Sparse graphs

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Graphs

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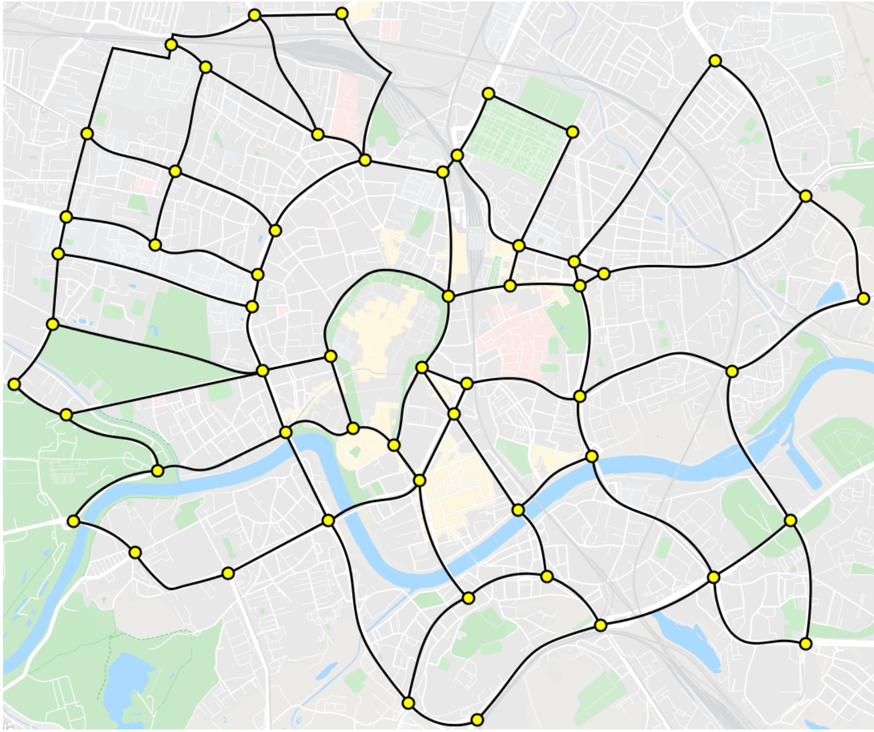

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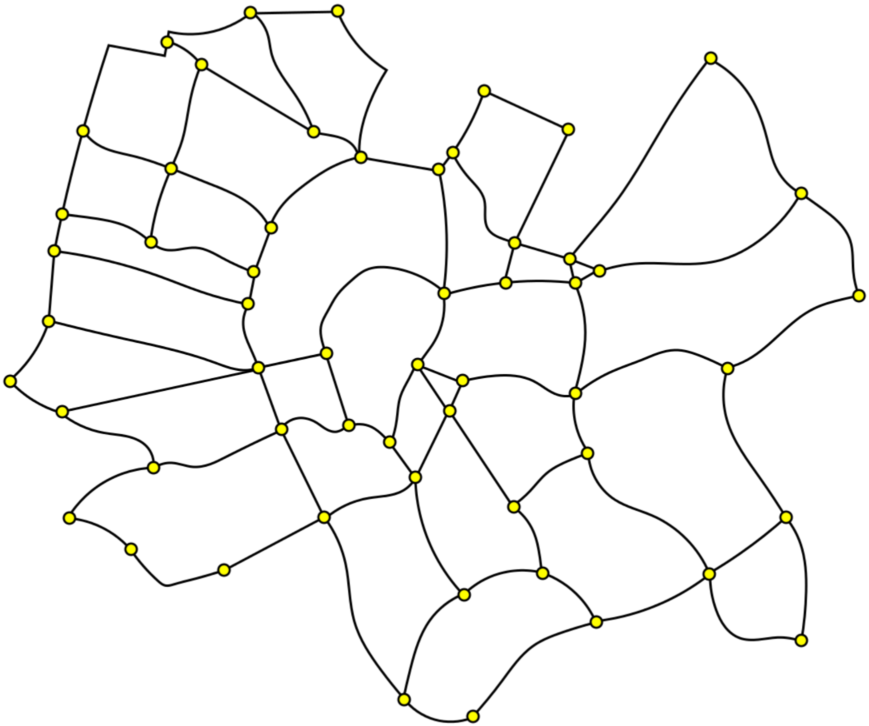

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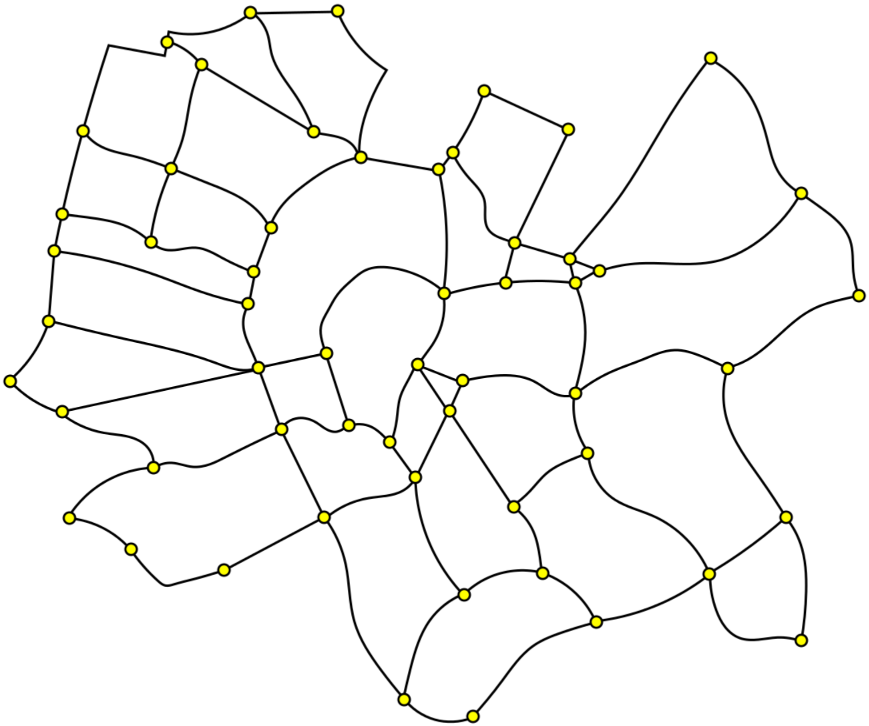

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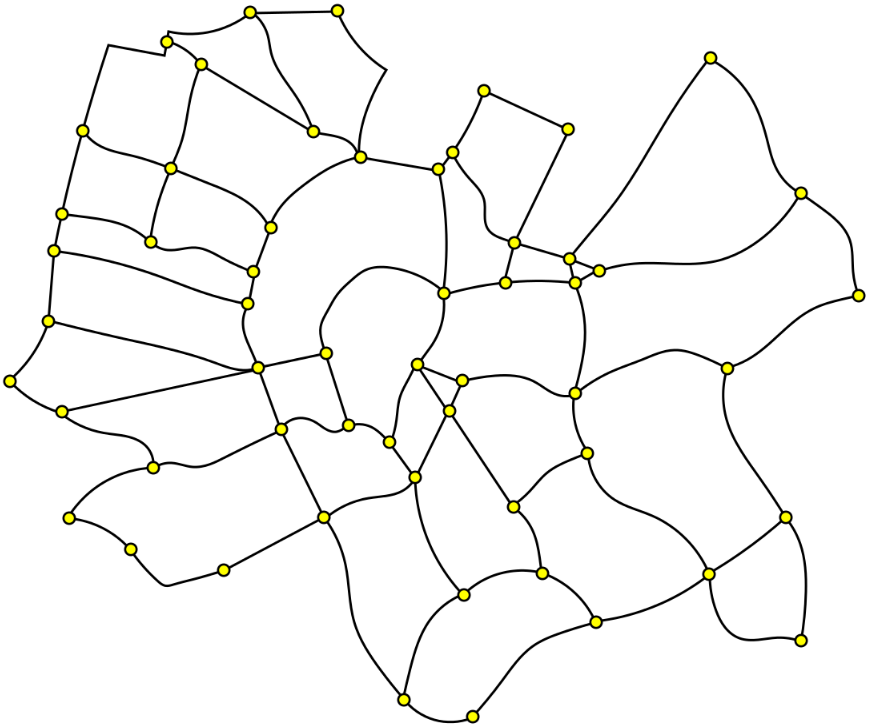

Graph: a set of vertices connected in pairs by edges

- **Finite** for the purpose of this talk.
- **Simple:** no two edges connect the same pair of vertices.

Graphs in practice

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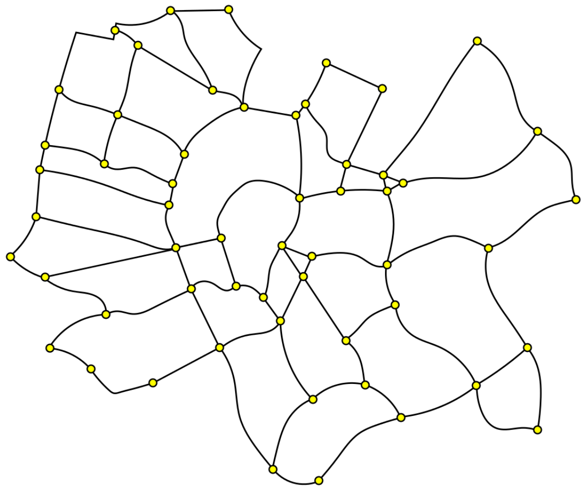
– Transportation networks



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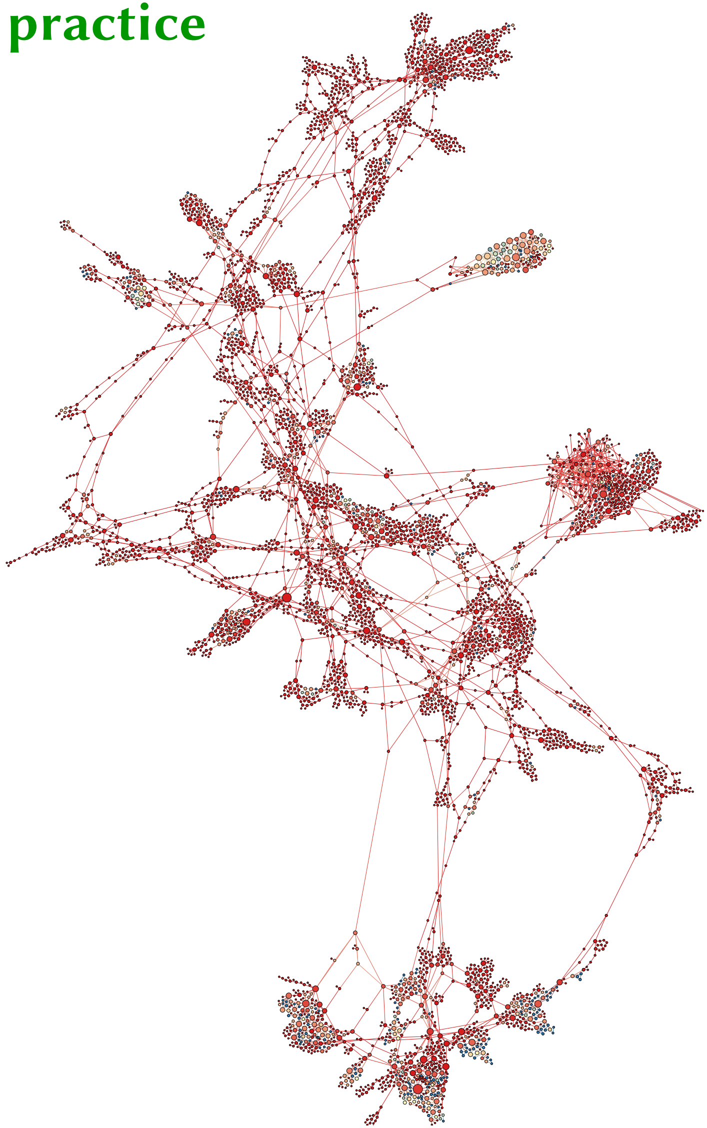

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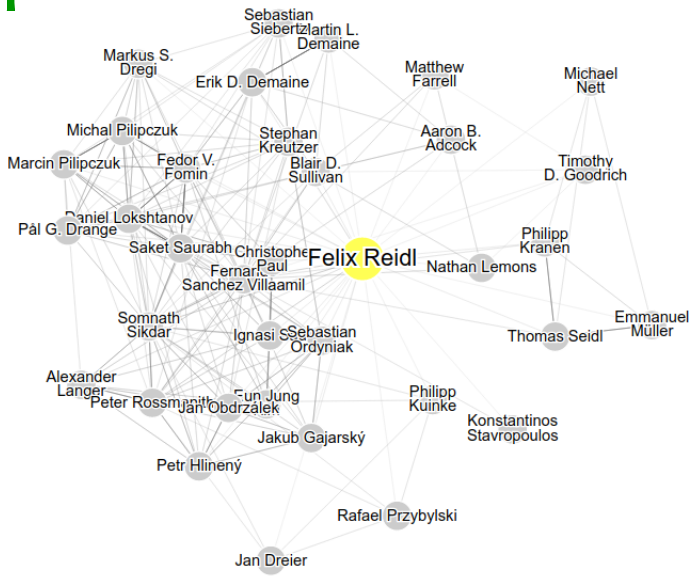

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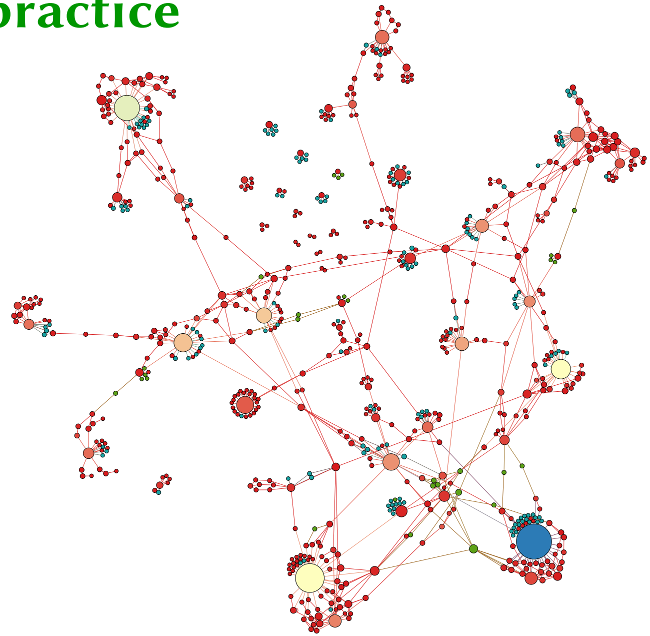

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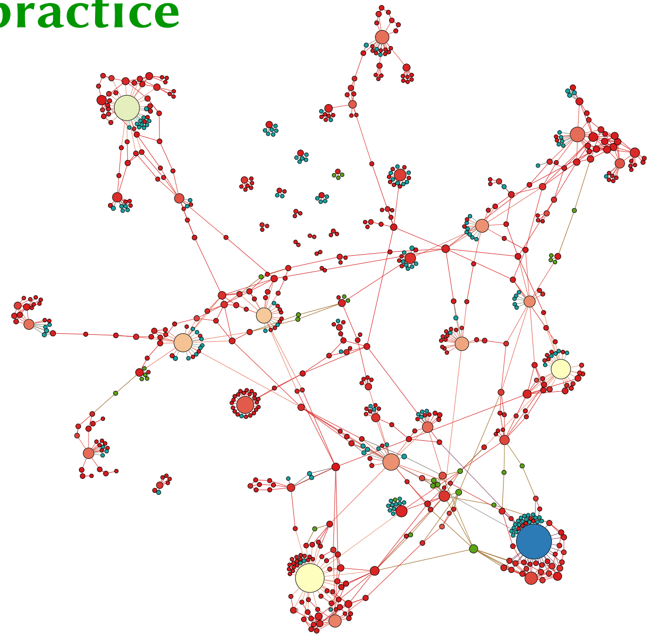

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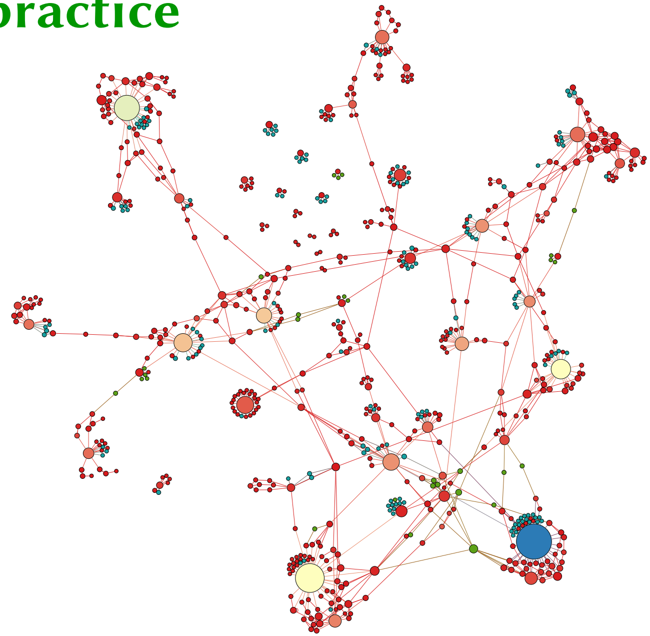

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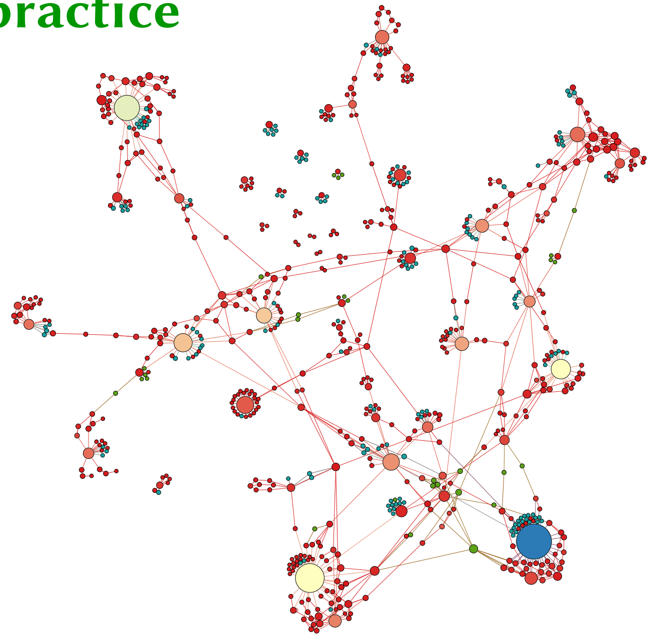

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- Facebook graph has average degree 338 and median degree 200.

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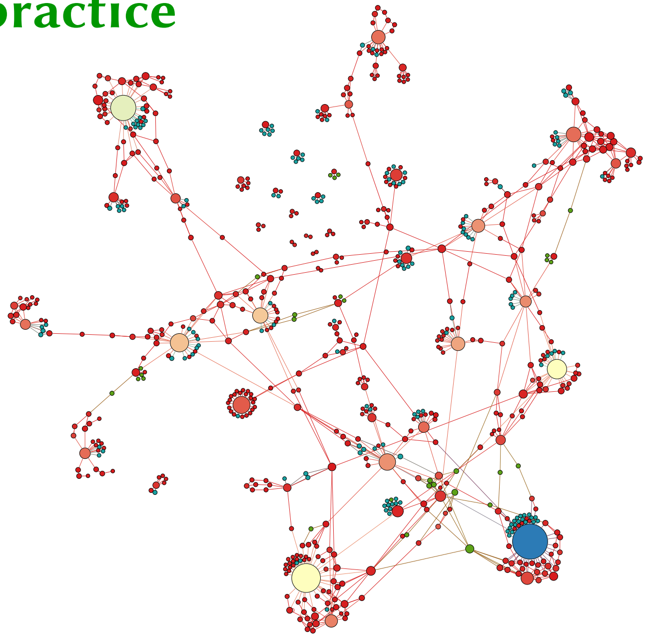

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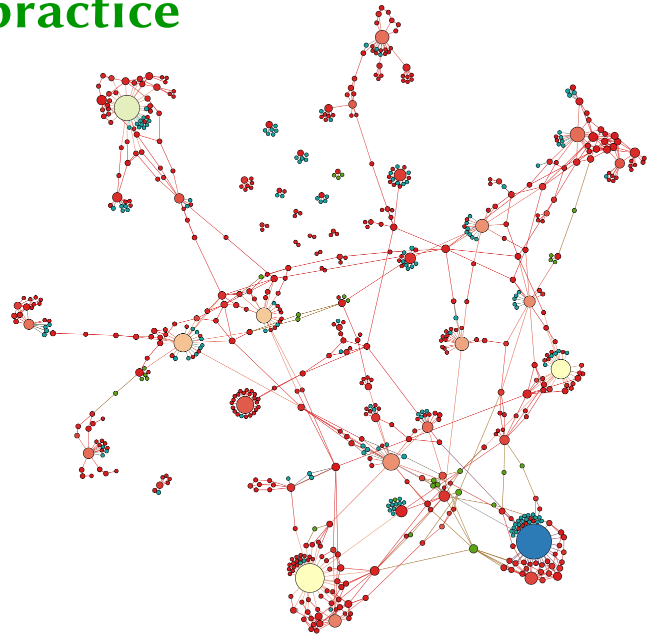

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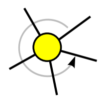

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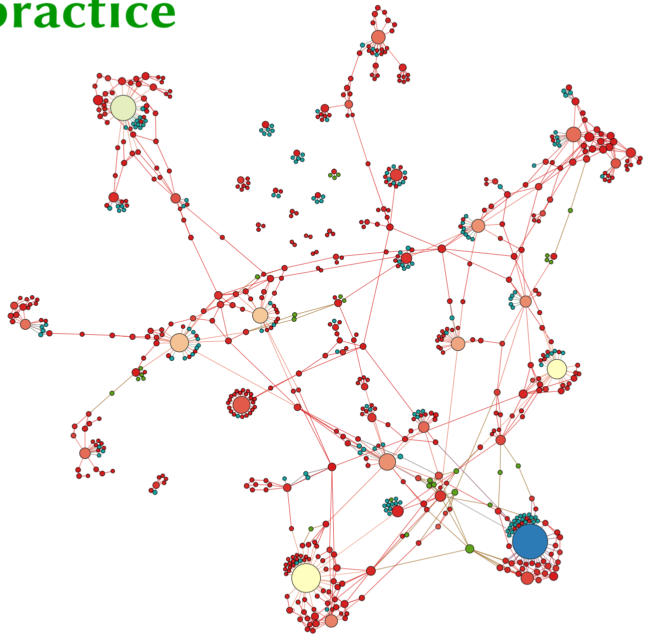

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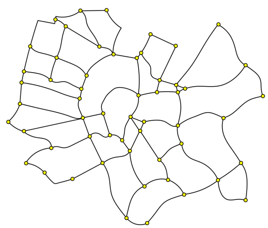

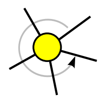

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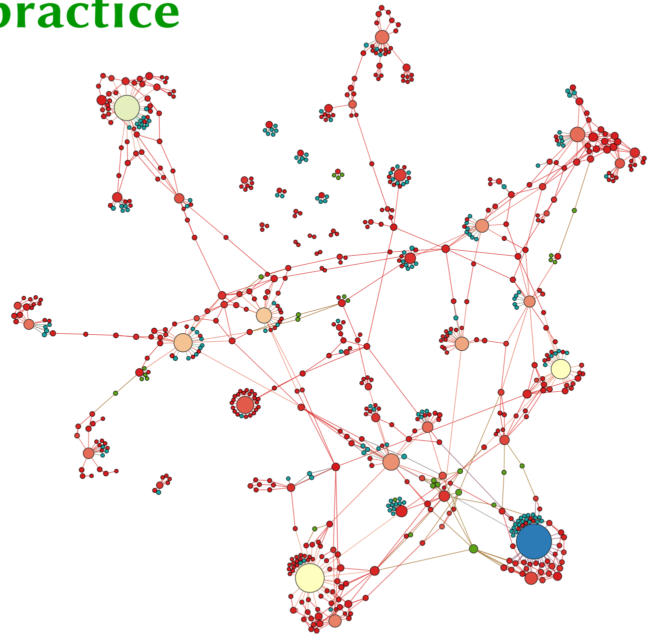

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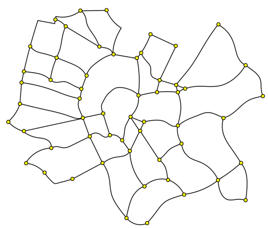

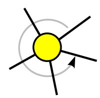

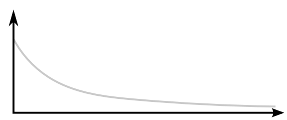

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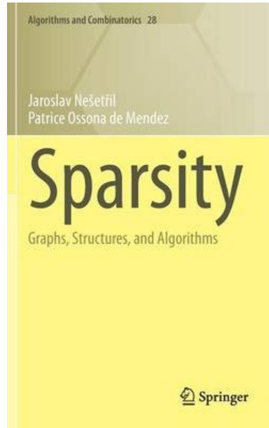

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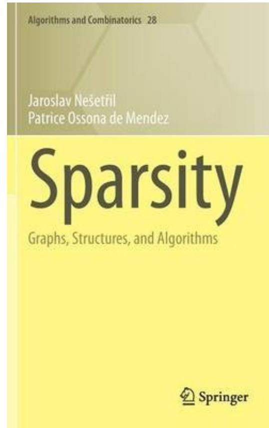

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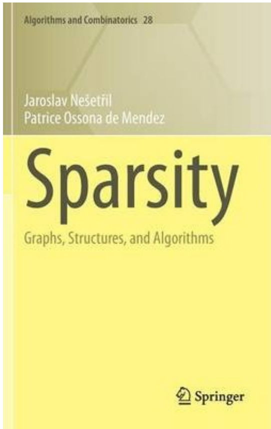

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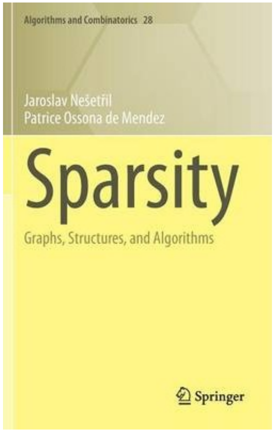

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Question: What does it mean that a graph is **sparse**?

Measuring sparsity

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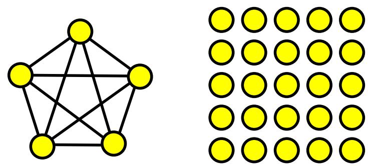

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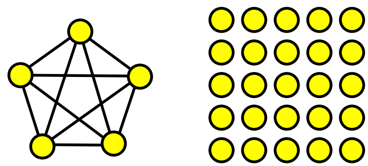

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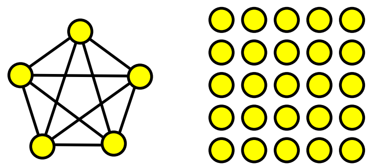

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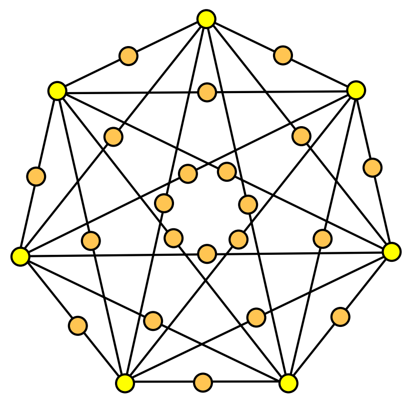

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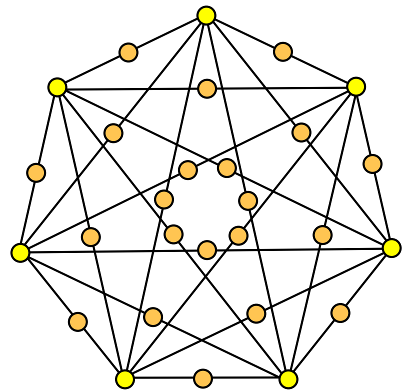

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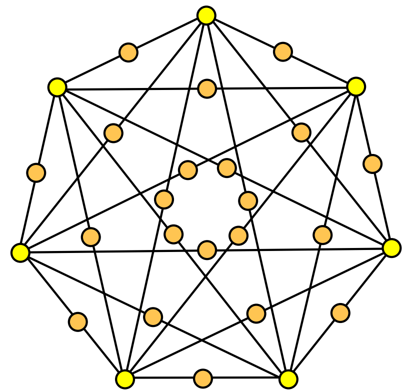

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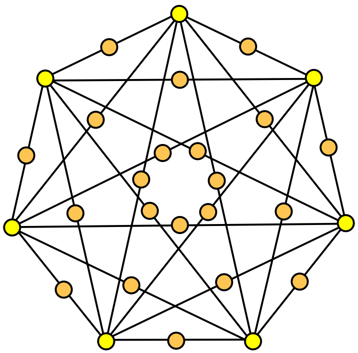

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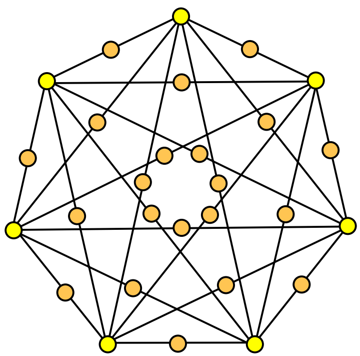

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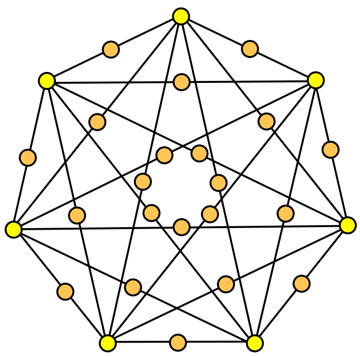

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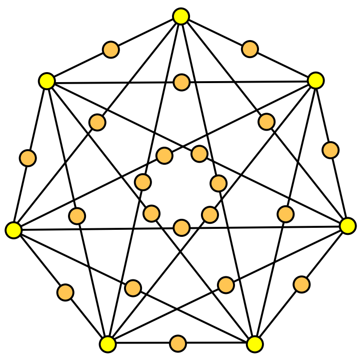

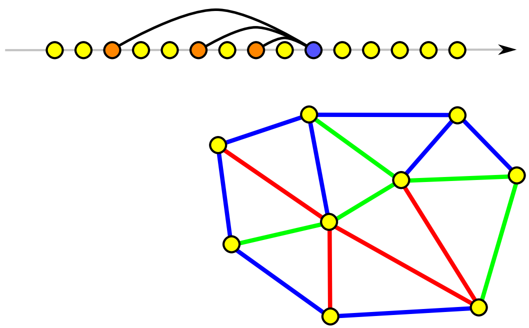

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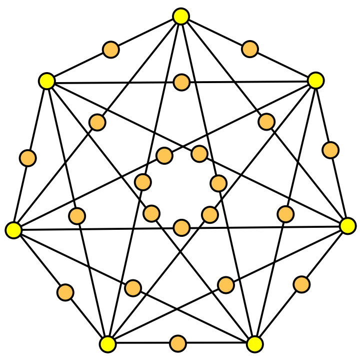
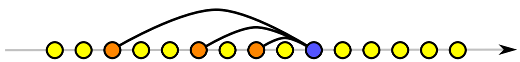
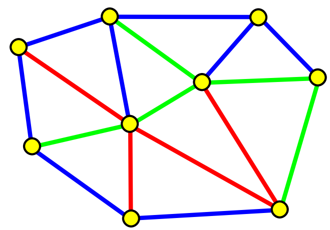
– These connections are useful, but not really very deep.







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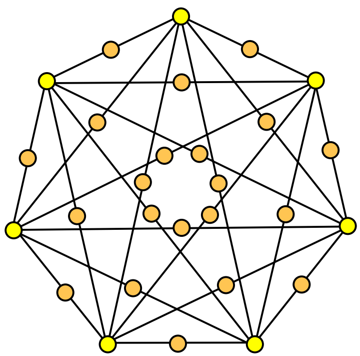

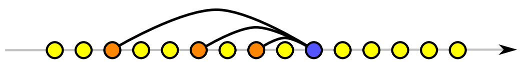

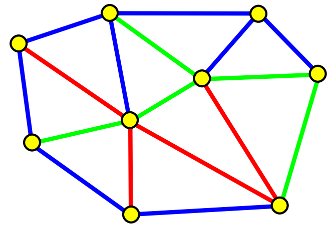

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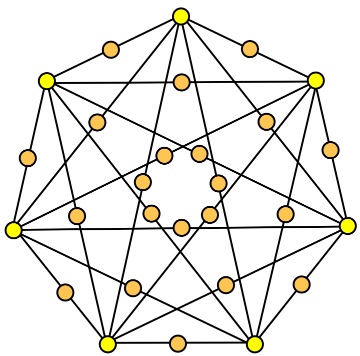

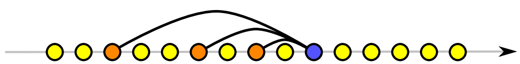

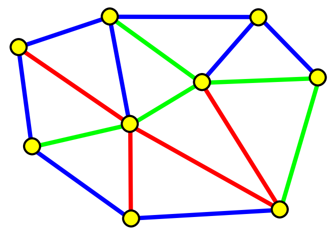

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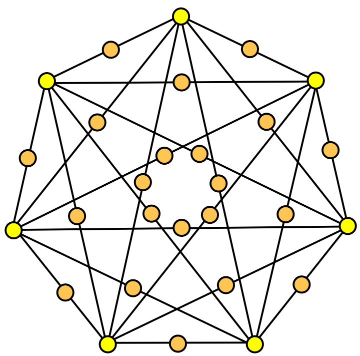

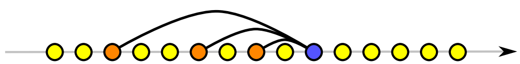

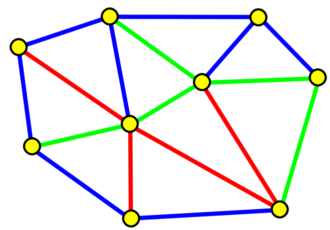

Minor order

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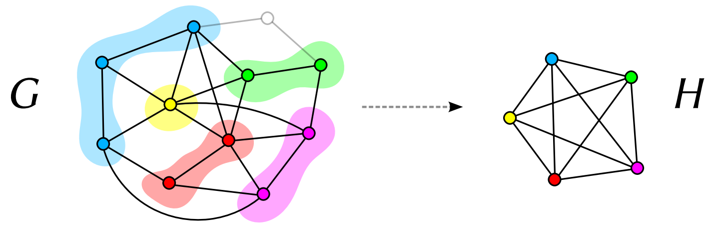

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H is obtained from a subgraph of G by contracting connected subgraphs

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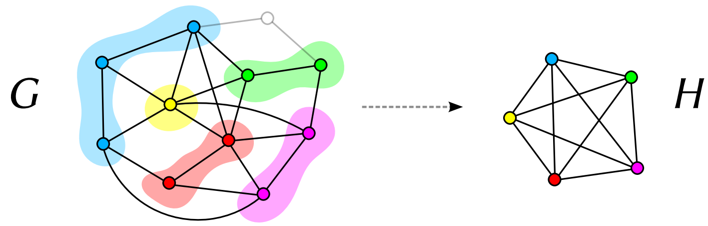

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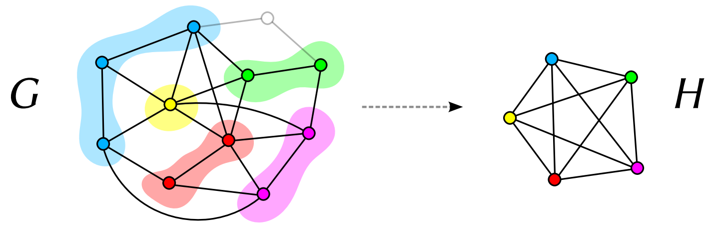

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Theorem (Robertson and Seymour)

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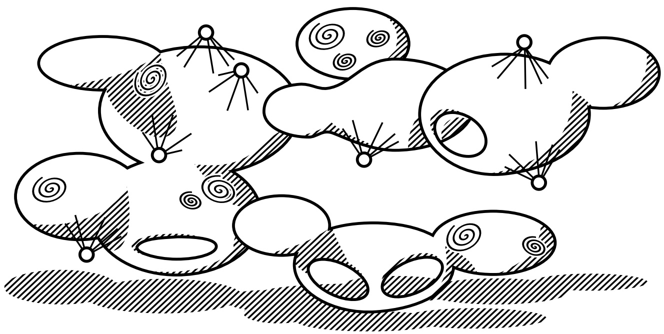

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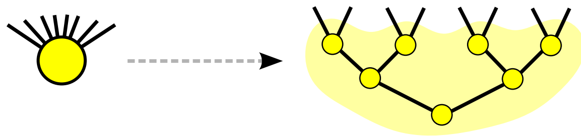

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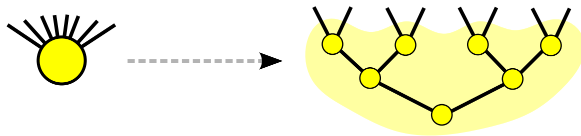

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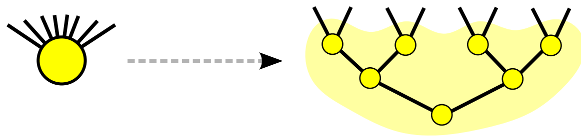

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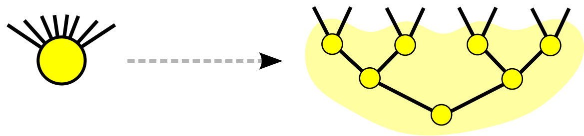

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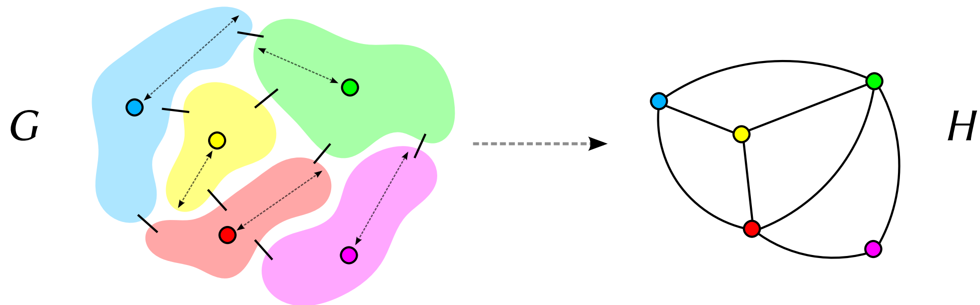

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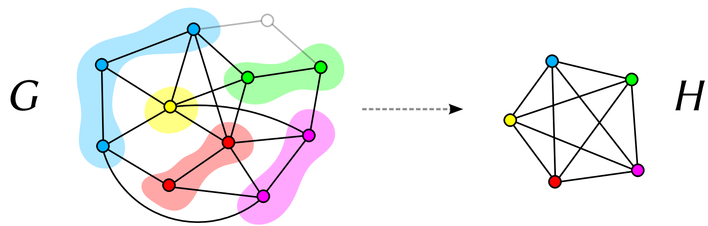

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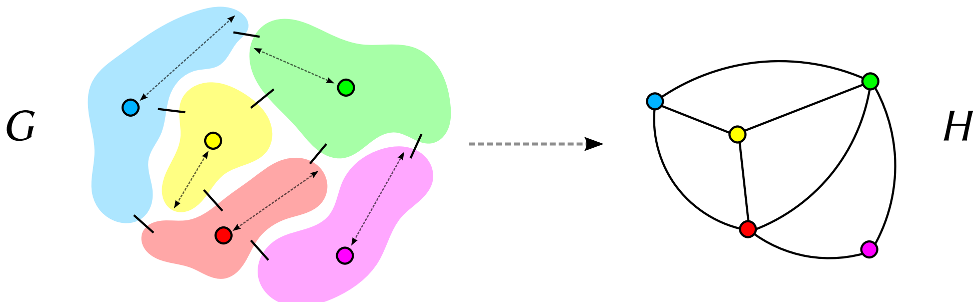

Notions of sparsity

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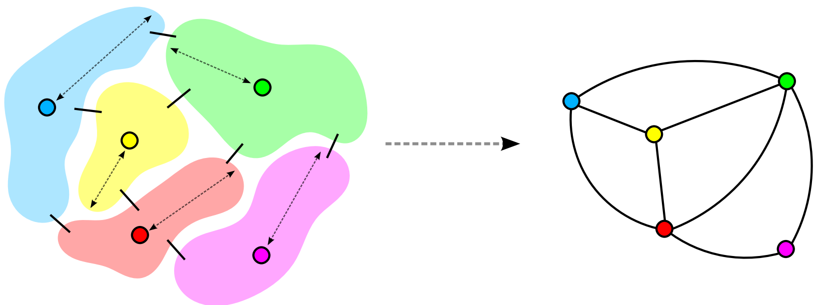

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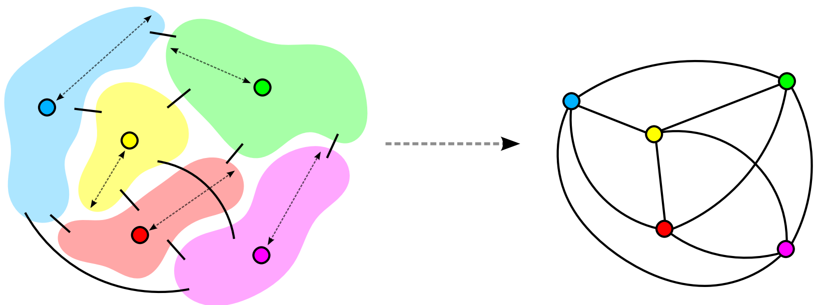

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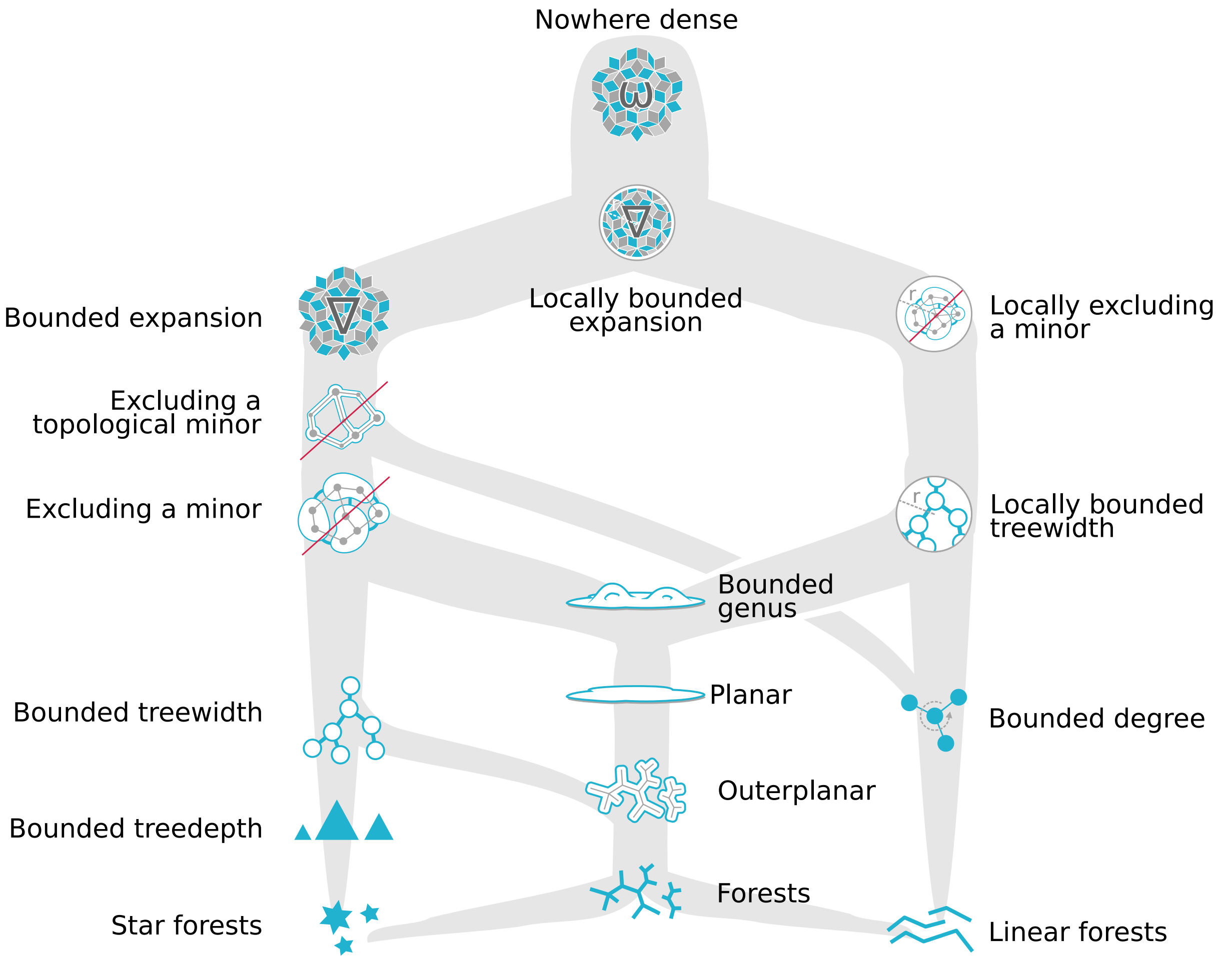
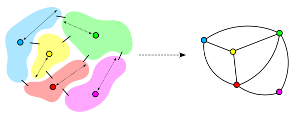
Graphs from nowhere dense classes are somewhat sparse w.r.t. $\nabla_d(\cdot)$.

The World of Sparsity



Equivalent characterizations

Sparsity of shallow minors



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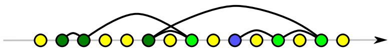
Generalized coloring numbers



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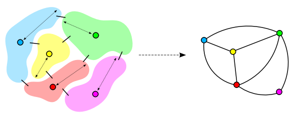
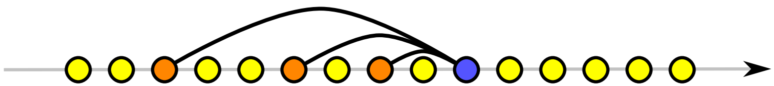
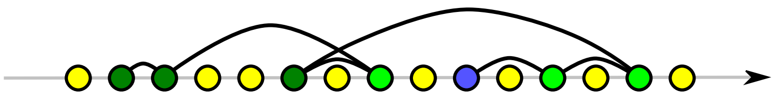




Degeneracy



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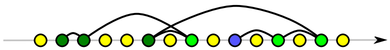

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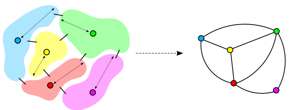
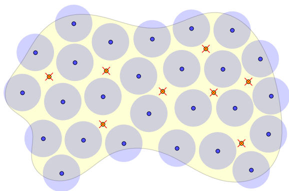
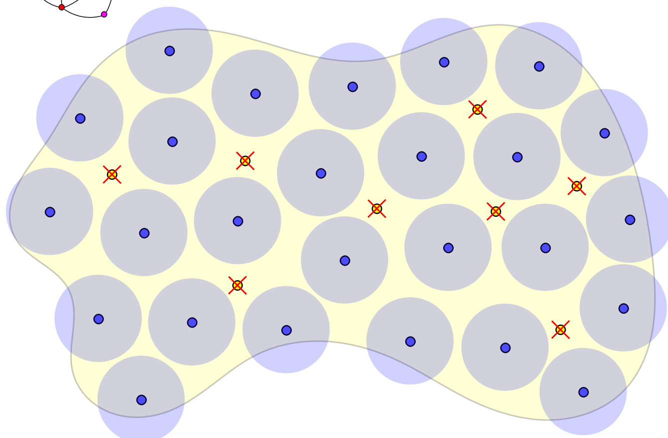


Uniform quasi-wideness





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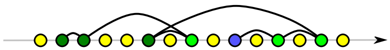

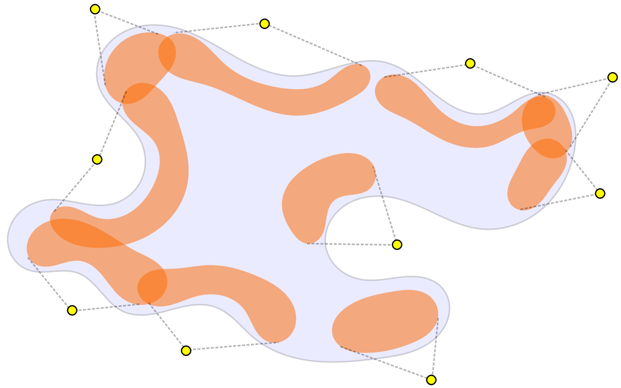

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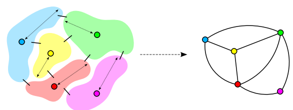

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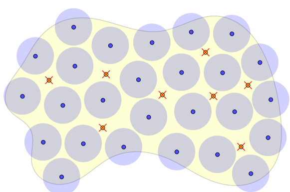

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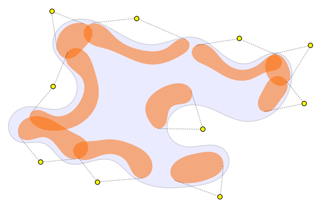

Equivalent characterizations

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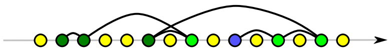


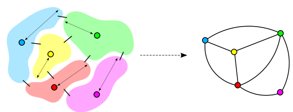
Sparsity of shallow top-minors

Fraternal augmentations

Neighborhood covers

Sparsity of shallow minors

Low treedepth colorings



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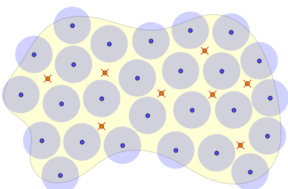
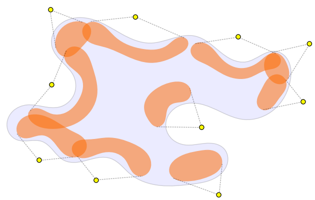


Splitter game



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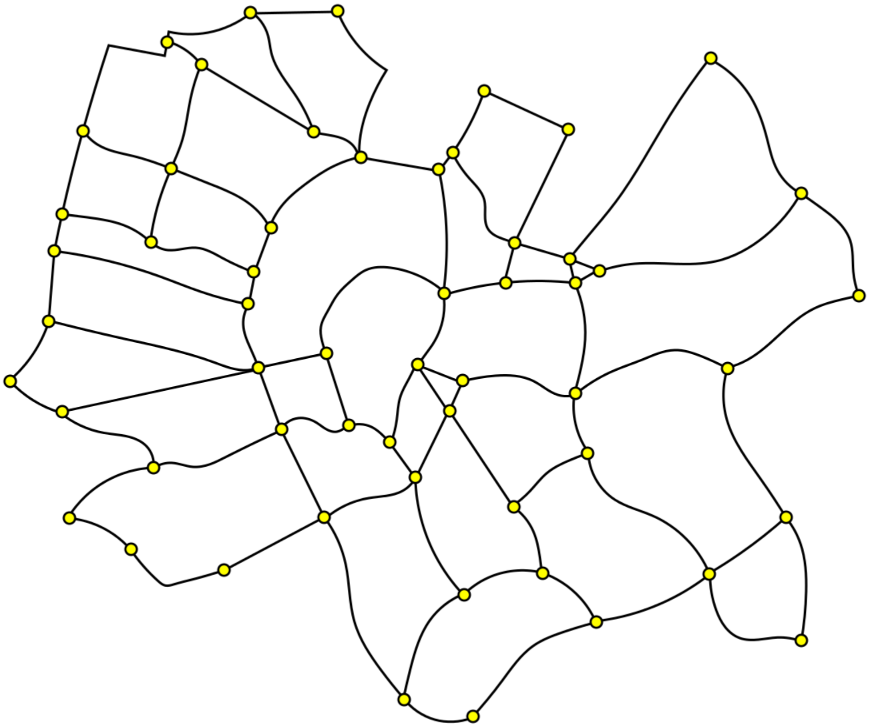
Now: Example **algorithmic** application.

Distance-d Dominating Set

Distance-*d* Dominating Set



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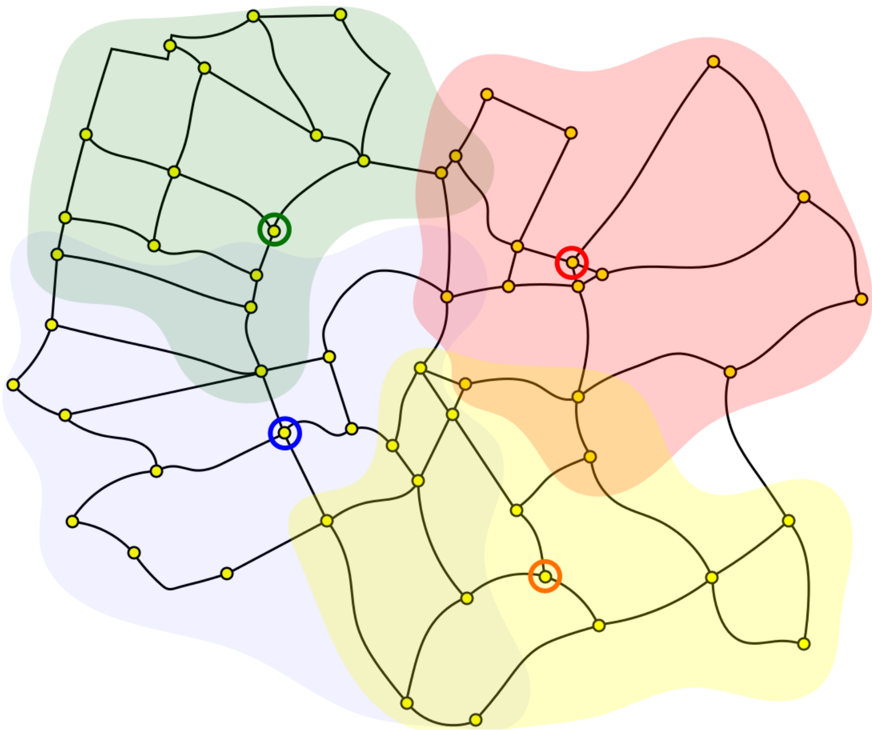

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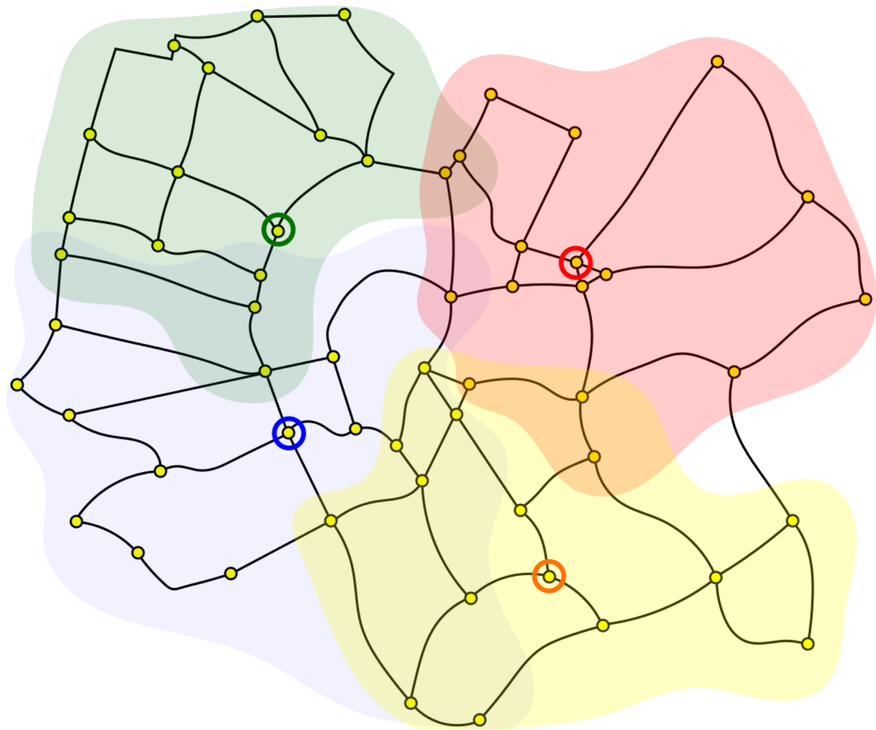

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Here: A distance-3 dominating set of size 4.

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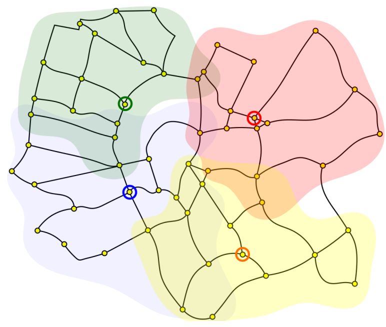

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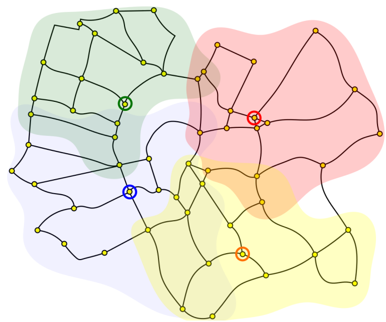

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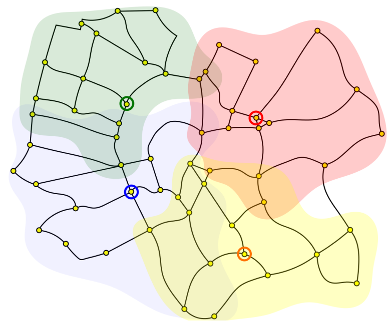

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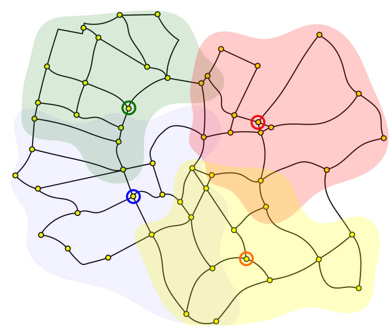

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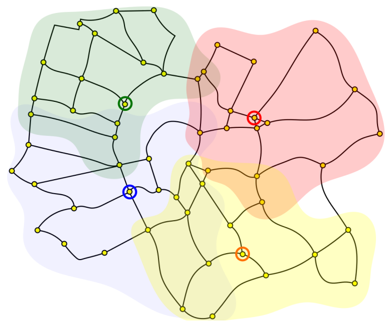

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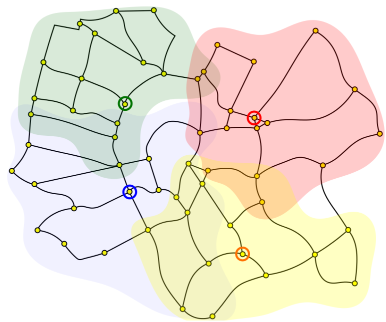

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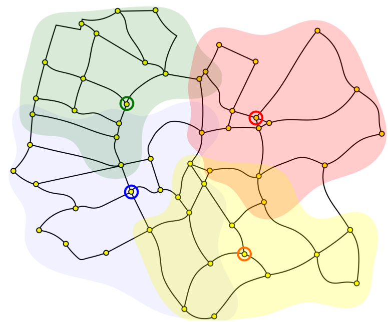

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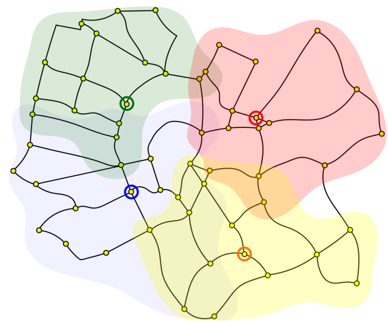
$\|G\| = |V(G)| + |E(G)|$



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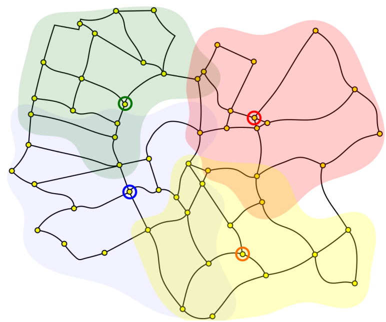

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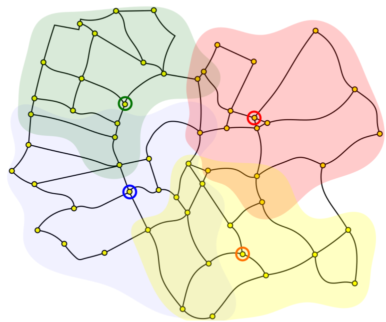

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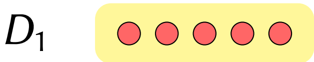

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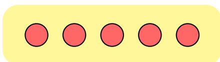
D_1 

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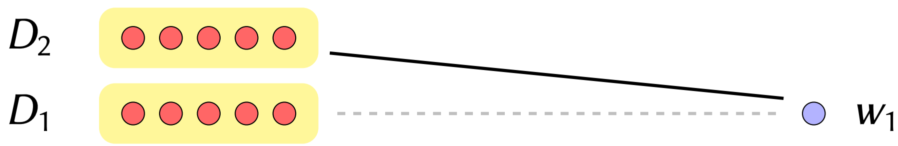

Round 2: Take any k-tuple of vertices D_2 that dominates w_1.

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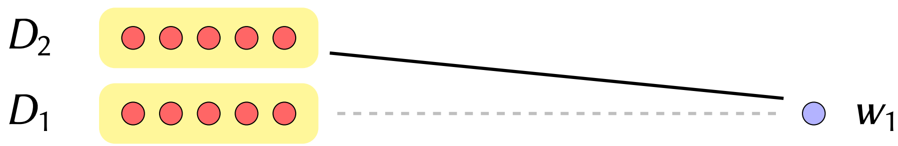

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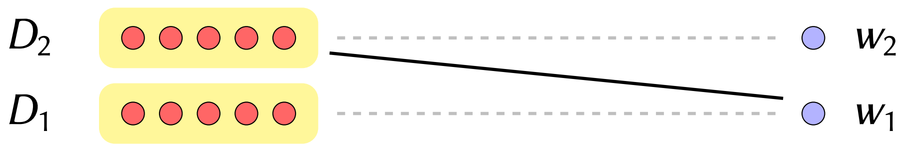

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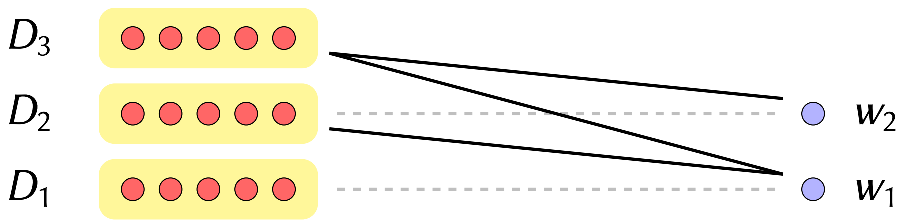

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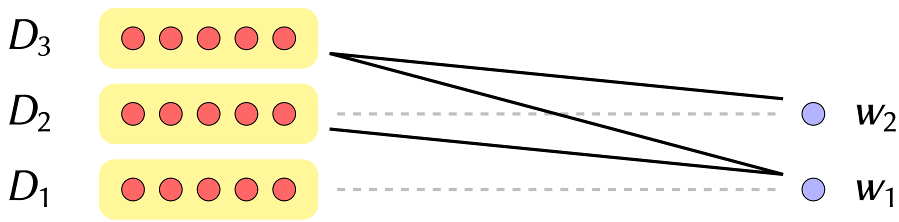

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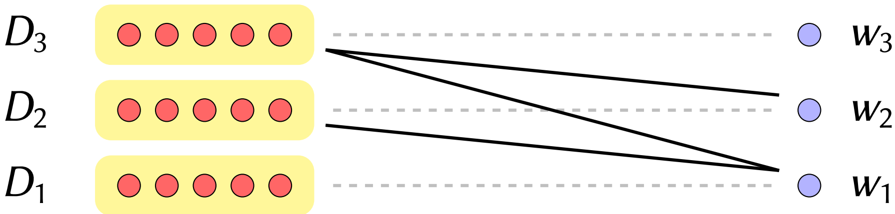

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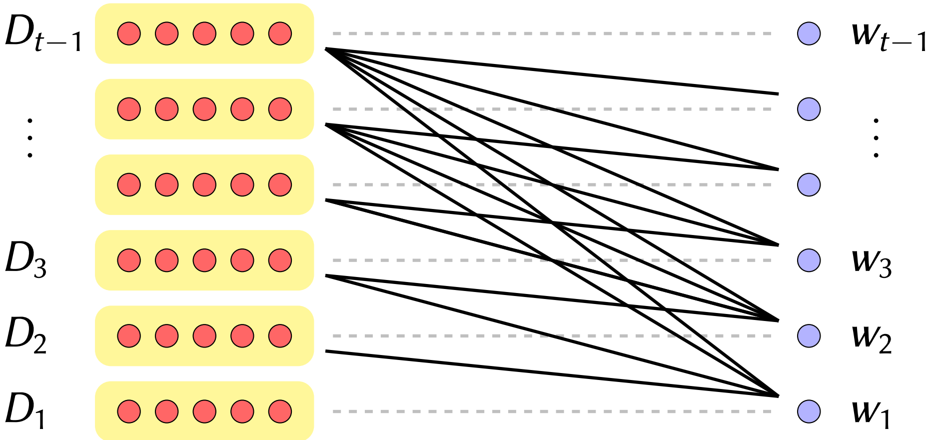

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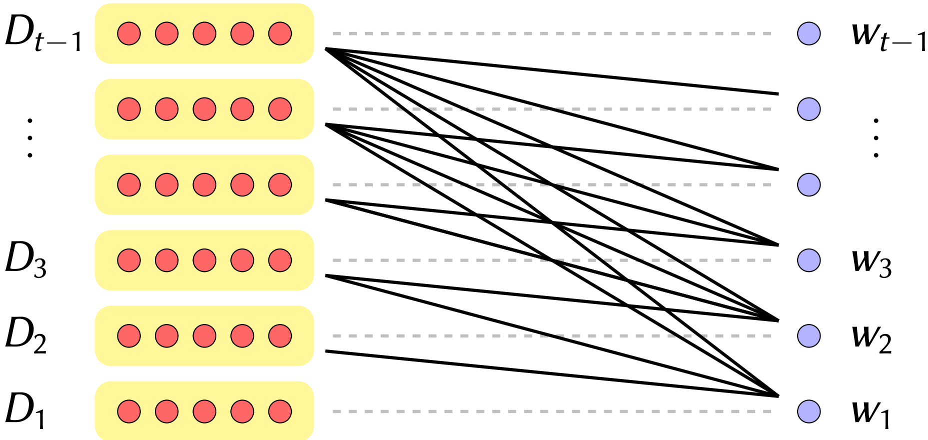

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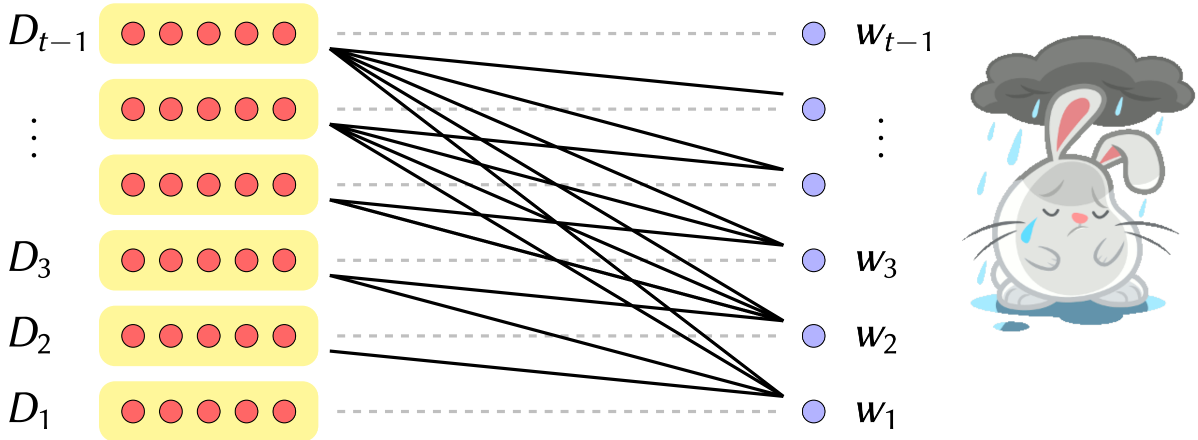

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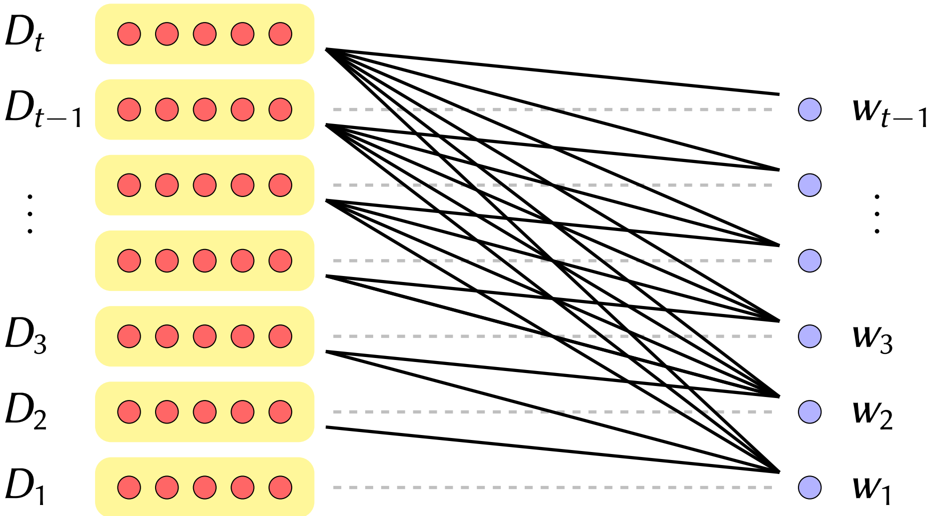

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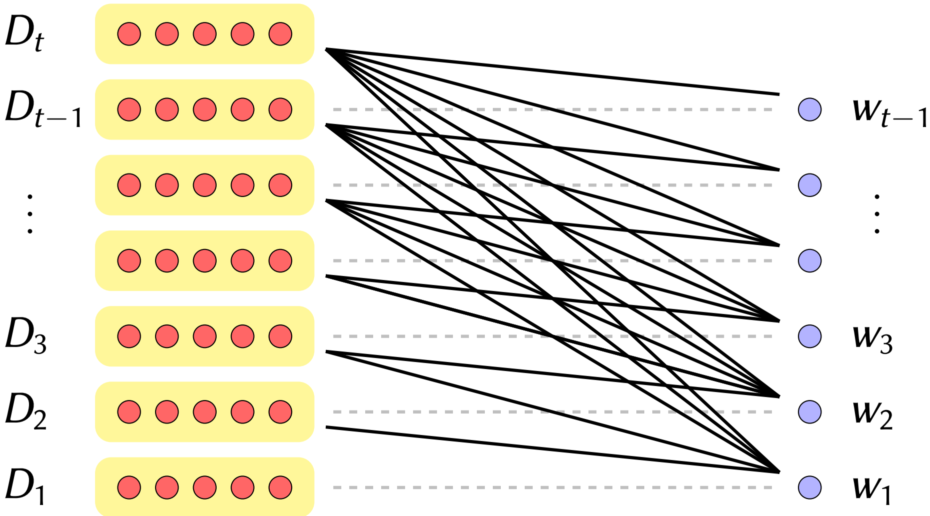

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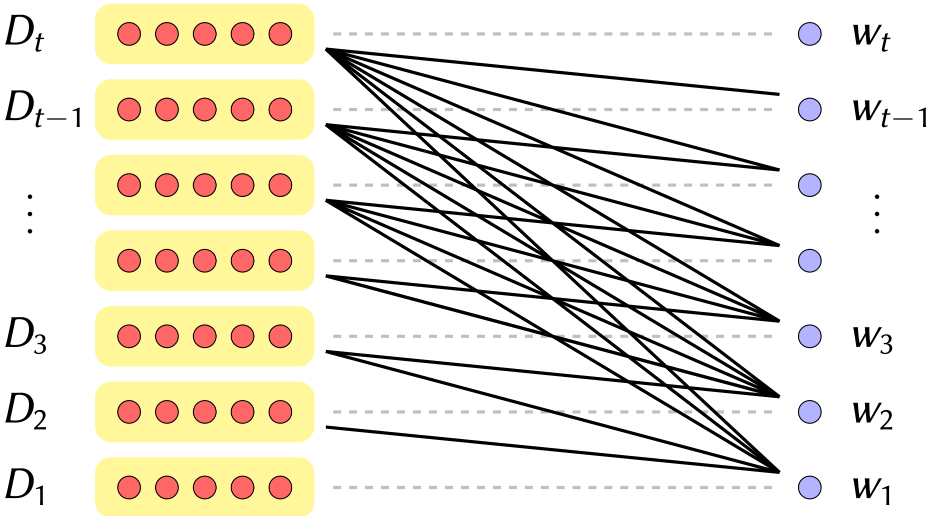

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Cor: Runtime $p(k)^{\mathcal{O}(k)} \cdot \|G\| = k^{\mathcal{O}(k)} \cdot \|G\|$ on any nowhere dense $\mathcal{C}$.

Proof sketch

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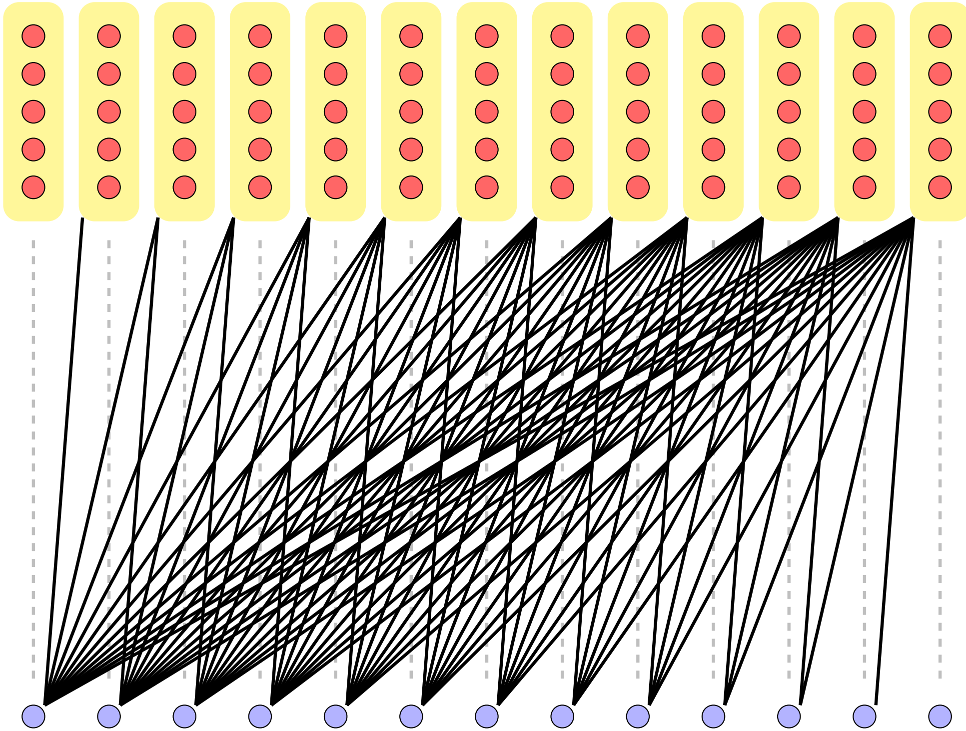
The Algorithm constructs a **semi-ladder** of length L.



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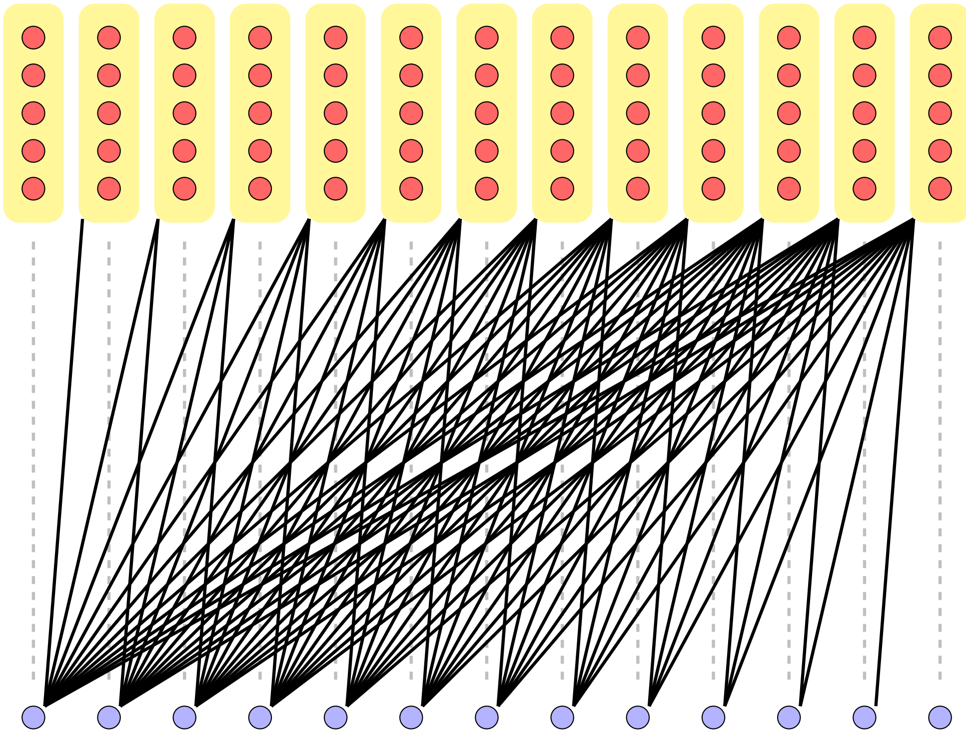

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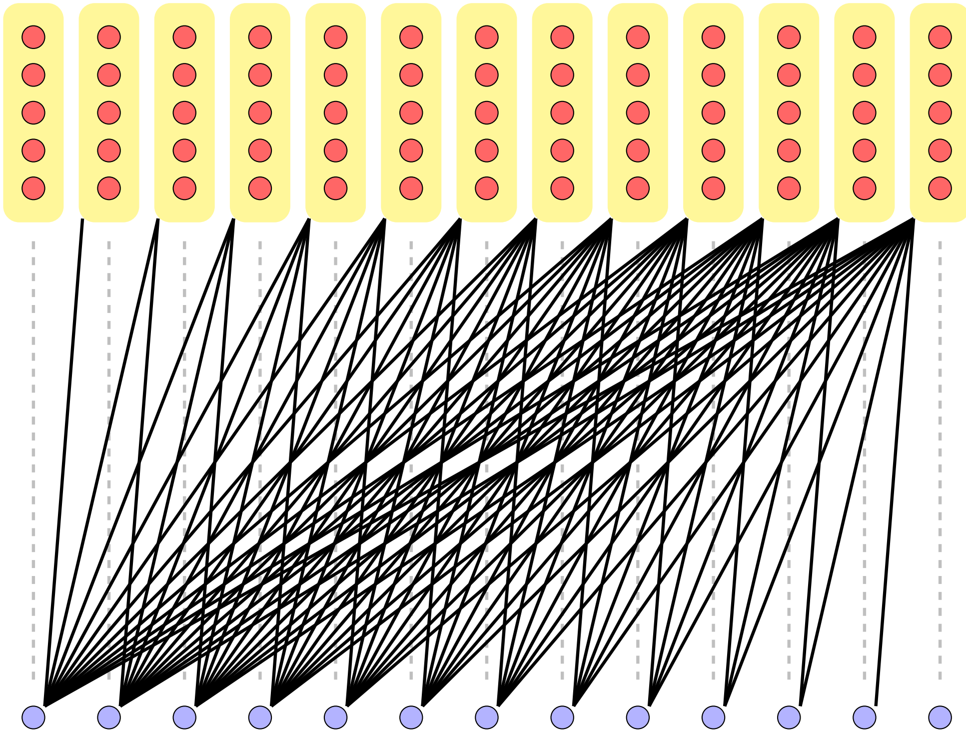
Uniform quasi-wideness $\Rightarrow$ Such a structure cannot be too long.



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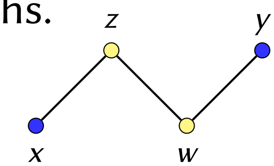

Ex 1. $\delta_d(x, y)$ expresses that $\mathrm{dist}(x, y) \leqslant d$.

$$\delta_3(x, y) \quad := \quad \delta_2(x, y) \vee \exists_z \exists_w \, \mathrm{adj}(x, z) \wedge \mathrm{adj}(z, w) \wedge \mathrm{adj}(w, y).$$

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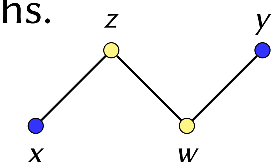

Ex 1. $\delta_d(x, y)$ expresses that $\mathrm{dist}(x, y) \leqslant d$.

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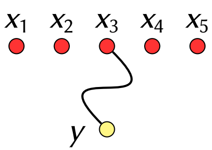

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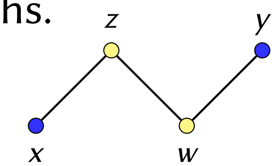

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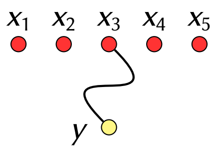

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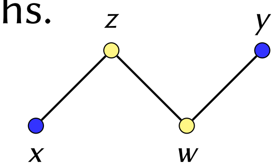

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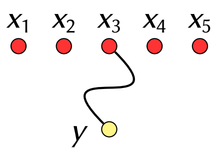

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Q: Can we solve efficiently all FO-expressible problems on sparse graphs?

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Cor: Nowhere denseness **exactly** delimits tractability of **First Order Logic**.

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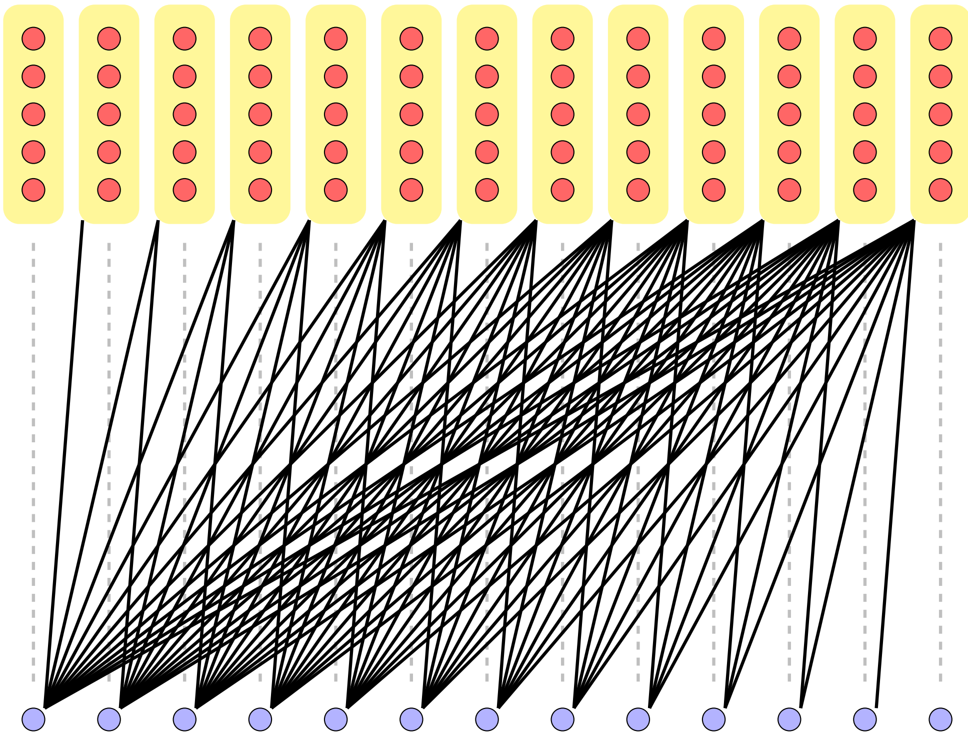
Idea: Generalize the bound on semi-ladders to other properties?



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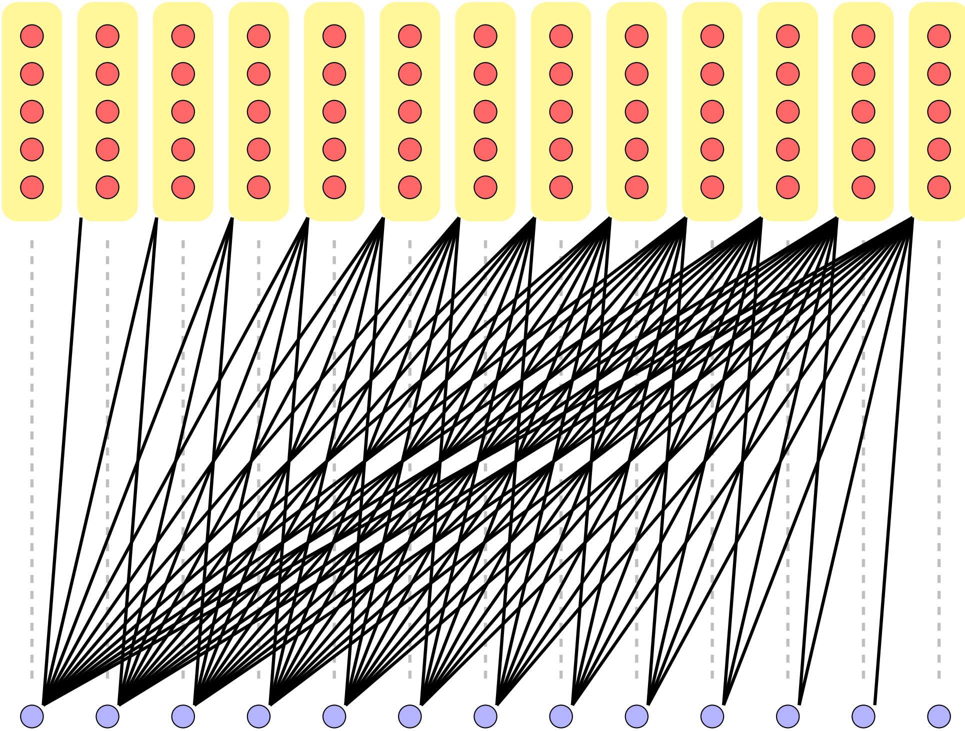

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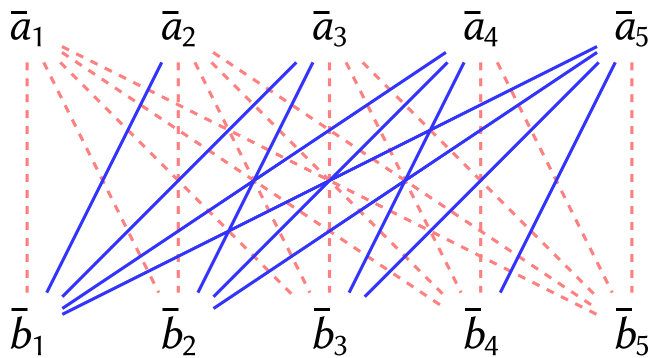

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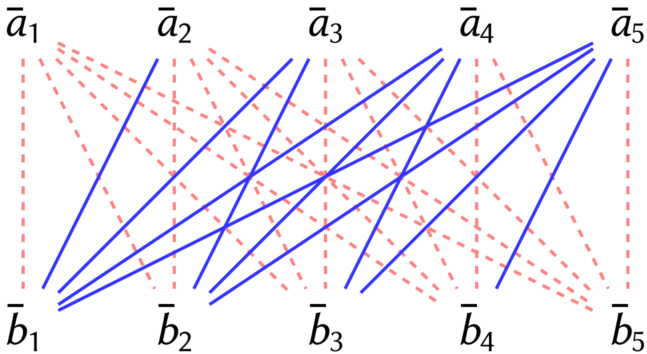
NB: More restrictive than semi-ladders!



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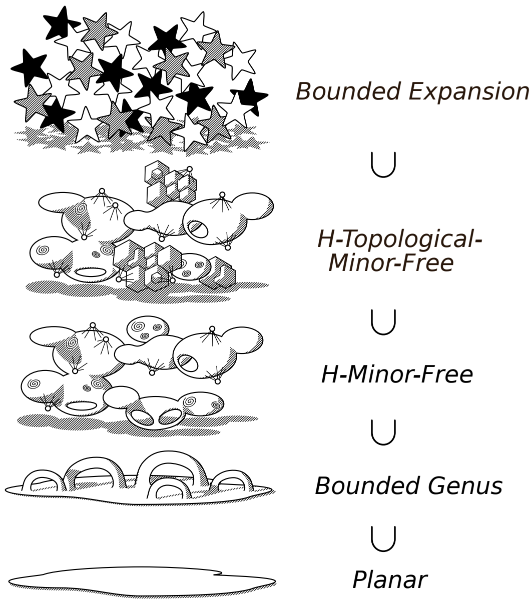

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