Small uncountable cardinals in Large-Scale Topology

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Home

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Erlangen Program

According to the Erlangen Program (1872) of Felix Klein various geometries study invariants of the corresponding transformation groups.

For example,

- topology studies properties, preserved by homeomorphisms;
- uniform topology studied properties preserved by uniform homeomorphisms (= microform bijections);
- large-scale topology is intersted in properties preserved by coarse isomorphisms (= macroform bijections).

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Let $\mathbb{R}_+$ denote the open half-line $(0, +\infty)$.

A function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called

• microform (= uniformly continuous) if
$\forall \varepsilon \in \mathbb{R}_+ \, \exists \delta \in \mathbb{R}_+ \, \forall x, x' \in X \; (d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon)$

• macroform (= coarse) if
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Example: Any Lipschitz map is both microform and macroform.

Remark: The half-line $\mathbb{R}_+$ has two ends.
Microform and macroform maps are interested each by its own end.

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Micro-bijections and macro-bijections

A bijective function $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called

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Structures preserved by corresponding isomorphisms

Which structure of a metric space is preserved by homeomorphisms?
The answer is known: the topology.

Which structure of a metric space is preserved by micro-bijections (=uniform homeomorphisms)?
The answer is also known: the uniform structure.

Which structure of a metric space is preserved by macro-bijections?
The answer: the coarse structure.

In fact, uniform and coarse structures are the two "ends" of a common structure called the duoform structure.

All these structures are introduced with the help of entourages.

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Definition

An entourage on a set X is any subset $E \subseteq X \times X$ containing the diagonal $\Delta_X = \{\langle x, x\rangle : x \in X\}$ of the square $X \times X$.

Entourages are subject to some algebra.

Namely, for two entourages E, F on a set X we can consider the inverse entourage

$$E^{-1} = \{\langle y, x\rangle : \langle x, y\rangle \in E\}$$

and the composition

$$E \circ F = \{\langle x, z\rangle : \exists y\ (\langle x, y\rangle \in E\ \wedge\ \langle y, z\rangle \in F)\}.$$

For an entourage U let

$$\uparrow U = \{W : U \subseteq W \subseteq X \times X\} \quad \text{and} \quad \downarrow U = \{W : \Delta_X \subseteq W \subseteq U\}$$

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Entourages and geometric intuition

For an entourage $U \subseteq X \times X$ and point $x \in X$ the set

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is called the ball of radius U around x.

For a subset $A \subseteq X$ the set

$$U[A] = \bigcup_{a \in A} U(a)$$

is called the U-neighborhood of A.

In fact, the entourage U can be recovered from its balls since

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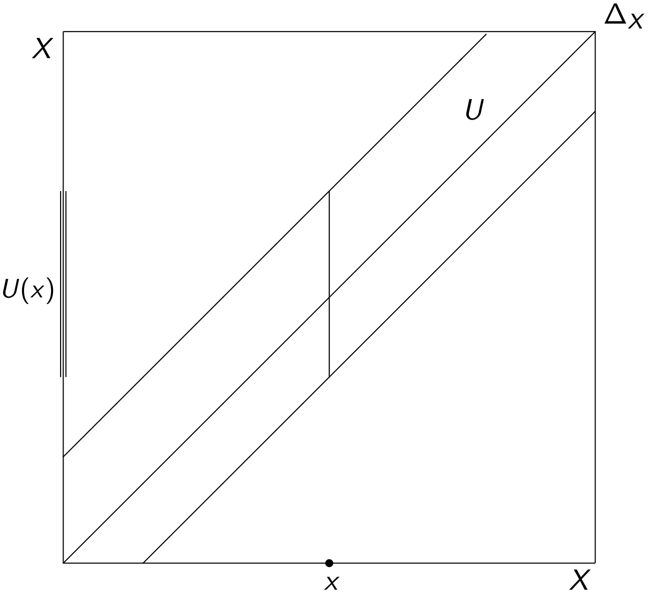
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Each base of a uniform structure has the property (B) and each family of entourages $\mathcal{B}$ that has property (B) is a base of a unique uniform structure, namely, ${\uparrow}\mathcal{B} = \bigcup_{B \in \mathcal{B}} {\uparrow}B$.

The weight $w(\mathcal{U})$ of a uniform structure $\mathcal{U}$ is the smallest cardinality $|\mathcal{B}|$ of a base $\mathcal{B} \subseteq \mathcal{U}$.

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Basic example

Every metric space (X, d) has the canonical uniform structure $\uparrow\mathcal{B}$ and the canonical coarse structure $\downarrow\mathcal{B}$, both generated by the base

$$\mathcal{B} = \big\{\{(x, y) \in X \times X : d(x, y) < \varepsilon\} : \varepsilon \in \mathbb{R}_+\big\}.$$

A uniform (coarse) structure on a set X is called metrizable if it coincides with the canonical uniform (coarse) structure of some metric d on X.

Theorem

(U) *A uniform structure is metrizable iff it has countable weight.*

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In fact, uniform and coarse structures are particular cases of a common structure, called the duoform structure.

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Theorem

(D) *A duoform structure is metrizable iff it has a countable base.*

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From now on we shall be interested only in coarse spaces

Subspaces of coarse spaces

For a coarse structure $\mathcal{E}$ on a set X and a subset $A \subseteq X$ the family

$$\mathcal{E}{\restriction}A = \{E \cap (A \times A) : E \in \mathcal{E}\}$$

is a coarse structure on A.
The pair $(A, \mathcal{E}{\restriction}A)$ is called a subspace of the coarse space $(X, \mathcal{E})$.

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The bornology of a coarse space

Let $\mathcal{E}$ be a coarse structure on a set X.

Definition

A subset $B \subseteq X$ is $\mathcal{E}$-bounded if B is contained in some ball, i.e.,

$$B \text{ is } \mathcal{E}\text{-bounded} \Leftrightarrow \exists E \in \mathcal{E}\ \exists x \in X\ (B \subseteq E(x)).$$

The family of all $\mathcal{E}$-bounded sets is called the bornology of the coarse space $(X, \mathcal{E})$.

The bornology is closed under taking subsets and unions.

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Definition

A coarse structure $\mathcal{E}$ on a set X is called

- locally finite if for every $E \in \mathcal{E}$ and point $x \in X$ the E-ball $E(x)$ is finite;
- finitary if for every $E \in \mathcal{E}$ the cardinal $\sup_{x \in X} |E(x)|$ is finite.

finitary $\Rightarrow$ locally finite.

What about the converse?

Example

- The metric space of integers $\mathbb{Z}$ is finitary.
- The space $\{\sqrt{n} : n \in \mathbb{N}\} \subset \mathbb{R}$ is locally finite but not finitary.

Remark: A coarse structure is locally finite if and only if its bornology consists of finite sets.

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Canonical example of a finitary space

Let X be a set and S_X be the permutation group of X.

Every subgroup $G \subseteq S_X$ induces the finitary coarse structure $\mathcal{E}_G$ on X, generated by the base consisting of the entourages

$$E = \Delta_X \cup (B \times B) \cup \{\langle x, y\rangle : x \in X,\ y \in Fx\}$$

where B is a finite subset of X and F is a finite subsets of G. So, the E-ball of a point $x \in X \setminus B$ is the set $\{x\} \cup Fx$.

Example

For the group of integers $\mathbb{Z}$, the coarse structure $\mathcal{E}_{\mathbb{Z}}$ on $\mathbb{Z}$ coincides with the coarse structure generated by the Euclidean metric.

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For the group of integers $\mathbb{Z}$, the coarse structure $\mathcal{E}_{\mathbb{Z}}$ on $\mathbb{Z}$ coincides with the coarse structure generated by the Euclidean metric.

Theorem (Protasov)

Every finitary coarse structure $\mathcal{E}$ on a set X coincides with the finitary coarse structure $\mathcal{E}_G$ generated by some subgroup $G \subseteq S_X$.

Maximal and minimal coarse structures

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The largest finitary coarse structure on a set X is generated by the whole permutation group S_X. This coarse structure consists of all possible entourages E on X which are finitary in the sense that the cardinal

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Discrete coarse structures

Definition

A coarse structure $\mathcal{E}$ on a set X is called *discrete* if for every entourage $E \in \mathcal{E}$ there exists an $\mathcal{E}$-bounded subset $B \subseteq X$ such that for every $x \in X \setminus B$ the E-ball $E(x)$ coincides with the singleton $\{x\}$.

Example: The subspace $\{n^2 : n \in \omega\}$ of $\mathbb{Z}$ is discrete.

Proposition

For a locally finite coarse structure $\mathcal{E}$ on a set X the following conditions are equivalent:

1. $\mathcal{E}$ *is discrete;*
2. $\forall E \in \mathcal{E}\ (|\{x \in X : |E(x)| \neq \{x\}\}| < \omega)$*;*
3. $\mathcal{E} = \mathcal{E}_{\{\mathrm{id}\}}$.

Remark: Countable discrete sets are antipods of Cauchy sequences.

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Indiscrete coarse spaces

Now let us look at the largest finitary coarse structure $\mathcal{E}_{S_X}$ on X.
Is it discrete? No! And in a very strong sense:

The coarse space $(X, \mathcal{E}_{S_X})$ contains no infinite discrete subspaces

So, $\mathcal{E}_{S_X}$ resembles $\beta\mathbb{N}$ which contains no convergent sequences.

Definition

A coarse structure $\mathcal{E}$ on a set X is called indiscrete if for any infinite subset $A \subseteq X$ the coarse structure $\mathcal{E}{\restriction}A$ is not discrete.

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The first set-theoretic question

What is the smallest possible weight of an indiscrete finitary coarse structure on ω?

As we shall see, this is a cardinal in the interval $[\omega_1, \mathfrak{c}]$, so is a typical cardinal characteristic of the continuum.
Let us denote this cardinal by Δ. More precisely:

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Let Δ denote the smallest weight $w(\mathcal{E})$ of an indiscrete finitary coarse structure on ω.

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Asymptotically separated sets

Definition

Let $\mathcal{E}$ be a coarse structure on a set X. Two sets $A, B \subseteq X$ are called asymptotically separated if for any $E \in \mathcal{E}$ the set $E[A] \cap E[B]$ is $\mathcal{E}$-bounded.

Remark: If a coarse structure $\mathcal{E}$ on a set X is discrete, then any disjoint subsets $A, B \subseteq X$ are asymptotically separated.

Observation: For the largest finitary coarse structure $\mathcal{E}_{S_X}$ on a set X, no infinite sets $A, B \subset X$ are $\mathcal{E}_{S_X}$-separated.

Definition

A coarse structure $\mathcal{E}$ on a set X is called inseparated if for any $\mathcal{E}$-separated sets $A, B \subseteq X$ one of the sets A or B is $\mathcal{E}$-bounded.

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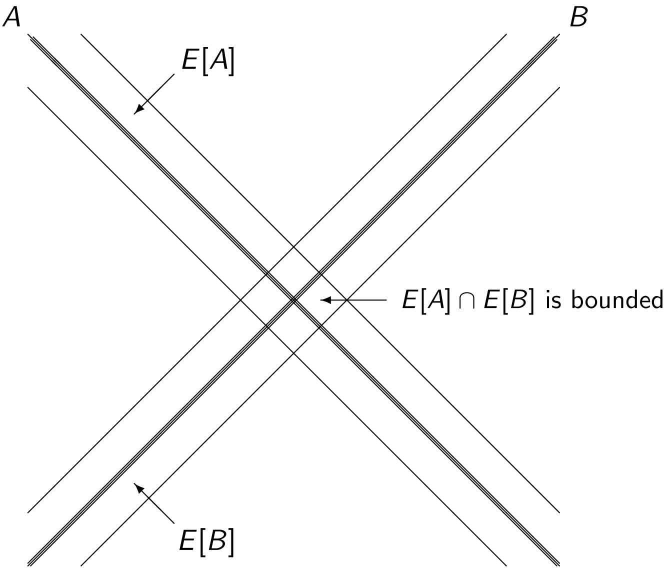
An example of asymptotically separated sets



The second set-theoretic question

Since the largest finitary coarse structure $\mathcal{E}_{S_X}$ is inseparated, we can ask about the smallest weight of an inseparated finitary coarse structure on ω.

Definition

Let Σ denote the smallest weight of an inseparated finitary coarse structure on ω.

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What is the value of Σ? Is Σ equal to some known cardinal characteristic of the continuum?

Since any disjoint sets in a discrete locally finite coarse space are asymptotically separated, each inseparated locally finite coarse space is indiscrete. This implies the inequality

$$\Delta \leq \Sigma.$$

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In fact the largest finitary coarse structure $\mathcal{E}_{S_X}$ has another exotic property: it is large in the following sense.

Definition

A coarse structure $\mathcal{E}$ on a set X is *large* if each $\mathcal{E}$-unbounded set $A \subseteq X$ is *$\mathcal{E}$-large* in the sense that $E[A] = X$ for some $E \in \mathcal{E}$.

Observation: The coarse structure $\mathcal{E}_{S_X}$ is large if and only if the set X is countable. In particular, $\mathcal{E}_{S_\omega}$ is large.

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Find the smallest weight of a large finitary coarse structure on ω.

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The main theorem on critical cardinalities

Theorem

The smallest weight of

1. *a large finitary coarse structure on ω is equal to $\mathfrak{c}$;*
2. *an inseparated finitary coarse structure on ω is equal to Σ;*
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All cardinals in this theorem except for Δ and Σ are well-known.

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Equivalent definitions of Δ and Σ

Proposition

1. *The cardinal Δ is equal to the smallest cardinality of a subgroup $G \subseteq S_\omega$ such that for any infinite set $A \subseteq \omega$ there exists $g \in G$ such that the set $\{a \in A : a \neq g(a) \in A\}$ is infinite.*
2. *The cardinal Σ is equal to the smallest cardinality of a subgroup $G \subseteq S_\omega$ such that for any infinite sets $A, B \subseteq \omega$ there exists $g \in G$ such that the set $A \cap g[B]$ is infinite.*

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Equivalent definitions of Δ, Σ via 2-to-1-maps

A function $\varphi : \omega \to \omega$ is called 2-to-1 if $\forall y \in Y\ (|\varphi^{-1}(y)| \le 2)$.

A function $f : X \to Y$ is called *almost injective* if for some finite set $F \subseteq X$ the restriction $f{\restriction}X \setminus F$ is injective.

Proposition

1. *The cardinal Δ is equal to the smallest cardinality of a subset $F \subseteq \omega^\omega$ such that for any infinite set $A \subseteq \omega$ there exists a 2-to-1-function $f \in F$ such that $f{\restriction}A$ is not almost injective.*
2. *The cardinal Σ is equal to the smallest cardinality of a subset $F \subseteq \omega^\omega$ such that for any infinite sets $A, B \subseteq \omega$ there exists a 2-to-1-function $f \in F$ such that $f[A] \cap f[B]$ is infinite*

Now we will discuss the location of the cardinals Δ and Σ among other known cardinal characteristics of the continuum.

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A function $f : X \to Y$ is called *almost injective* if for some finite set $F \subseteq X$ the restriction $f{\restriction}X \setminus F$ is injective.

Proposition

1. *The cardinal Δ is equal to the smallest cardinality of a subset $F \subseteq \omega^\omega$ such that for any infinite set $A \subseteq \omega$ there exists a 2-to-1-function $f \in F$ such that $f{\restriction}A$ is not almost injective.*
2. *The cardinal Σ is equal to the smallest cardinality of a subset $F \subseteq \omega^\omega$ such that for any infinite sets $A, B \subseteq \omega$ there exists a 2-to-1-function $f \in F$ such that $f[A] \cap f[B]$ is infinite*

Now we will discuss the location of the cardinals Δ and Σ among other known cardinal characteristics of the continuum.

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Some classical cardinal characteristics of the continuum

For a set X, let $[X]^{<\omega}$ be the family of all finite subsets of X, and $[X]^{\omega}$ be the family of all countably infinite subsets of X.

For two sets A, B, we write $A \subseteq^* B$ if $A \setminus B$ is finite.

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Let us recall that

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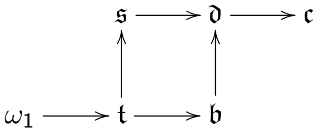
The order relations between $\omega_1, \mathfrak{t}, \mathfrak{b}, \mathfrak{d}, \mathfrak{s}, \mathfrak{c}$

The order relations between these cardinals are described by the following diagram in which for two cardinals κ, λ the arrow $\kappa \to \lambda$ indicates that $\kappa \leq \lambda$ in ZFC.

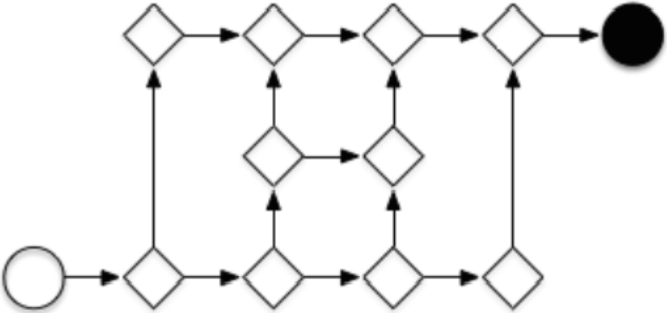


Each family of sets $\mathcal{I}$ with $\bigcup\mathcal{I} \notin \mathcal{I}$ has four basic cardinal characteristics:

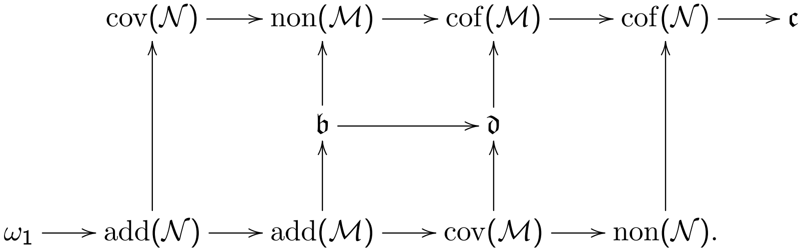
$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \ \wedge \ \bigcup\mathcal{A} \notin \mathcal{I}\};$$
$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \ \wedge \ \bigcup\mathcal{A} = \bigcup\mathcal{I}\};$$
$$\mathrm{non}(\mathcal{I}) = \min\{|A| : A \subseteq \bigcup\mathcal{I} \ \wedge \ A \notin \mathcal{I}\};$$
$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{J}| : \mathcal{J} \subseteq \mathcal{I} \ \wedge \ \forall I \in \mathcal{I}\ \exists J \in \mathcal{J}\ (I \subseteq J)\}.$$

These cardinal characteristics are usually considered for the σ-ideals $\mathcal{M}$ and $\mathcal{N}$ of meager sets and Lebesgue null sets on the real line, respectively.

Cichoń Diagram



The cardinal characteristics of the σ-ideals $\mathcal{M}$ and $\mathcal{N}$ are described by the famous Cichoń diagram:



Main Theorem: $\max\{\mathfrak{b}, \mathfrak{s}, \mathrm{cov}(\mathcal{N})\} \le \Delta \le \Sigma \le \mathrm{non}(\mathcal{M})$

Problem

Which of the strict inequalities are consistent with ZFC

$$\max\{\mathfrak{b}, \mathfrak{s}, \mathrm{cov}(\mathcal{N})\} < \Delta < \Sigma < \mathrm{non}(\mathcal{M})?$$

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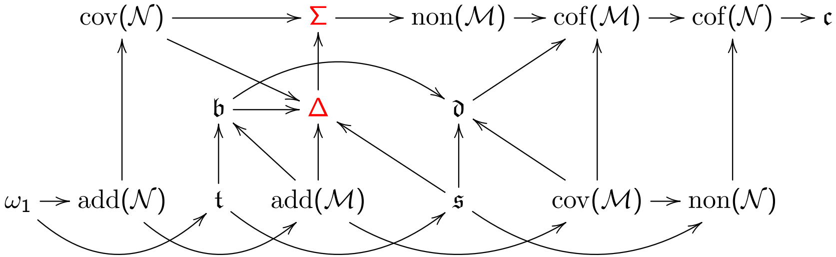

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The location of Δ, Σ

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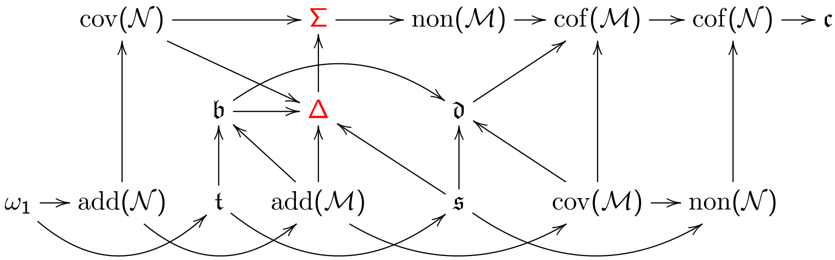

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$$\max\{\mathfrak{b}, \mathfrak{s}, \mathrm{cov}(\mathcal{N})\} < \Delta < \Sigma < \mathrm{non}(\mathcal{M})?$$

Def: An entourage E on a set X is called cellular if $E^{-1} = E = E \circ E$, which means that E is an equivalence relation. So, $\{E(x)\}_{x \in X}$ is a partition of a set X into pairwise disjoint E-balls.

Def:

A coarse structure is called *cellular* if it has a base consisting of cellular entourages.

Fact 1: A metrizable coarse structure is cellular if and only of it is generated by an ultrametric.

Fact 2: A coarse structure is cellular if and only if it has Gromov's asymptotic dimension zero.

Cellular coarse spaces

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Some critical weights of cellular coarse structures

$$\Delta^\circ_\omega = \min\big(\{\mathfrak c^+\}\cup\{w(\mathcal E) : \mathcal E \text{ is an indiscrete cellular finitary coarse structure on } \omega\}\big);$$

$$\Sigma^\circ_\omega = \min\big(\{\mathfrak c^+\}\cup\{w(\mathcal E) : \mathcal E \text{ is an inseparated cellular finitary coarse structure on } \omega\}\big);$$

$$\Lambda^\circ_\omega = \min\big(\{\mathfrak c^+\}\cup\{w(\mathcal E) : \mathcal E \text{ is a large cellular finitary coarse structure on } \omega\}\big);$$

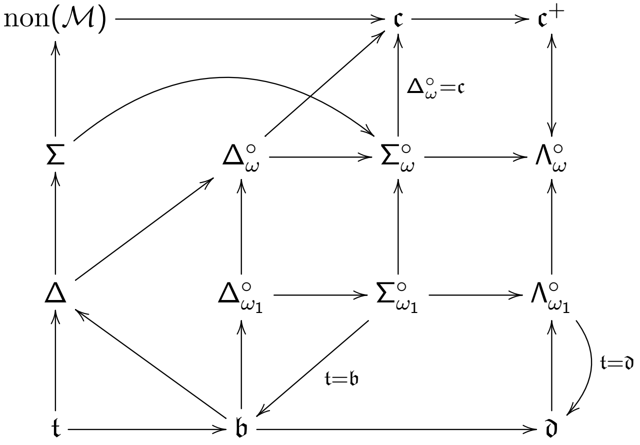
$$\Delta^\circ_{\omega_1} = \min\big(\{\mathfrak c^+\}\cup\{w(\mathcal E) : \mathcal E \text{ is an indiscrete cellular locally finite coarse structure on } \omega\}\big);$$

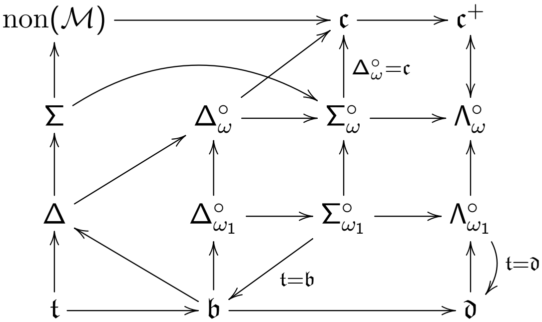
$$\Sigma^\circ_{\omega_1} = \min\big(\{\mathfrak c^+\}\cup\{w(\mathcal E) : \mathcal E \text{ is an inseparated cellular locally finite coarse structure on } \omega\}\big);$$

$$\Lambda^\circ_{\omega_1} = \min\big(\{\mathfrak c^+\}\cup\{w(\mathcal E) : \mathcal E \text{ is a large cellular locally finite coarse structure on } \omega\}\big).$$

The cardinal $\mathfrak c^+$ appears in the definitions of those cardinals in order to make these cardinals well-defined (this indeed is necessary for the cardinal Λ°_ω, which is equal to $\mathfrak c^+$ in ZFC).

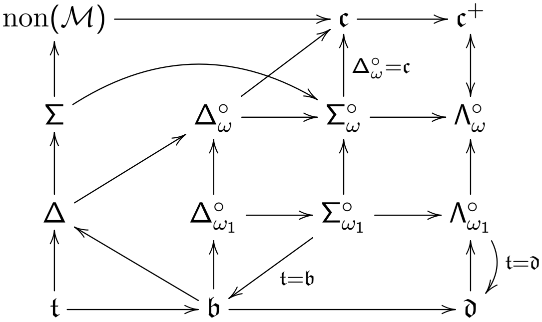
Available info on these cardinals





Theorem

1. $\Lambda^\circ_\omega = \mathfrak{c}^+$.
2. $\Delta^\circ_\omega \le \mathfrak{c}$.
3. $\Delta^\circ_\omega = \mathfrak{c}$ *implies* $\Sigma^\circ_{\omega_1} \le \Sigma^\circ_\omega = \mathfrak{c}$.
4. $\mathfrak{t} = \mathfrak{b}$ *implies* $\Delta^\circ_{\omega_1} = \Sigma^\circ_{\omega_1} = \mathfrak{b}$.
5. $\mathfrak{t} = \mathfrak{d}$ *implies* $\Lambda^\circ_{\omega_1} = \mathfrak{d}$.

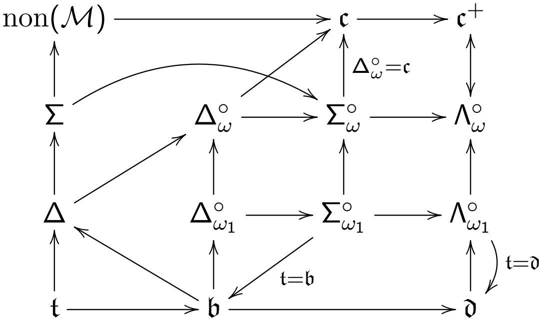


Problem: Is $\Sigma^\circ_\omega \le \mathfrak{c}$ in ZFC? This is equivalent to asking if there exists an inseparable cellular finitary coarse structure on ω in ZFC?

Positive answer to this problem follows from negative answer to

Problem: Is $\Delta^\circ_\omega < \mathfrak{c}$ consistent?

Some Open Problems



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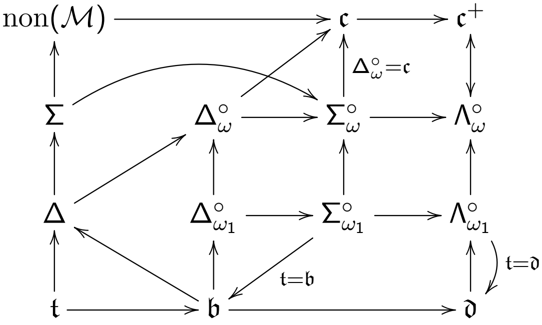

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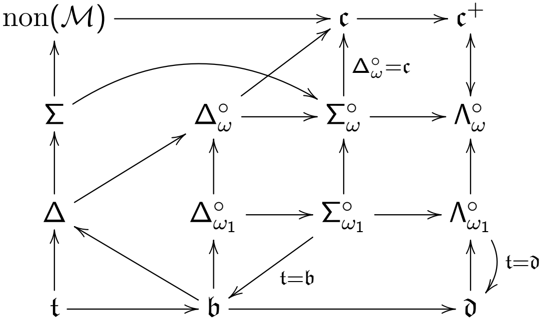

Problem: Does there exist a large cellular locally finite coarse structure on ω in ZFC? If yes, is $\mathfrak{d}$ equal to the smallest weight of a large cellular locally finite coarse structure on ω?

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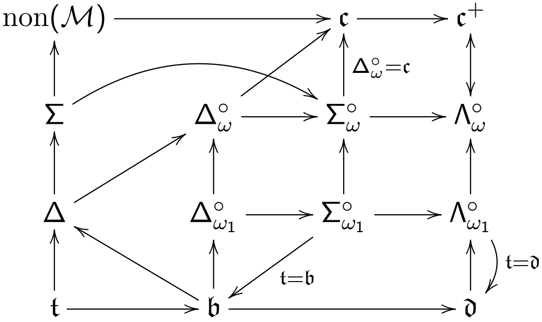

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Cardinal Characteristics

of the poset $\mathbb{E}^\bullet_\omega$

(of nontrivial cellular finitary entourages on ω)

Upper and lower sets in a poset

Let P be a *poset*, i.e., a set P endowed with a partial order $\leq$. For an element $x \in P$ let

$$\uparrow x = \{y \in P : x \leq y\} \quad \text{and} \quad \downarrow x = \{y \in P : y \leq x\}$$

be the *upper* and *lower sets* of x in the poset.

For a subset $A \subseteq P$ let

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The cofinality and coinitiality of a poset

The cardinal

$\downarrow(P) = \min\{|A| : A \subseteq P \ \wedge \ \downarrow A = P\}$ is called the *cofinality*, and

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of the poset P.

Example 1: $\downarrow(\omega^\omega, \le) = \mathfrak{d}$ and $\uparrow(\omega^\omega, \le) = 1$.

Example 2: $\downarrow([\omega]^\omega, \subseteq) = 1$ and $\uparrow([\omega]^\omega, \subseteq) = \mathfrak{c}$.

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$\downarrow(P)=\min\{|A| : A\subseteq P\ \wedge\ \downarrow A=P\}$ is called the *cofinality*, and
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Example 1: $\downarrow(\omega^\omega,\le)=\mathfrak{d}$ and $\uparrow(\omega^\omega,\le)=1$.

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The cofinality and coinitiality of a poset

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Iterated upper and lower sets

The transitivity of the partial order $\le$ implies that for any subset $A \subseteq P$

$$\uparrow\uparrow A = \uparrow A \quad \text{and} \quad \downarrow\downarrow A = \downarrow A.$$

On the other hand, we can consider the lower-upper and upper-lower sets of A: $\uparrow\downarrow A$ and $\downarrow\uparrow A$ and also the corresponding cardinal characteristics of the poset P:

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Iterating, we obtain the cardinal characteristics

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$$\upuparrows(P) = \sup\{|A| : A \subseteq P \ \wedge\ \forall x, y \in A \ \ (x \neq y \ \Rightarrow\ {\uparrow}x \cap {\uparrow}y = \emptyset)\},$$
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which are counterparts of the cellularity in topological spaces.

The order relation between these cardinals characteristics are described in the following diagram (in which an arrow $\kappa \to \lambda$ between cardinals indicates that $\kappa \leq \lambda$):

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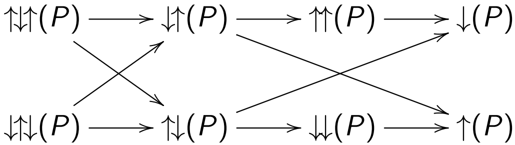

Cardinal characteristics of the poset $\mathbb{E}^\bullet_\omega$

We shall be interested in these cardinal characteristics for the poset $\mathbb{E}^\bullet_\omega$ of nondiscrete cellular finitary entourages on ω.

An entourage E on a set X is called

- *nondiscrete* if $\{x \in X : E(x) \neq \{x\}\}$ is infinite;
- *cellular* if $E^{-1} = E = E \circ E$;
- *finitary* if $\sup_{x \in X} |E(x) \cup E^{-1}(x)| < \omega$.

Theorem

1. $\downarrow\!\uparrow\!\downarrow(\mathbb{E}^{\bullet}_{\omega}) = \uparrow\!\downarrow\!\uparrow(\mathbb{E}^{\bullet}_{\omega}) = 1$.
2. $\upuparrows(\mathbb{E}^{\bullet}_{\omega}) = \downdownarrows(\mathbb{E}^{\bullet}_{\omega}) = \downarrow(\mathbb{E}^{\bullet}_{\omega}) = \uparrow(\mathbb{E}^{\bullet}_{\omega}) = \mathfrak{c}$.
3. $\mathrm{cov}(\mathcal{M}) \le \downarrow\!\uparrow(\mathbb{E}^{\bullet}_{\omega}) \le \mathfrak{c}$.
4. $\Sigma \le \uparrow\!\downarrow(\mathbb{E}^{\bullet}_{\omega}) \le \mathrm{non}(\mathcal{M})$.

Problem: Which of the following strict inequalities is consistent?

1. $\Sigma < \uparrow\!\downarrow(\mathbb{E}^{\bullet}_{\omega})$;
2. $\uparrow\!\downarrow(\mathbb{E}^{\bullet}_{\omega}) < \mathrm{non}(\mathcal{M})$;
3. $\mathrm{cov}(\mathcal{M}) < \downarrow\!\uparrow(\mathbb{E}^{\bullet}_{\omega})$;
4. $\downarrow\!\uparrow(\mathbb{E}^{\bullet}_{\omega}) < \mathfrak{c}$.

Remark: Since the strict inequality $\mathrm{cov}(\mathcal{M}) < \mathfrak{c}$ is consistent, either (3) or (4) in the above Problem is consistent.
But which one? Or both?

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Special Ultrafilters in Large-Scale Topology

Thin ultrafilters

Definition

A free ultrafilter $\mathcal{U}$ on ω is called *thin* if for any metrizable finitary coarse structure $\mathcal{E}$ on ω there exists a set $U \in \mathcal{U}$ such that the coarse structure $\mathcal{E}{\restriction}U$ on U is discrete.

Problem (Protasov)

Is it true that thin ultrafilters exist in ZFC?

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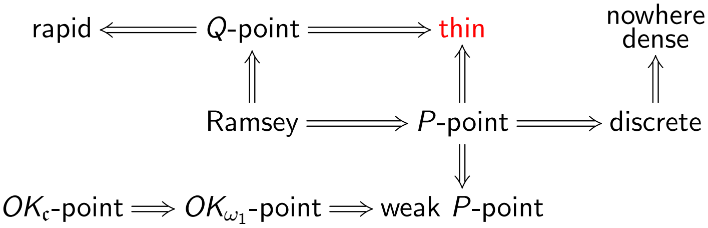
Some known types of ultrafilters

A free ultrafilter $\mathcal{U}$ on ω is called

- a *P-point* if for any sequence of sets $\{U_n\}_{n\in\omega} \subseteq \mathcal{U}$ there exist a set $U \in \mathcal{U}$ such that $U \subseteq^* U_n$ for every $n \in \omega$;
- a *Q-point* if for any locally finite cellular entourage E on ω there exists a set $U \in \mathcal{U}$ such that $|U \cap E(x)| \le 1$ for any $x \in \omega$;
- *Ramsey* if for any map $f : \omega \to \omega$ there exists $U \in \mathcal{U}$ such that $f{\restriction}U$ is either constant or injective;
- a *weak P-point*, if for any sequence $(\mathcal{U}_n)_{n\in\omega}$ of free ultrafilters that are not equal to $\mathcal{U}$ there exists a set $U \in \mathcal{U} \setminus \bigcup_{n\in\omega} \mathcal{U}_n$;
- an *OK_κ-point* for a cardinal κ if for any sequence of sets $\{U_n\}_{n\in\omega} \subseteq \mathcal{U}$ there exists a family $(V_\alpha)_{\alpha\in\kappa} \subseteq \mathcal{U}$ such that for any ordinals $\alpha_1 < \cdots < \alpha_n$ in κ we have $\bigcap_{i=1}^n V_{\alpha_i} \subseteq^* U_n$;
- *rapid* if for any function $f \in \omega^\omega$ there exists a function $g \in \omega^\omega$ such that $f \le g$ and $\{g(n) : n \in \omega\} \in \mathcal{U}$;
- *discrete* if for any injective function $\varphi : \omega \to \mathbb{R}$ there exists a set $U \in \mathcal{U}$ whose image $f(U)$ is a discrete subspace of $\mathbb{R}$;
- *nowhere dense* if for any injective function $\varphi : \omega \to \mathbb{R}$ there exists a set $U \in \mathcal{U}$ whose image $f(U)$ is nowhere dense in $\mathbb{R}$.

Thin ultrafilter vs other types of ultrafilters



By a famous result of Kunen, $OK_{\mathfrak{c}}$-points exist in ZFC.

Shelah constructed a model without nowhere dense ultrafilters.

Miller constructed a model containing no rapid ultrafilters.

It is unknown if there is a model without P-points and Q-points.

So the problem of constructing a model containing no thin ultrafilters is more difficult than the (already difficult) problem of constructing a model without P-points and Q-points.

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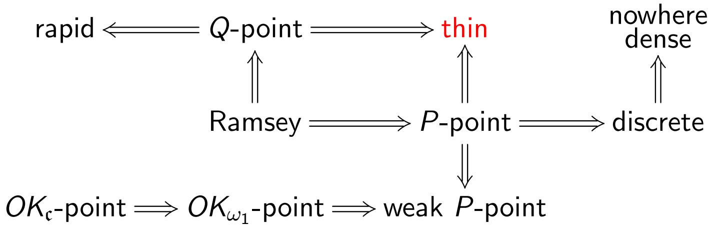

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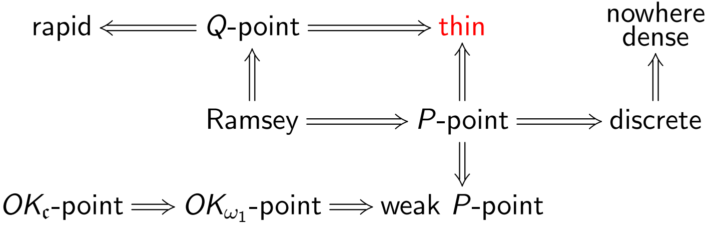

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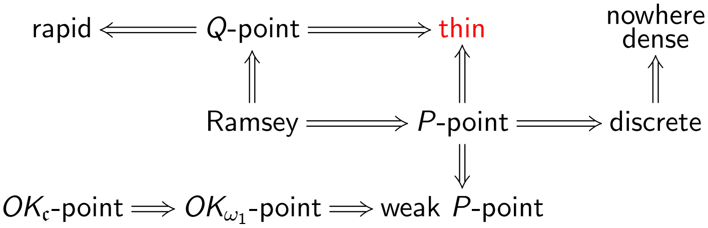

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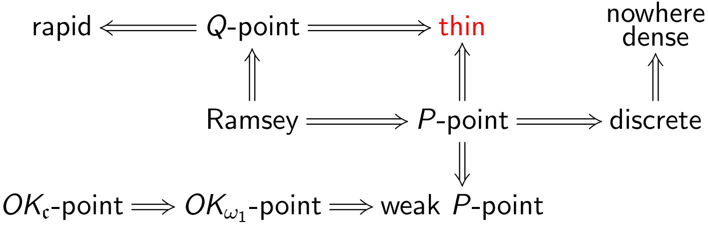

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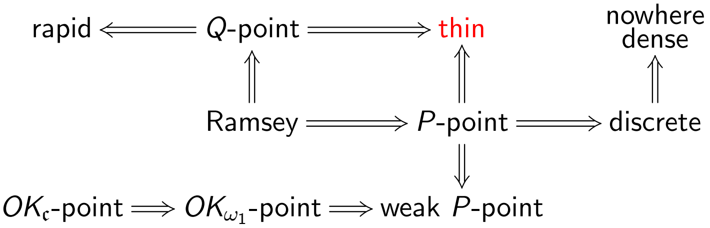

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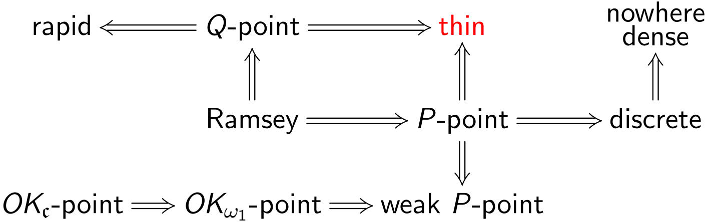

Problem

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- Is each discrete ultrafilter thin?

Thin ultrafilter vs other types of ultrafilters

$$
\begin{array}{ccccccc}
\text{rapid} & \Longleftarrow & Q\text{-point} & \Longrightarrow & \text{thin} & & \begin{array}{c}\text{nowhere}\\ \text{dense}\end{array} \\
 & & \Uparrow & & \Uparrow & & \Uparrow \\
 & & \text{Ramsey} & \Longrightarrow & P\text{-point} & \Longrightarrow & \text{discrete} \\
 & & & & \Downarrow & & \\
OK_{\mathfrak{c}}\text{-point} \Longrightarrow & & OK_{\omega_1}\text{-point} & \Longrightarrow & \text{weak } P\text{-point} & &
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Thank You!