A strongly rigid countable Hausdorff space

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Rigid and strongly rigid spaces

Definition

A topological space X is

- rigid if every homeomorphism $h : X \to X$ coincides with the identity map of X;
- strongly rigid if every non-constant continuous map $f : X \to X$ coincides with the identity map of X.

Trivial examples of (strongly) rigid spaces are topological spaces of cardinality ≤ 1. So, from now on, all (strongly) rigid spaces are assumed to contain more than one point.

Rigid continua

Many examples of (strongly) rigid spaces appear in Theory of Continua.

For example, it is easy to construct a rigid dendrite X (containing a dense set $\{x_n\}_{n\in\mathbb{N}}$ of points such that for every $n \in \mathbb{N}$ the complement $X \setminus \{x_n\}$ has exactly n connected components).

However dendrites never are strongly rigid because they contain arcs (=topological copies of $[0,1]$), and functionally Hausdorff spaces containing arcs are not strongly rigid (because such spaces admit many non-constant continuous maps to $[0,1]$).

Definition

A topological space X is functionally Hausdorff if for every distinct points $x, y \in X$, there exists a continuous function $f : X \to \mathbb{R}$ with $f(x) \neq f(y)$.

Strongly rigid continua

Nonetheless strongly rigid continus do exist.

Theorem (Cook, 1967)

There exists a strongly rigid plane continuum K.

The Cook continuum has even a stronger rigidity property: any non-constant continuous map $f : X \to K$ defined on a subcontinuum X of K is the identity embedding of K into X.

Strongly rigid spaces are connected

Simple Fact:

Every strongly rigid topological space X is connected.

Indeed, if $X = U \cup V$ for two disjoint nonempty clopen sets U, V, then we can choose any points $u \in U$ and $v \in V$ and consider a non-constant non-identity continuous map

$$f : X \to X, \quad f : x \mapsto \begin{cases} u & \text{if } x \in V; \\ v & \text{if } x \in U; \end{cases}$$

witnessing that X is not strongly rigid.

Corollary

Every strongly rigid functionally Hausdorff space X has cardinality $|X| \geq \mathfrak{c}$.

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Strongly rigid countable Hausdorff spaces?

Problem (Kannan, Rajagopalan, 1978)

Is there a strongly rigid countable Hausdorff space?

Problem (`Patric Rabau` from `MathOverflow`; Nov 2022)

Is there a connected countable Hausdorff space X which is anticompact ($=$ compact sets in X are finite)?

Main Theorem (B., Stelmakh, 2023)

A strongly rigid anticompact countable Hausdorff space exists.

Such a countable Hausdorff space is necessarily connected.
One of the most known connected countable Hausdorff spaces is the Bing plane.

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The Bing plane

Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{Q}_+ := \{x \in \mathbb{Q} : x \geq 0\}$.
The Bing plane is the set $\mathbb{Q} \times \mathbb{Q}_+$ endowed with the topology generated by the countable base consisting of the sets

$$B\big((x,y),\varepsilon\big) = \{(x,y)\} \cup \{(z,0) : |z - (x \pm \tfrac{y}{\sqrt{3}})| < \varepsilon\}$$

where $(x,y) \in \mathbb{Q} \times \mathbb{Q}_+$ and $\varepsilon \in \mathbb{Q}_+ \setminus \{0\}$.

Theorem (Bing, 1953)

The Bing plane is countable, second-countable, Hausdorff and connected.

The connectedness of the Bing plane follows from the fact that the closures of any two nonempty open subsets of the Bing plane have infinite intersection.
Topological spaces with this connectedness property are called Brown.

Brown spaces

Definition

A topological space X is Brown if X for every nonempty open sets $U, V \subseteq X$, the intersection $\overline{U} \cap \overline{V}$ is infinite.

The first example of a Brown space was constructed by Brown in 1953. This space is well-known in Topological Arithmetics as the Golomb space.

The Golomb space $\mathbb{G}$ is the set $\mathbb{N}$ of positive integers endowed with the topology generated by the base consisting of the arithmetic sequences $S_{a,b} := \{a + bn : n \in \omega\}$ with coprime numbers $a, b \in \mathbb{N}$.

The Golomb space

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Theorem (Brown, 1953)

The Golomb space $\mathbb{G}$ is countable, second-countable, Hausdorff and superconnected in the sense that for any non-empty open sets $U_1, \dots, U_n \subseteq \mathbb{G}$ the intersection $\overline{U}_1 \cap \cdots \cap \overline{U}_n$ is infinite.

Superconnected spaces are Brown but not vice versa:
the Bing plane is Brown but contains three nonempty open sets U, V, W with $\overline{U} \cap \overline{V} \cap \overline{W} = \varnothing$.

The homogeneity of the Bing plane

Since the Bing plane and the Golomb space are examples of connected Hausdorff topological spaces, it is natural to ask whether those spaces are (strongly) rigid.

Theorem (Banakh, B., Hryniv, Stelmakh, 2021)

The Bing plane $\mathbb{B}$ is topologically homogeneous, moreover, any bijection $f : A \to B$ between finite subsets A and B in $\mathbb{B}$ extends to a homeomorphism of the Bing plane.

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The rigidity of the Golomb space

Theorem (B., Spirito, Turek; 2021)

The Golomb space is rigid.

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The Golomb space is not strongly rigid, because it admits continuum many continuous self-maps.

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Transforming Brown to strongly rigid

Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X. The topology $\mathcal{T}'$ is called an open refinement of the topology $\mathcal{T}$ if for any open sets $V \in \mathcal{T}$, $V' \in \mathcal{T}'$ and point $x \in V \cap V'$, there exists an open set $U \in \mathcal{T}$ such that $U \subseteq V \cap V'$ and $U \cup \{x\} \in \mathcal{T}'$.
This definition implies $\mathcal{T} \subseteq \mathcal{T}'$, so $\mathcal{T}'$ is indeed a refinement of $\mathcal{T}$.

Example

The Sorgenfrey topology on the real line is an open refinement of the Euclidean topology.

Theorem (B., Stelmakh; 2023)

The topology $\mathcal{T}$ of any second-countable Brown Hausdorff space X has an open refinement $\mathcal{T}'$ such that the topological space $(X, \mathcal{T}')$ is Brown, Hausdorff, anticompact and strongly rigid.

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The idea of the proof

We should change the topology $\mathcal{T}$ of X at each point $x \in X$ in order to make those points strongly topologically different. To ensure such a strong topological difference we shall exploit Rudin–Keisler incomparable ultrafilters on $\mathbb{N}$.

Definition

Two ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$ are Rudin–Keisler incomparable if $\beta f(\mathcal{U}) \neq \mathcal{V}$ and $\beta f(\mathcal{V}) \neq \mathcal{U}$ for every function $f : \mathbb{N} \to \mathbb{N}$.
Here $\beta f : \beta\mathbb{N} \to \beta\mathbb{N}$ is the unique continuous extension of f to the Stone-Čech compatification of the discrete space $\mathbb{N}$.

Theorem (Kunen, 1972)

There exist continuum many Rudin–Keisler incomparable ultrafilters on $\mathbb{N}$.

Let $(X, \mathcal{T})$ be a second-countable Brown Hausdorff space and $\{U_n\}_{n\in\omega}$ be a base of the topology $\mathcal{T}$ with $U_n \neq \varnothing$ for all n. For every $x \in X$ and $n \in \omega$, consider the set

$$O_n(x) := \bigcap\{U_i : x \in U_i,\ i \le n\}$$

and observe that $\big(O_n(x)\big)_{n\in\omega}$ is a neighborhood base of the topology $\mathcal{T}$ at x.

For every $n \in \mathbb{N}$ choose a point

$$b_n \in (\overline{U}_k \cap \overline{U}_m) \setminus \{b_i\}_{i<n},$$

where $n = 2^k(2m+1)$. The point b_n exists because $(X, \mathcal{T})$ is Brown and hence the set $\overline{U}_k \cap \overline{U}_m$ is infinite.

The sketch of the proof: preparation 1

Let $(X, \mathcal{T})$ be a second-countable Brown Hausdorff space and $\{U_n\}_{n\in\omega}$ be a base of the topology $\mathcal{T}$ with $U_n \neq \varnothing$ for all n. For every $x \in X$ and $n \in \omega$, consider the set

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The sketch of the proof: preparation 2

For every $x \in X$, choose an injective function $s_x : \mathbb{N} \to X \setminus \{x\}$ such that

- $s_x(n) \in O_n(x)$ for every $n \in \mathbb{N}$;
- if $x = b_{2^k(2m+1)}$ for some $k, m \in \omega$, then $s_x[3\mathbb{N}+1] \subseteq U_k$ and $s_x[3\mathbb{N}-1] \subseteq U_m$.

Next, choose a function $S_x : \mathbb{N} \to \mathcal{T}$ such that

- $s_x(n) \subseteq S_x(n) \subseteq O_n(x)$ for every $n \in \mathbb{N}$;
- $S_x(n) \cap S_x(m) = \varnothing$ for any distinct $n, m \in \mathbb{N}$;
- if $x = b_{2^k(2m+1)}$ for some $k, m \in \omega$, then $\bigcup_{n\in\mathbb{N}} S_x(3n+1) \subseteq U_k$ and $\bigcup_{n\in\mathbb{N}} S_x(3n-1) \subseteq U_m$.

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Rudin-Keisler ultrafilters appear and finish the game!

Since $(X, \mathcal{T})$ is second-countable and Hausdoff, $|X| \le \mathfrak{c}$ and hence there exists a family $(\mathcal{F}_x)_{x\in X}$ of Rudin–Keisler incomparable ultrafilters on $\mathbb{N}$.

Let $\mathcal{T}'$ be the topology on X consisting of sets $W \subseteq X$ such that for every $x \in X$ there exists a set $F \in \mathcal{F}_x$ such that $S_x[3F \pm 1] \subseteq W$, where $3F \pm 1 := \{3n + k : n \in F,\ k \in \{-1, 1\}\}$.

It is can be shown that $\mathcal{T}'$ is an open refinement of $\mathcal{T}$ and the space $(X, \mathcal{T})$ is Brown, Hausdorff, anticompact, and strongly rigid.

The End!

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Thank you!