) homeomorphism group of the Cantor set

# EPPA (Hrushovski property)

**Definition**

A family $\mathcal{F}$ of finite structures in a given signature has the extension property for partial automorphisms (EPPA) if for every $A \in \mathcal{F}$, there exists $B \in \mathcal{F}$ containing $A$ as a substructure such that every partial automorphism of $A$ extends to an automorphism of $B$.

**Theorem (Siniora-Solecki, 2019)**

*Suppose that L is a finite relational language. Then any free amalgamation class of finite L-structures has coherent EPPA.*

The proof uses the Herwig–Lascar theorem.

**Corollary**

*Automorphism groups of relational free amalgamation structures have ample generics.*

# Ample generics 2

**Theorem (Kwiatkowska-Malicki, 2019)**

*Let $M$ be a countable structure such that for any finite $X \subseteq M$ the stabilizer $\mathrm{Aut}_X(M)$ fixes only finitely many points, and let $G = \mathrm{Aut}(M)$. Suppose that $G$ has ample generics. Then for every $n$ and a generic $n$-tuple $(f_1, \ldots, f_n)$ in $G^n$:*

1. $\overline{\langle f_1, \ldots, f_n \rangle}$ *is discrete, or*
2. $\overline{\langle f_1, \ldots, f_n \rangle}$ *is compact.*

**Example**

1. discrete case: homeomorphism group of the Cantor set
2. compact case: EPPA

# The criterion

Let $M$ be an ultrahomogeneous structure and let $\mathcal{K} = \mathrm{Age}(M)$. Let

$\mathcal{K}_n = \{(A, p_1^A, \ldots, p_n^A) \colon A \in \mathcal{K}$ and $p_i^A$ is a partial automorphism of $A\}$

**Theorem (Ivanov 1999, Kechris-Rosendal 2007)**

*There exists a comeager n-conjugacy class in $\mathrm{Aut}(M)$ iff $\mathcal{K}_n$ has JEP and WAP.*

# JEP and WAP

**Definition (JEP)**

For every $\bar{p} = (p_1, \ldots, p_n)$ and $\bar{q} = (q_1, \ldots, q_n)$ there exists $\bar{r} = (r_1, \ldots, r_n)$ which embeds $\bar{p}$ and $\bar{q}$.

**Definition (no 2-WAP)**

There is $\bar{p} = (p_1, p_2)$ such that for every $\bar{q} = (q_1, q_2)$ and an embedding $\delta \colon \bar{p} \to \bar{q}$ there are embeddings $\alpha_1 \colon \bar{q} \to \bar{r}_1$ and $\alpha_2 \colon \bar{q} \to \bar{r}_2$ such that we cannot amalgamate $\bar{r}_1$ and $\bar{r}_2$ over $\bar{p}$. That is, there is no $\bar{s}$ and $\beta_1 \colon \bar{r}_1 \to \bar{s}$ and $\beta_2 \colon \bar{r}_2 \to \bar{s}$ such that $\beta_1 \circ \alpha_1 \circ \delta = \beta_2 \circ \alpha_2 \circ \delta$

# The automorphism group of the random poset

**Theorem (Glass-McCleary-Rubin, 1993)**

*The automorphism group of the random poset is simple.*

**Theorem (Kuske-Truss, 2001)**

*The automorphism group of the random poset has a comeager conjugacy class.*

# The generic automorphism of the random poset

Let $f \in \mathrm{Aut}(\mathbb{P})$. Let $\sim_f$ be the binary relation on $\mathbb{P}$:

$$x \sim_f y \iff \exists i, j \in \mathbb{Z} \text{ such that } f^i(x) \leq y \leq f^j(x).$$

Equivalence classes are called orbitals. The orbital of $x \in \mathbb{P}$ denote by $\mathcal{O}_f(x)$. We have the partial order on orbitals given by:

$$\mathcal{O}_f(x) <_f^s \mathcal{O}_f(y) \iff \forall x' \sim_f x \forall y' \sim_f y (x' < y').$$

**Theorem (Ihli)**

*For the generic $f \in \mathrm{Aut}(\mathbb{P})$, the partial order $<_f^s$ on $\mathcal{O}_f(\mathbb{P})$ is isomorphic to the partial order on $\mathbb{P}$. Moreover, for every $\sigma \in \{-1, 1\}$ and $1 \leq n \leq \infty$, the sets $\{\mathcal{O}_f(x) \colon par(x, f) = \sigma\}$ and $\{\mathcal{O}_f(x) \colon par(x, f) = 0 \wedge sp(x, f) = n\}$ are dense in $\mathcal{O}_f(\mathbb{P})$.*

# Main question

**Definition**

Let $G$ be a topological group. A pair $(f_1, f_2) \in G^2$ is a generic pair if the conjugacy class $\{(gf_1g^{-1}, gf_2g^{-1}) \colon g \in G\}$ of $(f_1, f_2)$ is comeager in $G^2$.

**Question (Truss 2007, Kuske-Truss 2001)**

*Does the automorphism group of the random poset has a generic pair?*

# Main Theorem

**Theorem (Kwiatkowska-Panagiotopoulos, 2020)**

*The automorphism group* $\mathrm{Aut}(\mathbb{P})$ *of the random poset* $\mathbb{P}$ *does not have a generic pair. In fact, for every* $(f_1, f_2) \in \mathrm{Aut}(\mathbb{P})^2$ *the diagonal conjugacy class* $\{(gf_1g^{-1}, gf_2g^{-1})\colon g \in \mathrm{Aut}(\mathbb{P})\}$ *of* $(f_1, f_2)$ *is meager in* $\mathrm{Aut}(\mathbb{P})^2$.

# Main Lemma

Let $(B, <_B, f_B)$ be a partial automorphism and let $a, b \in B$ with $a <_B b$. We say that $f_B$ is free in $(a, b)$, if whenever $(B, <_B) \preceq (C, <_C)$ and $c_1, \ldots, c_\ell \in C$, with $a <_C c_1 <_C \cdots <_C c_\ell <_C b$, then $(C, <_C, f_C)$ is a partial automorphism, where

$$f_C := f_B \cup \{(a, c_1), (c_1, c_2), \ldots, (c_{\ell-1}, c_\ell)\}.$$

**Lemma (Kwiatkowska-Panagiotopoulos, 2020)**

*Let $(A, <_A, f_A)$ be a partial automorphism and let $s \in A$ with $s <_A f_A(s)$. Then, there is an extension $(A, <_A, f_A) \preceq (B, <_B, f_B)$, some $n \in \mathbb{N}$, and $a, b \in B$ with $a <_B b$, so that $f_B^n(s) = a$ and $f_B$ is free in $(a, b)$.*

# Questions

## Definition

A Polish group $G$ has the automatic continuity property if for every Polish group $H$ every abstract homomorphism $\phi \colon G \to H$ is continuous.

## Question

- *Does the automorphim group of the random poset has the automatic continuity property or the small index property?*