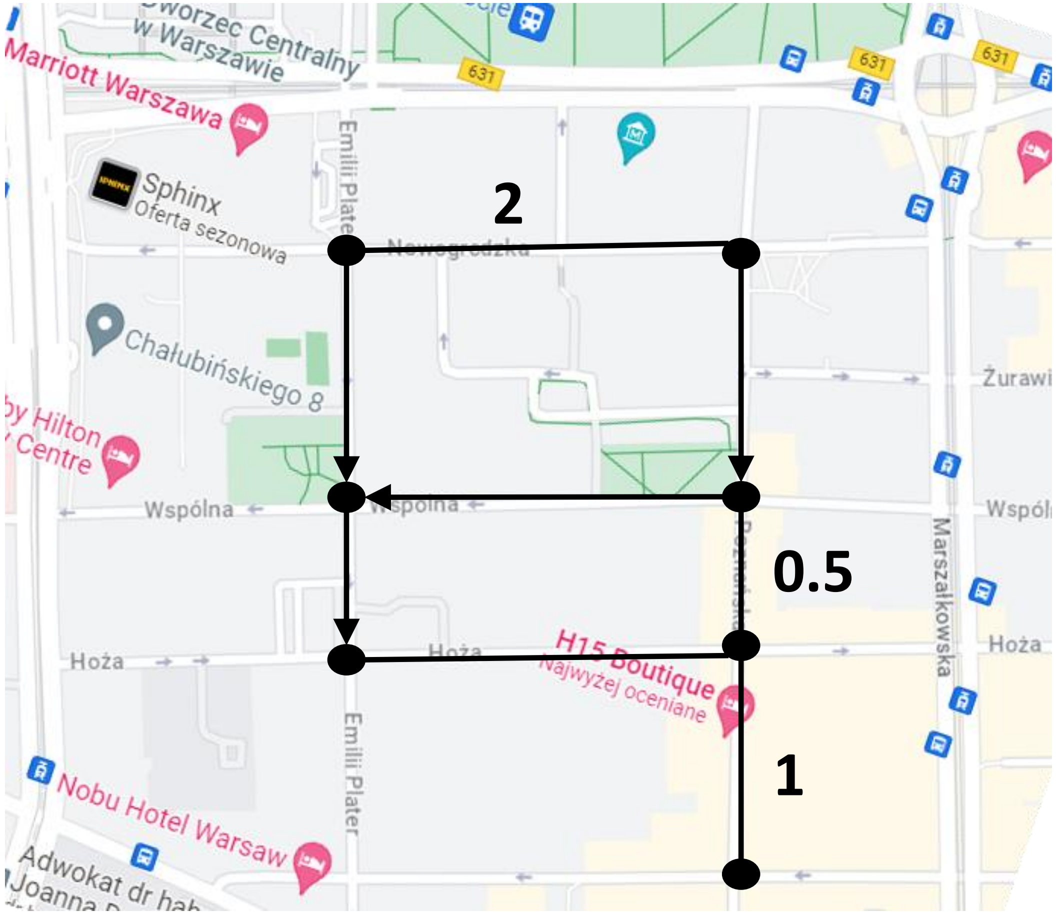
Shortest paths, edge weights, and models of computation

[Adam Karczmarz]

22/01/2026

Plan

1. Problem definition.
2. Recap of some basic techniques.
3. Recent breakthroughs.
4. On the way:
    - models of computations,
    - how restricted edge weight domain is exploited,
5. Briefly about one related result of ours.



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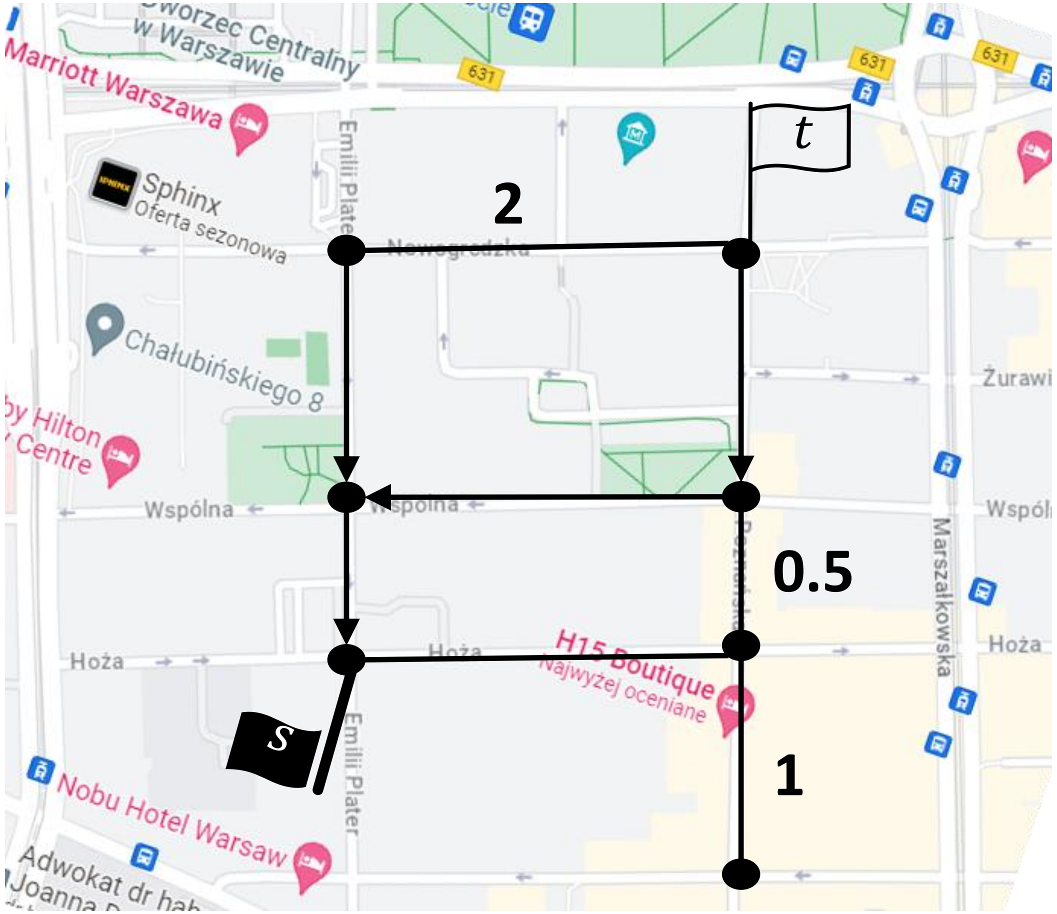
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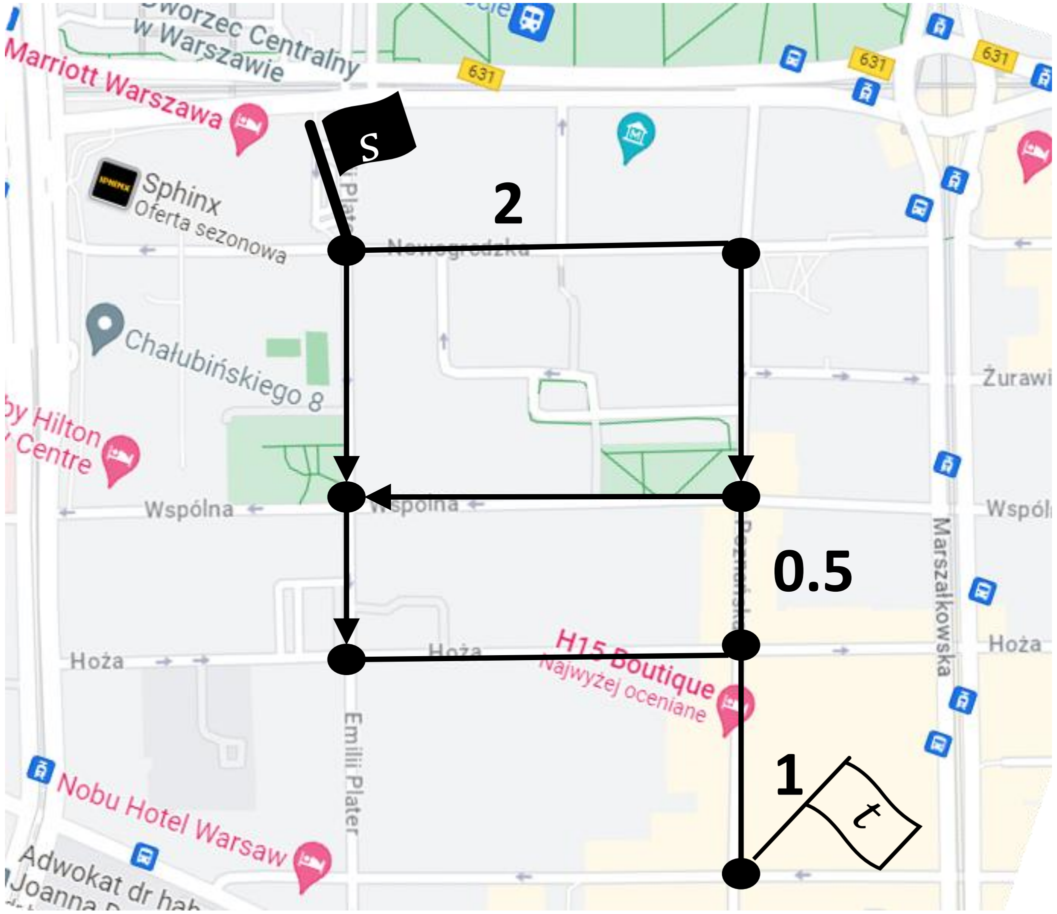
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The shortest path problem

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- Two vertices: source/target $s, t \in V$.

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- fast in the worst case (asymptotically).

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Notation:

- $\text{poly}(n)$, $\text{poly}(n, m)$; $\text{polylog}(n) = \log^{O(1)} n$,
- $\tilde{O}\big(f(n, m)\big) := O\big(f(n, m)\text{polylog}(n)\big)$.
- $w(uv) :=$ weight of a (directed) edge uv.

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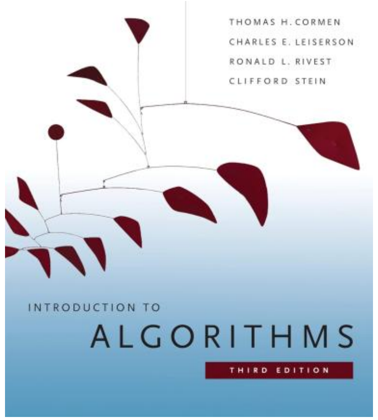

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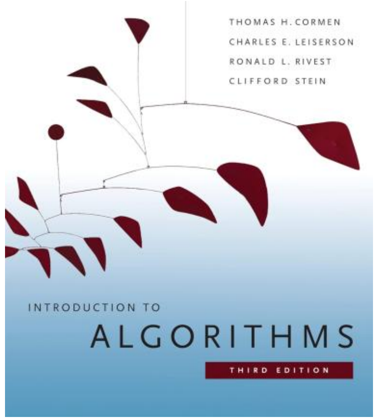

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No? Use Bellman-Ford algorithm ('56).
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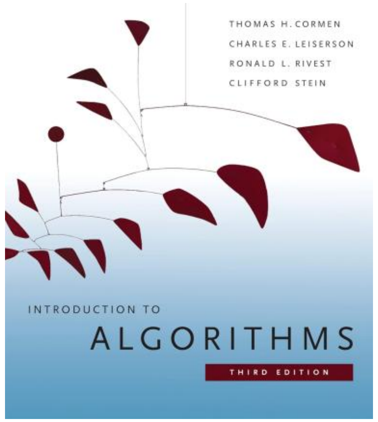

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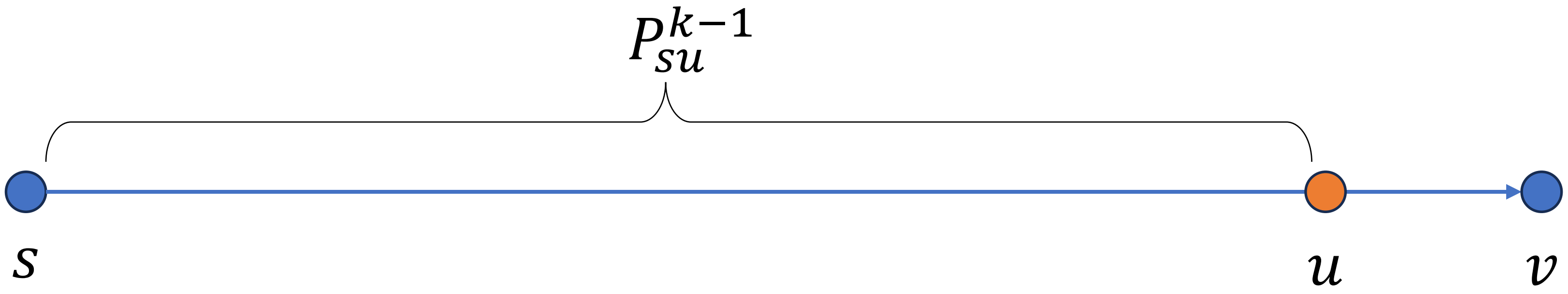

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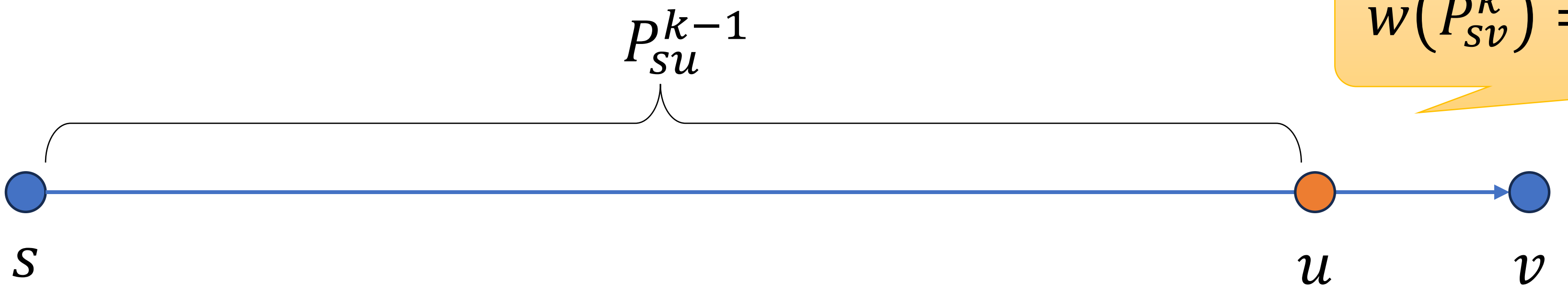

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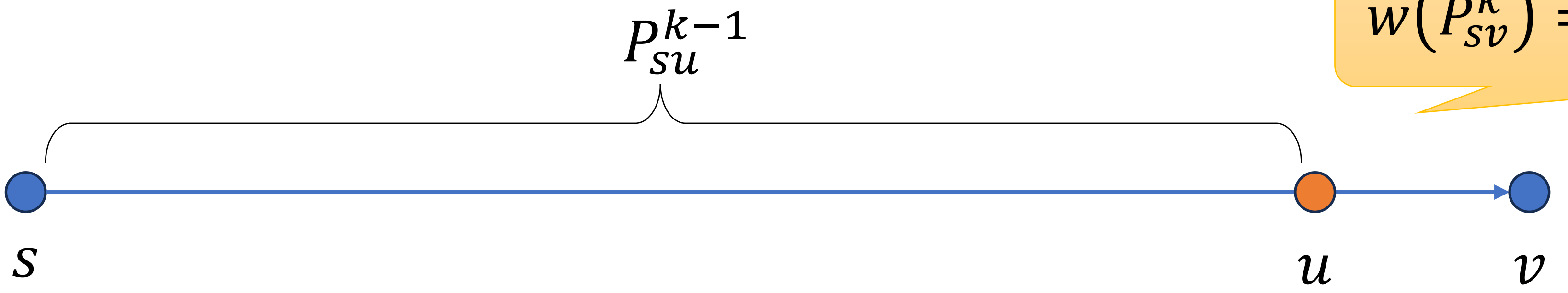

All weights of P_{st}^{k} for $t \in V$, $k = 0, \dots, n$ and can be computed in $O(m \cdot n)$ time.

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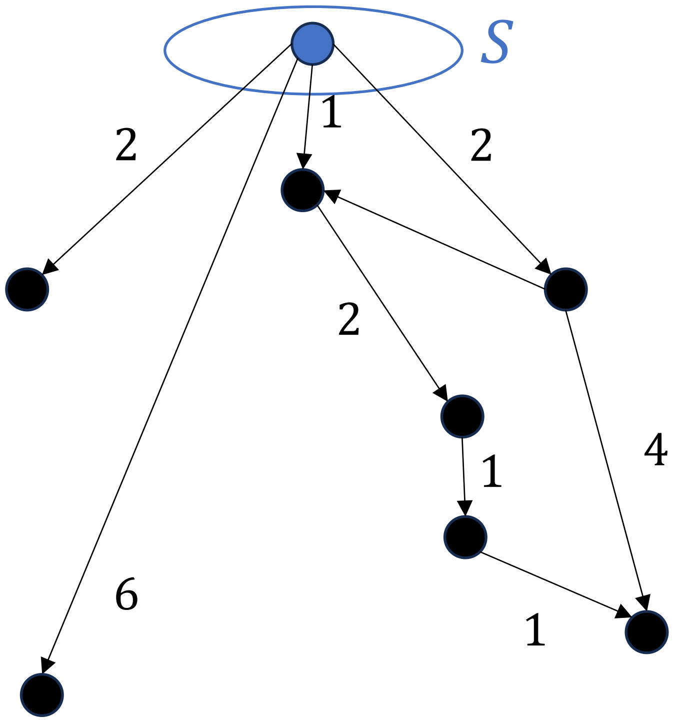

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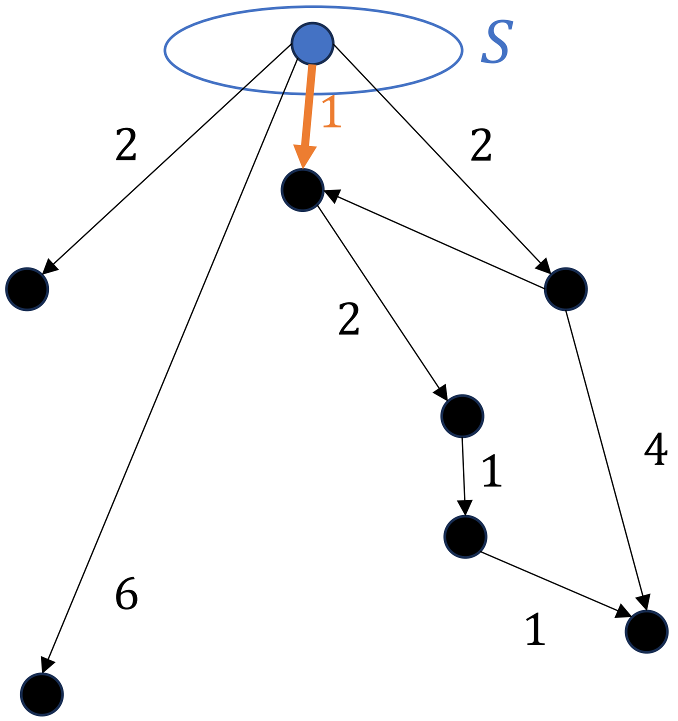

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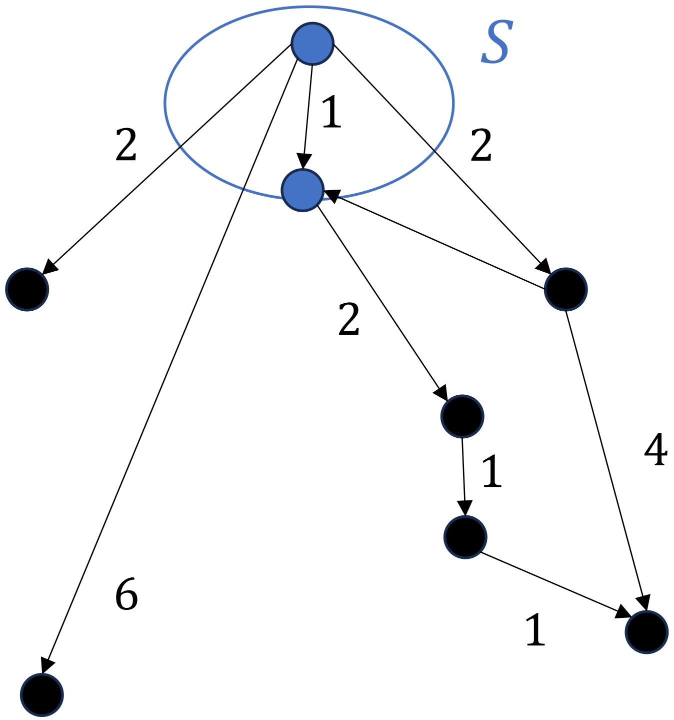

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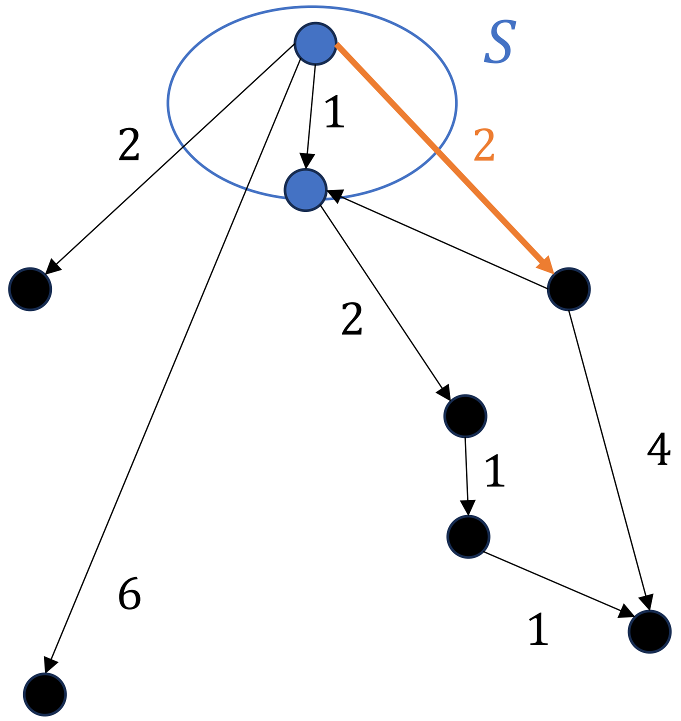

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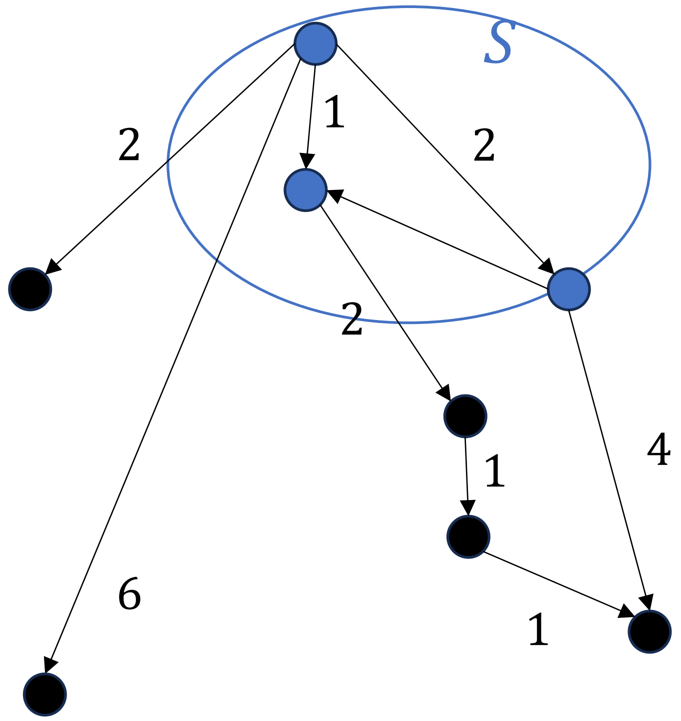

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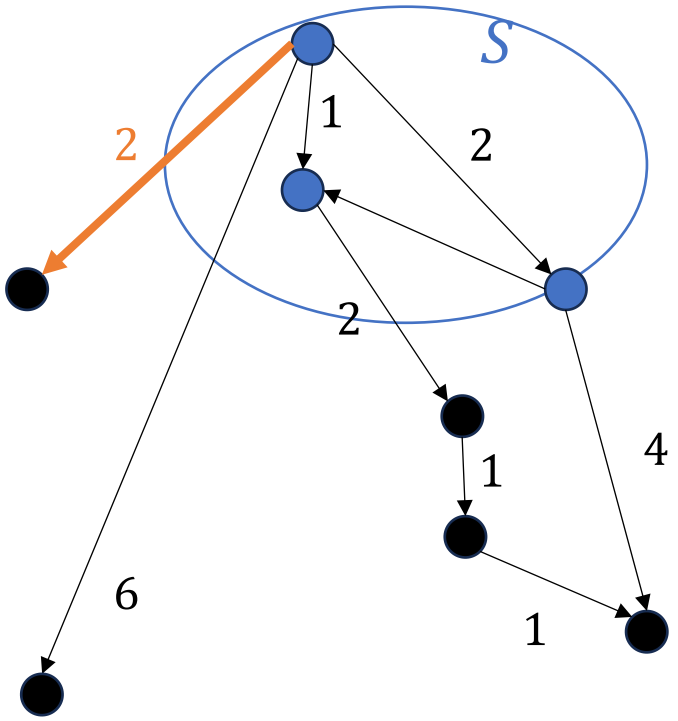

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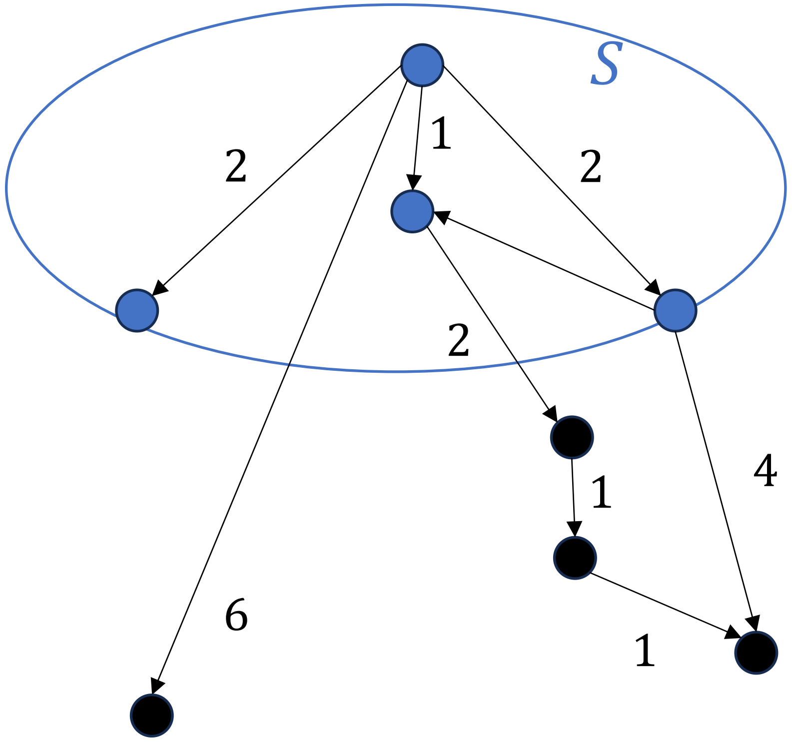

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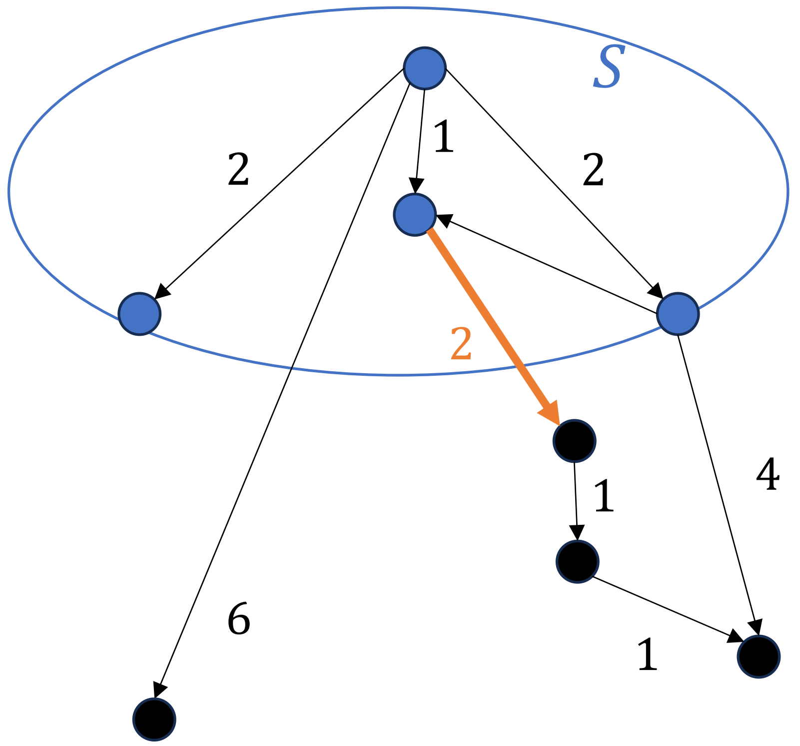

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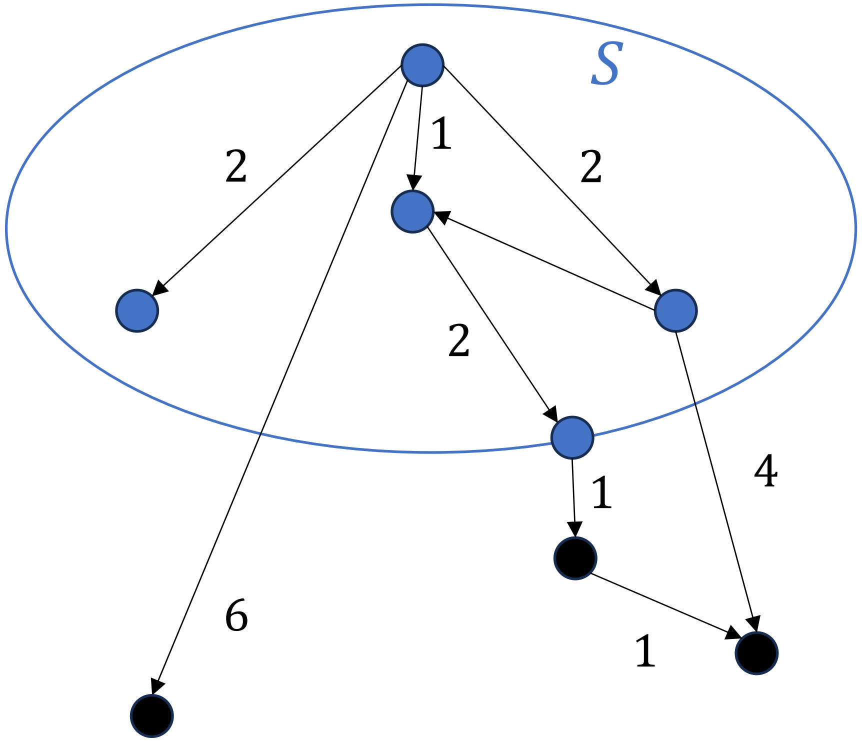

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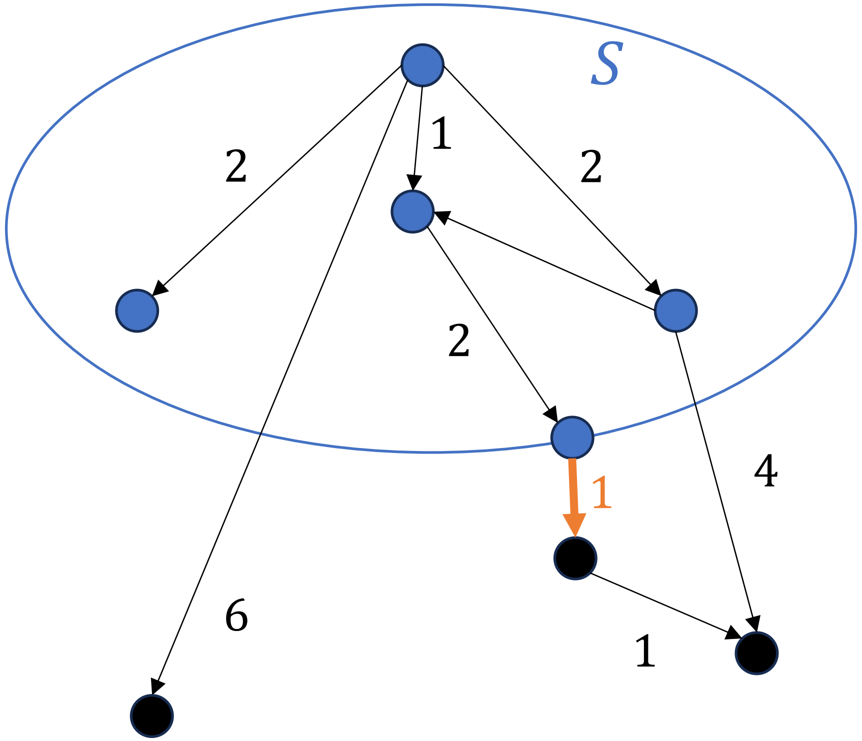

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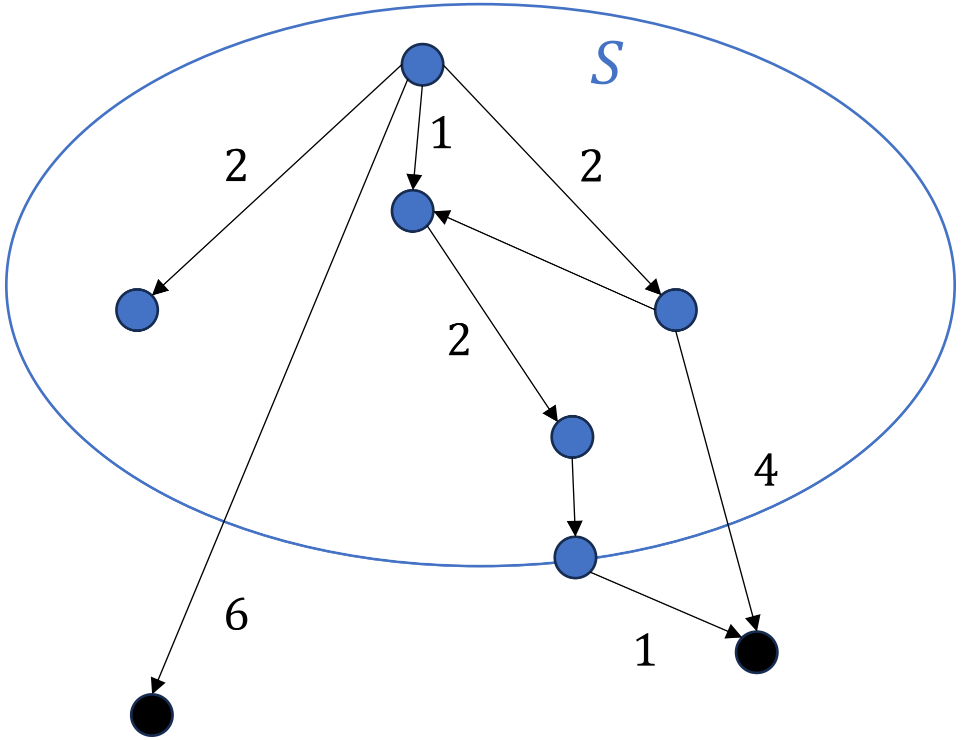

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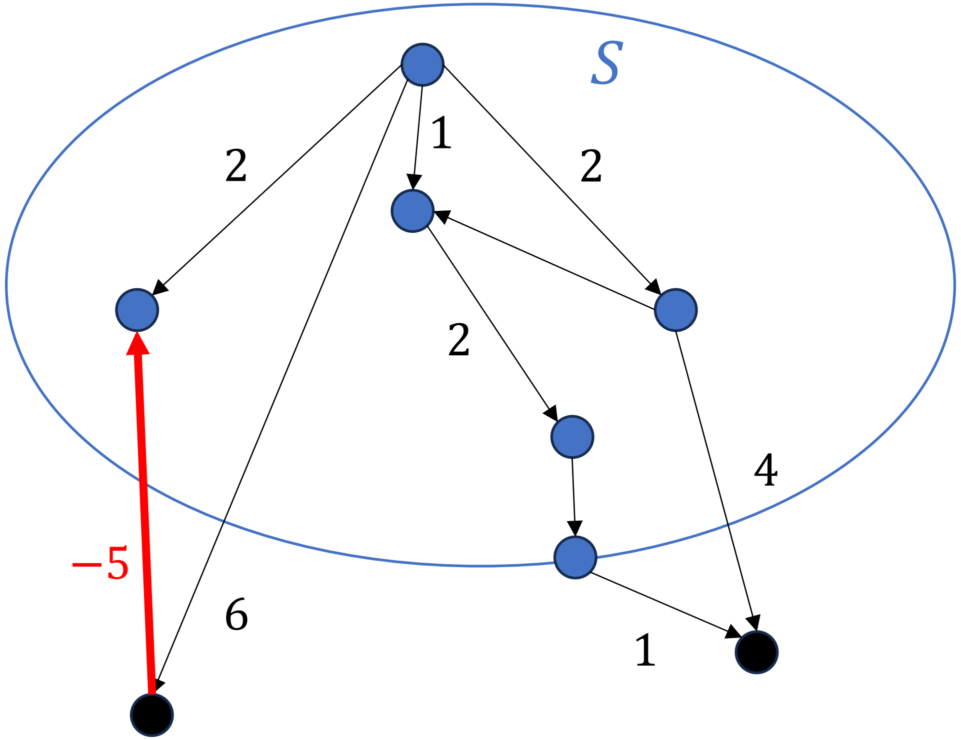

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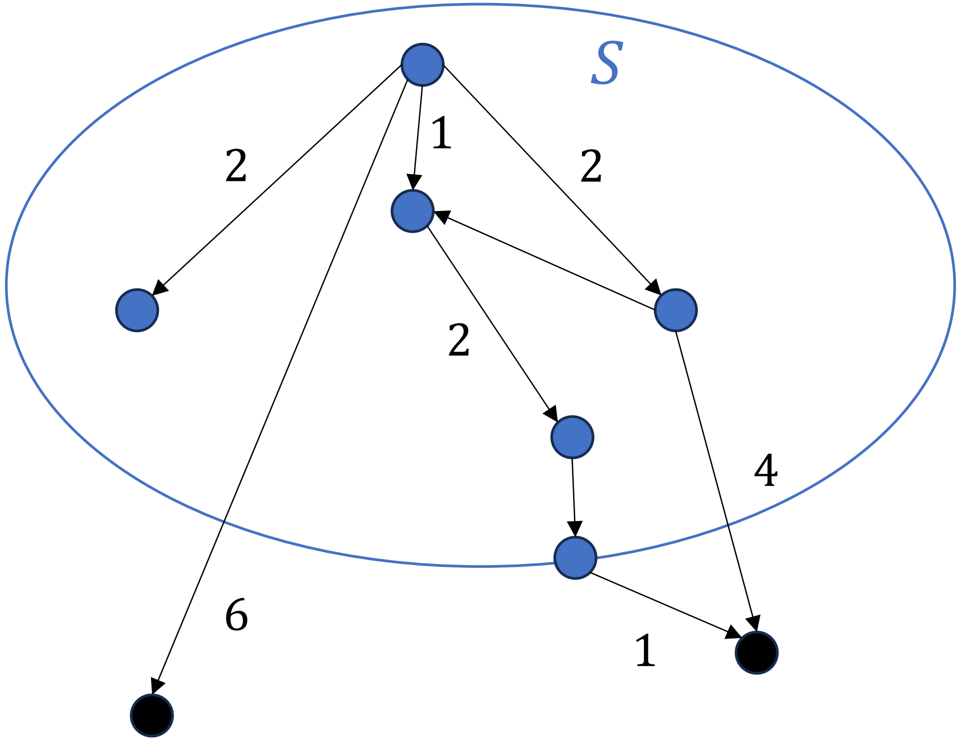

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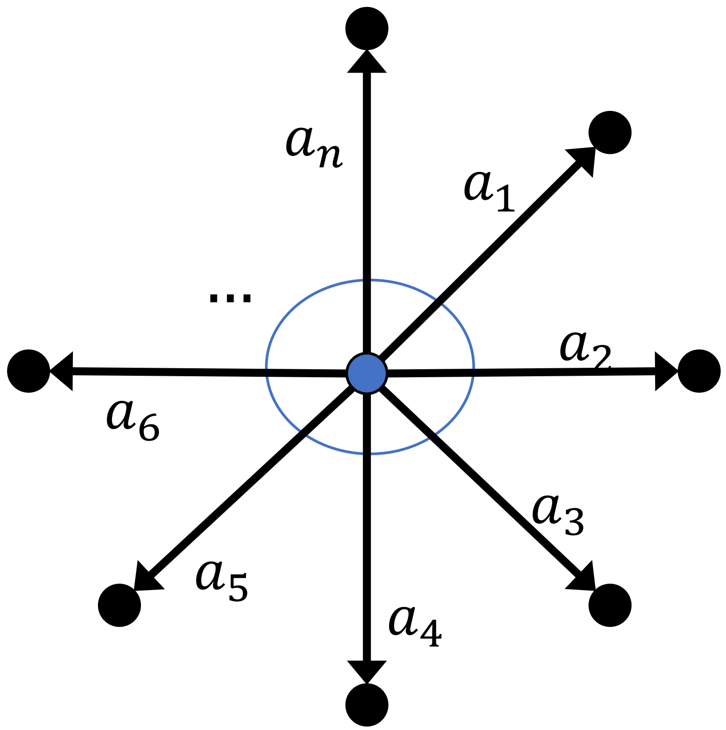

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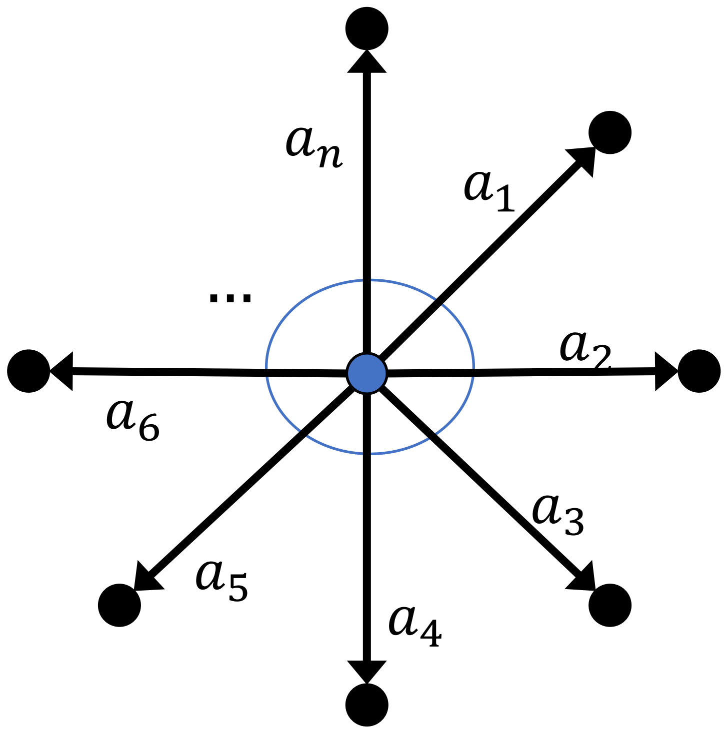

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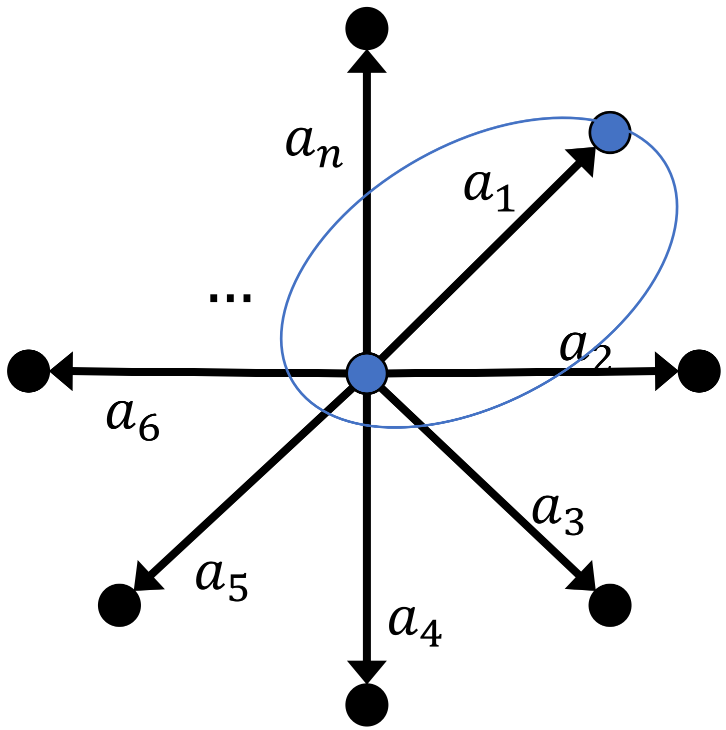

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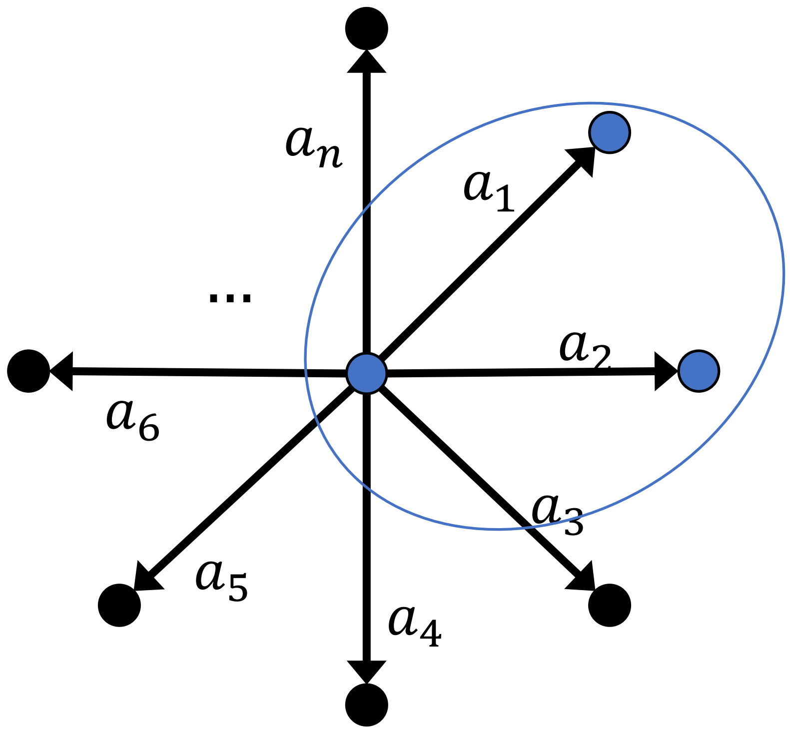

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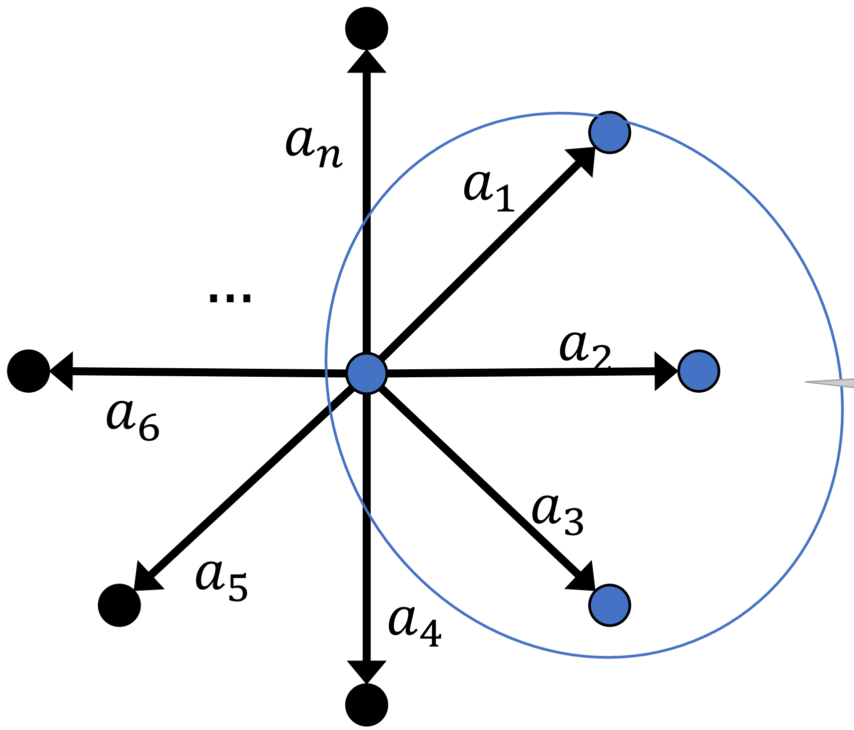

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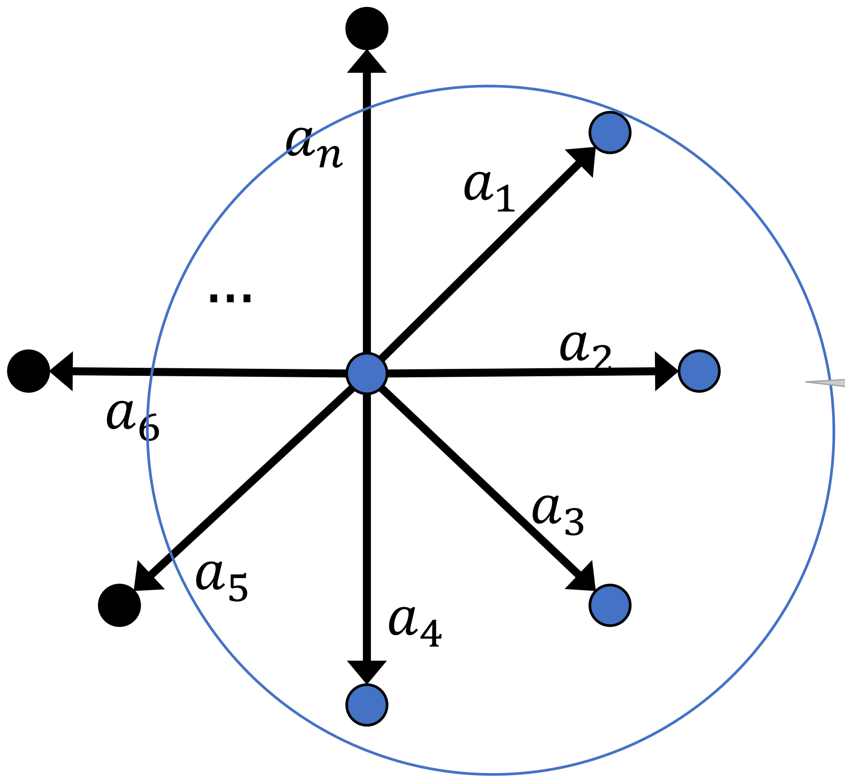

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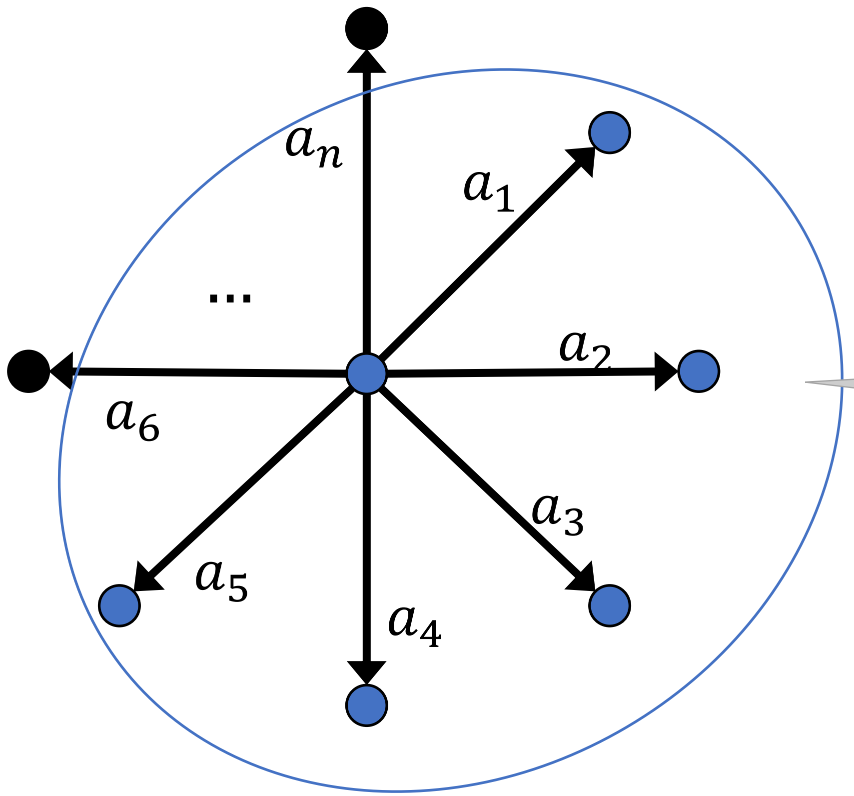

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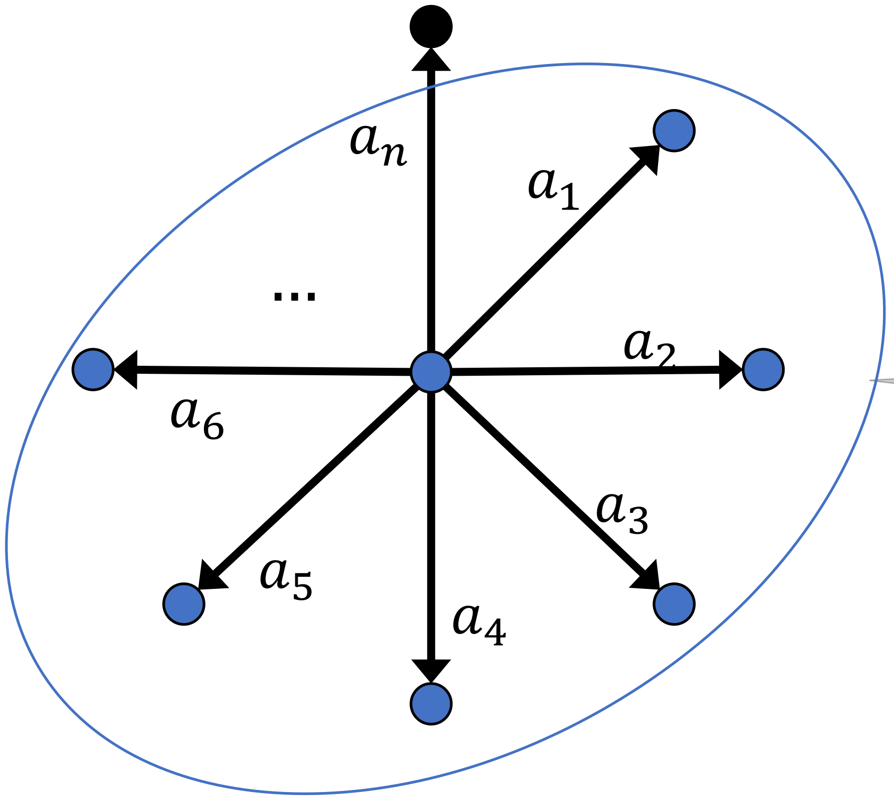

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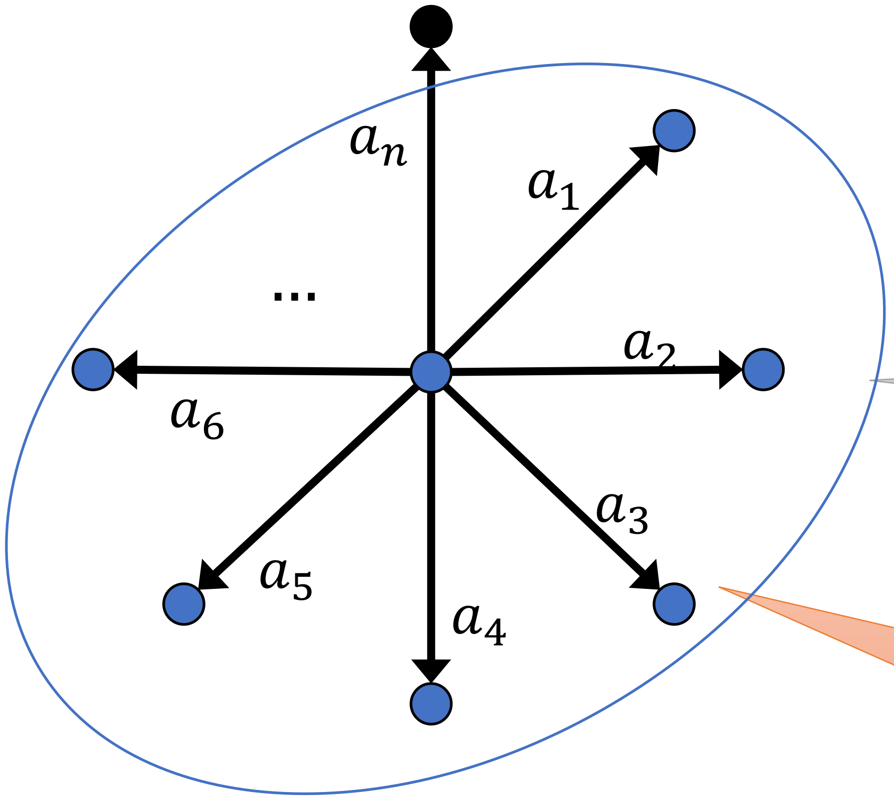

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Sorting with binary comparisons requires $\log_2 n! = \Omega(n \log n)$ comparisons.

Recent breakthroughs

1) FOCS 2022 Best Paper Award (Bernstein, Nanongkai, Wulff-Nilsen):
   "Negative-Weight Single-Source Shortest Paths in Near-linear Time"
2) STOC 2024 Best Paper Award (Fineman):
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Non-negative real weights.

- Real/integer regimes very different for SSSP.
- What does it even mean to solve optimization problems on real numbers?

Exact optimization on real-weighted graphs

Real RAM:

- Integer-indexed memory cells store infinite-precision real numbers,
- basic arithmetic operations ($+$, $-$, $\cdot$, $\div$) and comparisons performed in $O(1)$ time in a black-box way.

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- Integer division $\lfloor x/a \rfloor$ in $O(1)$ time ⇒ solve NP-hard problems in poly. time.
- Only $(+, <)$ ⇒ e.g., count triangles in a graph in $\tilde{O}(m)$ time.

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- Green flag: running within the same time bound on a realistic model.

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Non-negative.

Runs in $O(m \log^{2/3} n)$ time using $(+, <)$, thus cannot sort via comparisons!

Baseline: $O(m + n \log n)$

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Word RAM

Think C language, $w = 64$.
$n =$ "problem size"

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For simplicity, let's assume $w = \Theta(\log n) \Rightarrow$ absolute edge weights $\leq \text{poly}(n)$.

(obsolete) Manual for solving integer SSSP

Single-Source Shortest Paths (SSSP): [integer]
Given a directed graph $G = (V, E)$ with n vertices and m edges whose weights are integers fitting in words, and a source $s \in V$, compute $\mathrm{dist}_G(s, t)$ for all $t \in V$.

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No? Use a "scaling" algorithm. E.g. Gabow's $\tilde{O}(mn^{3/4})$ ('83) Or Goldberg's $\tilde{O}(m\sqrt{n})$ ('93)

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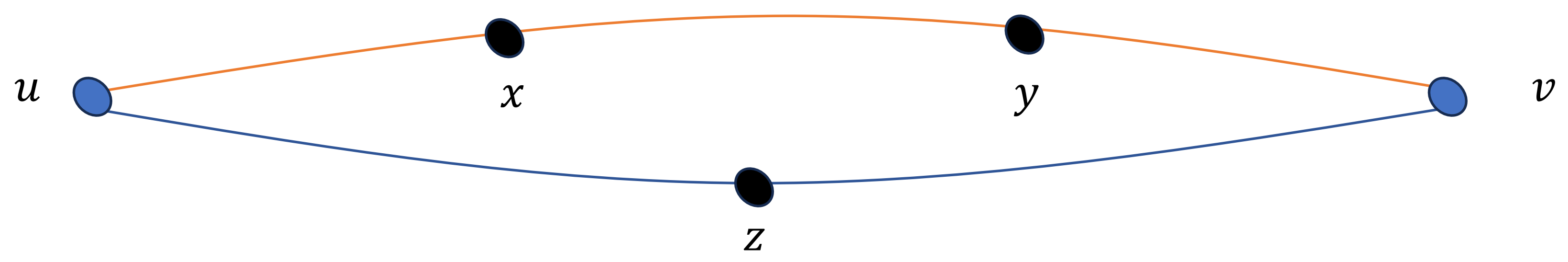

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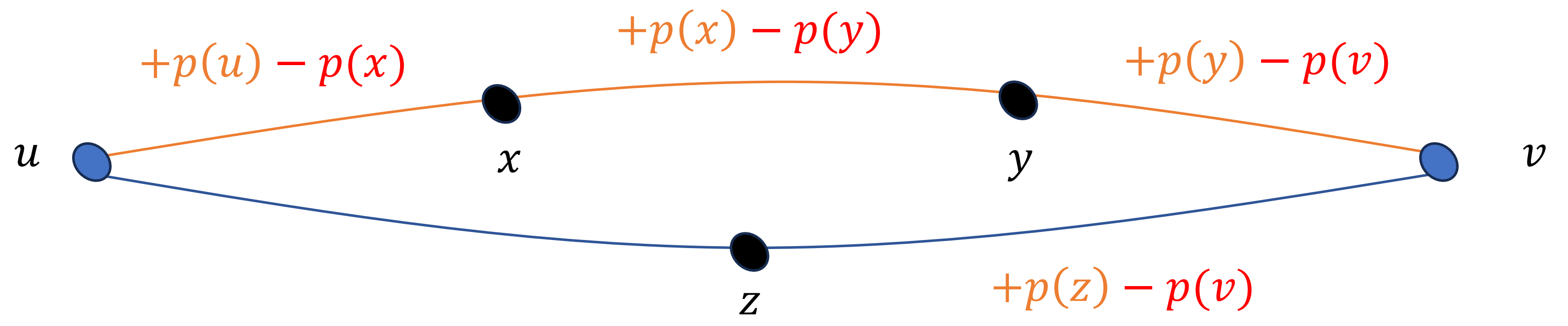

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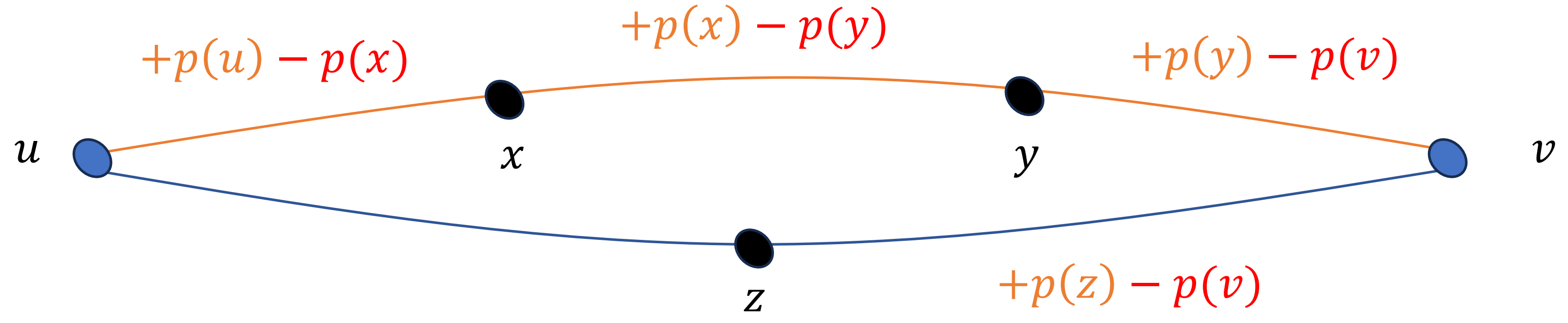

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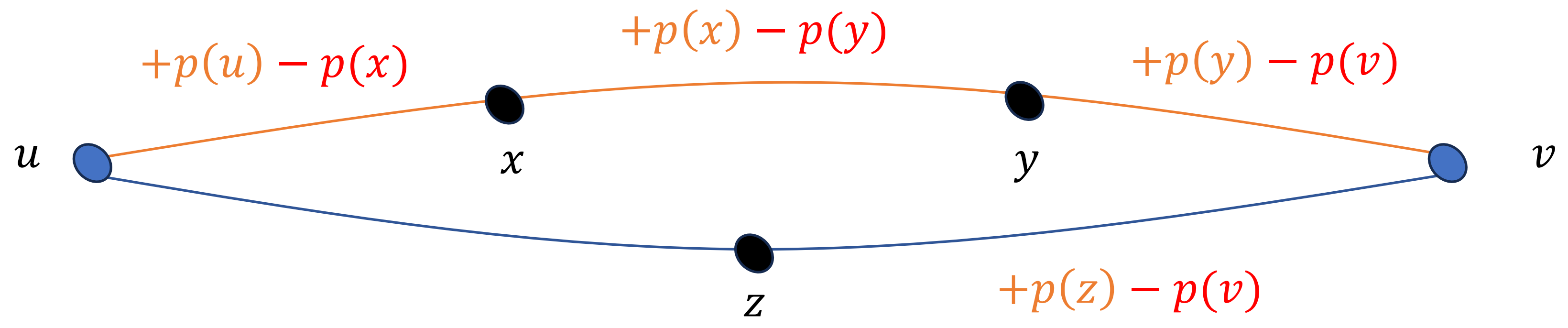

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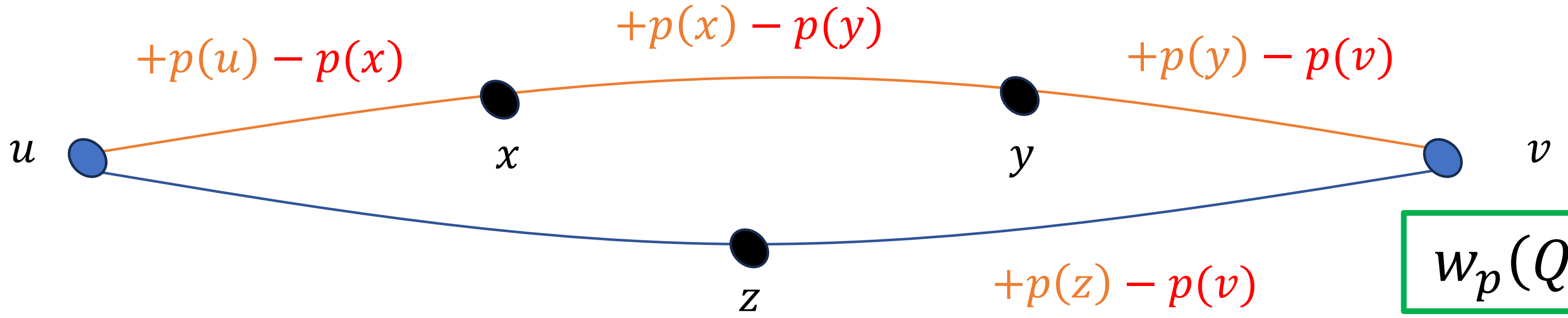

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1) does not change the shortest paths structure,
2) makes the problem amenable to Dijkstra.

Approximation scheme

1) Relax the inequalities with an error parameter ϵ that you can initialize easily

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$|w(uv)| \leq \text{poly}(n)$, so start with $p \equiv 0$, $\epsilon = \text{poly}(n)$.

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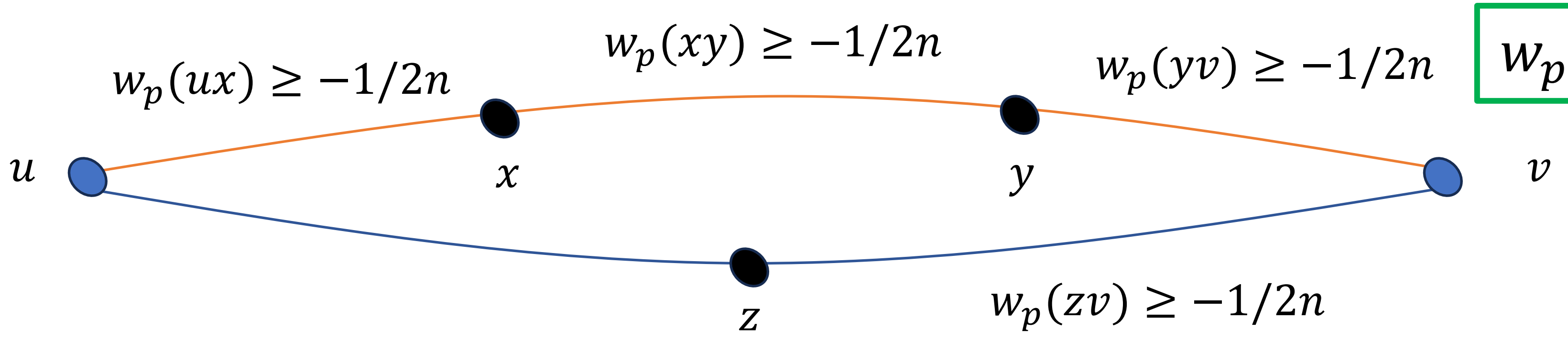

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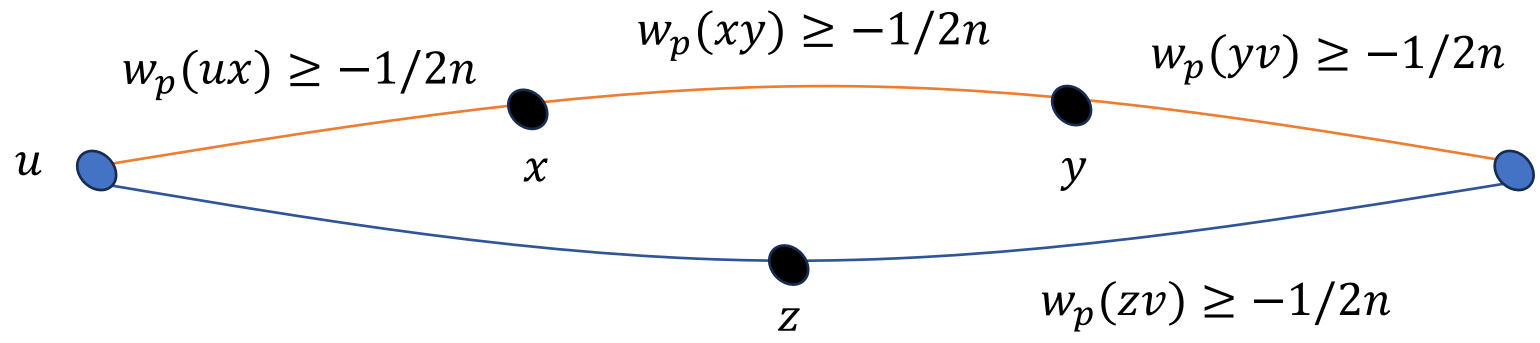

$w_p(Q) - w_p(R) = w(Q) - w(R)$

Increase weights by $\frac{1}{2}n$.

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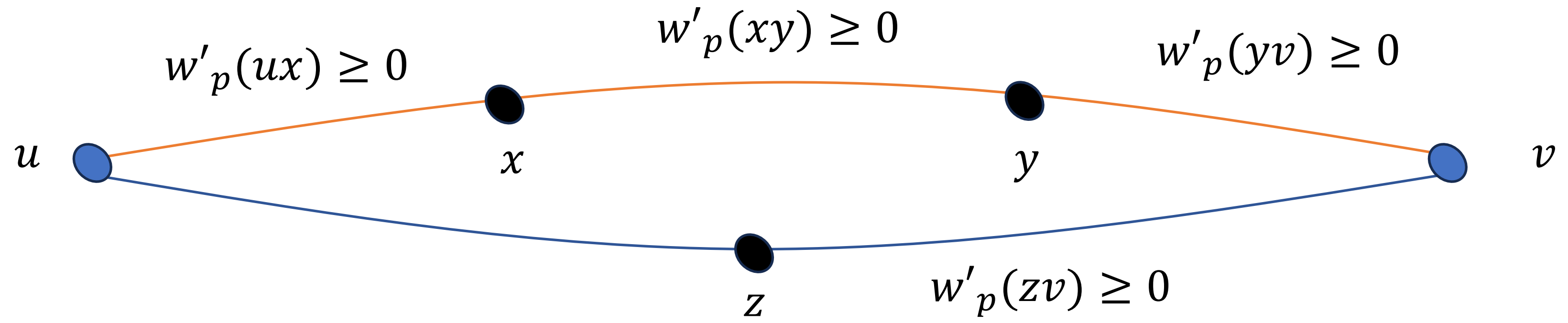

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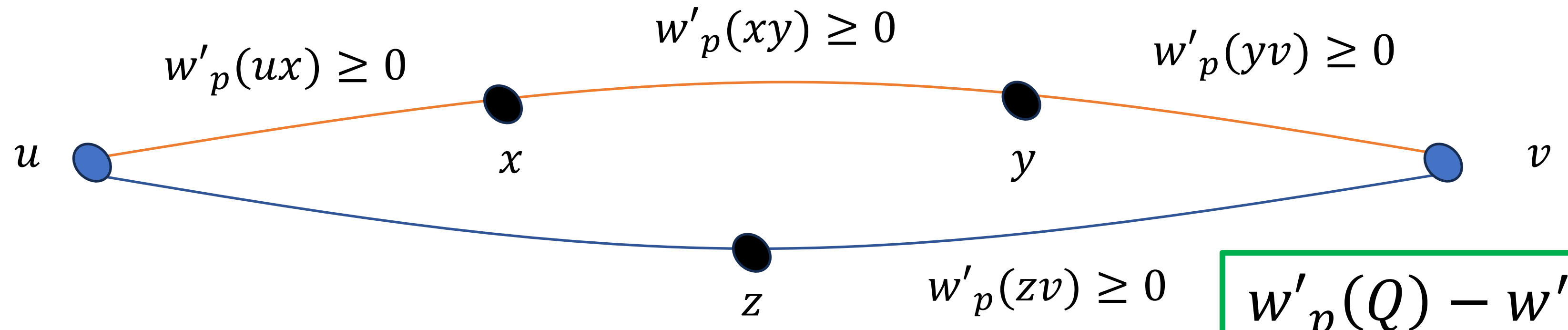

$$w'_p(Q) - w'_p(R) \leq w(Q) - w(R) + 1/2$$

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Scaling framework = very efficient approximation scheme!

- Single ``refinement” iteration: decrease error by a constant factor.
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- ... but would never terminate with an exact solution in $\mathrm{poly}(n, m)$ time.
- Integrality $\Rightarrow$ accuracy $\epsilon^{-1} = \mathrm{poly}(n)$ enough to correctly round.

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- max flow with integer weights in $m^{1+o(1)}$ time [Chen et al. ‘22].

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Best real RAM bound:
$O(nm)$ [Orlin ‘13]

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"Breaking the So … er for Directed Single-Source Shortest Paths"

Given a price function $p: V \to \mathbb{R}$ such that k vertices have adjacent negative edges, in $\tilde{O}\left(mk^{2/9}\right)$ *time one can compute a price fun.* p' *with* $k^{1/3}$ *fewer such negative vertices.*

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Can optimization problems on rational-weighted graphs be solved on the word RAM **exactly** and as efficiently as on integer-weighted graphs?

Rational SSSP

Single-Source Shortest Paths (SSSP): [rational]
Given a directed graph $G = (V, E)$ with n vertices and m edges whose weights are word-fitting rationals, and a source $s \in V$, compute $\mathrm{dist}_G(s, t)$ for all $t \in V$.

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Trivial adaptation of Dijkstra runs in $\tilde{O}(mn)$ time on rationals ≥ 0!

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At least $\Theta(n)$-factor time slowdown compared to integer data!

Our results

Joint work with W. Nadara and M. Sokołowski (SODA 2024):

Theorem:
SSSP with non-negative word-fitting rational weights can be solved in $\tilde{O}(n+m)$ time on the word RAM.

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- … even though arithmetic operations on k-bit rationals take $\tilde{O}(k)$ time.
- Indeed, almost matching the best-known integer bound possible for $\mathrm{SSSP}_{\geq 0}$.

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- Beats scaling with exponential accuracy for very dense graphs $m = \Omega(n^{2.51})$.
- No reason to believe near-linear time is impossible.

Conclusion

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1) Model choice and low-level details can make a huge difference even for very basic polynomial algorithmic optimization problems.
2) What we've been taught about computing single-source shortest paths is now completely obsolete at last.
3) Very fast-converging approximation schemes can be considered exact algorithms in realistic models of computation.
4) Studying truly exact computation in unrealistic models is okay and timely, but don't abuse them!

Open problem

See https://en.wikipedia.org/wiki/Smale's_problems

Big open problem in Real RAM vs. Word RAM optimization:

Can Linear Programming be solved exactly on a Real RAM in polynomial time (as a function of $\#(\text{variables} + \text{constraints})$) without model abuse?

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Khachiyan'79: Linear programming with rational data can be solved in polynomial time (in $\#(\text{variables} + \text{constraints})$, on the word RAM).