Shelah's Proof of Diamond

Jakub Andruszkiewicz

Faculty of Mathematics, Informatics and Mechanics, University of Warsaw

May 5, 2021

Definitions

Definition

By $\diamondsuit$ ("diamond principle") we denote the sentence: "there is a sequence $(A_\alpha)_{\alpha<\omega_1}$ of subsets of ω_1 such that for every $\alpha < \omega_1$ we have $A_\alpha \subseteq \alpha$ and for every subset $X \subseteq \omega_1$ the set

$$\{\alpha < \omega_1 : A_\alpha = X \cap \alpha\}$$

is stationary."

Definitions

Definition

By $\diamondsuit$ ("diamond principle") we denote the sentence: "there is a sequence $(A_\alpha)_{\alpha<\omega_1}$ of subsets of ω_1 such that for every $\alpha < \omega_1$ we have $A_\alpha \subseteq \alpha$ and for every subset $X \subseteq \omega_1$ the set

$$\{\alpha < \omega_1 : A_\alpha = X \cap \alpha\}$$

is stationary."

Definition

By $\clubsuit$ ("club principle") we denote the sentence: "there is a sequence $(A_\alpha)_{\alpha \in L_1}$ of subsets of ω_1 such that for each $\alpha \in L_1$ we have $\sup(A_\alpha) = \alpha$ and for every uncountable subset $X \subseteq \omega_1$ the set

$$\{\alpha \in L_1 : A_\alpha \subseteq X\}$$

is stationary (or just non-empty)."

Some well-known results

Theorem

(Jensen) $V = L \Rightarrow \diamondsuit \Rightarrow \neg\mathrm{SH}$.

Some well-known results

Theorem

(Jensen) $V = L \Rightarrow \diamondsuit \Rightarrow \neg \mathrm{SH}$.

Theorem

(Ostaszewski) $\clubsuit \Rightarrow$ *"there is an Ostaszewski space".*
$\diamondsuit \Rightarrow$ *"there is a countably compact Ostaszewski space".*

Some well-known results

Theorem

(Jensen) $V = L \Rightarrow \diamondsuit \Rightarrow \neg \mathrm{SH}$.

Theorem

(Ostaszewski) $\clubsuit \Rightarrow$ *"there is an Ostaszewski space".*
$\diamondsuit \Rightarrow$ *"there is a countably compact Ostaszewski space".*

Theorem

(Devlin) $\diamondsuit \Leftrightarrow \clubsuit + \mathrm{CH}$

Some well-known results

Theorem

(Jensen) $V = L \Rightarrow \diamondsuit \Rightarrow \neg \mathrm{SH}$.

Theorem

(Ostaszewski) $\clubsuit \Rightarrow$ *"there is an Ostaszewski space".*
$\diamondsuit \Rightarrow$ *"there is a countably compact Ostaszewski space".*

Theorem

(Devlin) $\diamondsuit \Leftrightarrow \clubsuit + \mathrm{CH}$

Theorem

(Jensen) $GCH \not\Rightarrow \diamondsuit$

Some well-known results

Theorem

(Jensen) $V = L \Rightarrow \diamondsuit \Rightarrow \neg \mathrm{SH}$.

Theorem

(Ostaszewski) $\clubsuit \Rightarrow$ *"there is an Ostaszewski space".*
$\diamondsuit \Rightarrow$ *"there is a countably compact Ostaszewski space".*

Theorem

(Devlin) $\diamondsuit \Leftrightarrow \clubsuit + \mathrm{CH}$

Theorem

(Jensen) $GCH \not\Rightarrow \diamondsuit$

Theorem

(Shelah) $\clubsuit \not\Rightarrow \diamondsuit$

Definition

For a uncountable regular cardinal κ *and stationary set* $S \subseteq \kappa$ *by* $\diamondsuit_\kappa(S)$ *we denote the sentence: "there is a sequence* $(A_\alpha)_{\alpha\in S}$ *of subsets of* κ *such that for every* $\alpha \in S$ *we have* $A_\alpha \subseteq \alpha$ *and for every subset* $X \subseteq \kappa$ *the set*

$$\{\alpha \in S : A_\alpha = X \cap \alpha\}$$

is stationary."

Definition

For a uncountable regular cardinal κ and stationary set $S \subseteq \kappa$ by $\diamondsuit_\kappa(S)$ we denote the sentence: "there is a sequence $(A_\alpha)_{\alpha \in S}$ of subsets of κ such that for every $\alpha \in S$ we have $A_\alpha \subseteq \alpha$ and for every subset $X \subseteq \kappa$ the set

$$\{\alpha \in S : A_\alpha = X \cap \alpha\}$$

is stationary."

Note that for stationary $S_1 \subseteq S_2 \subseteq \kappa$ we have $\diamondsuit_\kappa(S_1) \Rightarrow \diamondsuit_\kappa(S_2)$

Main theorem

Definition

Let κ be an uncountable cardinal. Then we denote
$T_\kappa = \{\alpha < \kappa^+ : cf(\alpha) = cf(\kappa)\}$

Main theorem

Definition

Let κ be an uncountable cardinal. Then we denote $T_\kappa = \{\alpha < \kappa^+ : cf(\alpha) = cf(\kappa)\}$

Theorem

(Shelah) *Let κ be an uncountable cardinal and let stationary set $S \subset \kappa^+$ be such that $S \cap T_\kappa = \emptyset$. Furthermore, suppose that $2^\kappa = \kappa^+$. Then $\diamondsuit_{\kappa^+}(S)$ holds.*

Other results

Theorem

(Shelah) *Let κ be a regular uncountable cardinal. Then it is consistent with* ZFC *that $2^\kappa = \kappa^+$, but $\diamondsuit_{\kappa^+}(T_\kappa)$ fails (no large cardinals required).*

Other results

Theorem

(Shelah) *Let κ be a regular uncountable cardinal. Then it is consistent with* ZFC *that* $2^\kappa = \kappa^+$*, but* $\diamondsuit_{\kappa^+}(T_\kappa)$ *fails (no large cardinals required).*

Theorem

(Zeman) *Let κ be a singular cardinal and let stationary set* $S \subseteq T_\kappa$ *be reflecting stationarily often. Furthermore, suppose that* $2^\kappa = \kappa^+$ *and* $\square^*_\kappa$ *holds. Then* $\diamondsuit_{\kappa^+}(S)$ *holds.*

Other results

Theorem

(Komjáth) *Let κ be a singular cardinal such that $cf(\kappa) > \aleph_0$ and let $T \subseteq T_\kappa$ be stationary. Suppose that $2^\kappa = \kappa^+$ and that $\diamondsuit_{\kappa^+}(T)$ fails. Then there is a function*

$$h : \{\alpha < \kappa^+ : cf(\alpha) < cf(\kappa)\} \to cf(\kappa)$$

such that for almost every (i.e. for club many) $\delta \in T$ there is a closed unbounded set $C \subseteq \delta$, such that $h|_C$ is strictly increasing.

The proof

Notation: for $A \subseteq \kappa \times \kappa^+$ we define $(A)_\alpha = \{\beta < \kappa^+ : (\alpha, \beta) \in A\}$.

The proof

Notation: for $A \subseteq \kappa \times \kappa^+$ we define
$(A)_\alpha = \{\beta < \kappa^+ : (\alpha, \beta) \in A\}$.
For every $\delta < \kappa^+$ fix an increasing decomposition
$\delta = \bigcup\{A^\delta_\alpha : \alpha < cf(\kappa)\}$ with $|A^\delta_\alpha| < \kappa$ for every $\alpha < cf(\kappa)$.

The proof

Key lemma: $\exists_{\gamma<cf(\kappa)}\exists_{(X_\alpha)_{\alpha<\kappa^+}:X_\alpha\subseteq\kappa\times\kappa^+}\forall_{Z\subseteq\kappa\times\kappa^+}$ there are stationarily many $\delta\in S$ such that $\forall_{\alpha_0<\delta}\exists_{\alpha\in(\alpha_0,\delta)}\exists_{\beta<\delta}\ \alpha,\beta\in A^\delta_\gamma$ and $Z\cap(\kappa\times\alpha)=X_\beta$.

The proof

Key lemma: $\exists_{\gamma<cf(\kappa)}\exists_{(X_\alpha)_{\alpha<\kappa^+}:X_\alpha\subseteq\kappa\times\kappa^+}\forall_{Z\subseteq\kappa\times\kappa^+}$ there are stationarily many $\delta\in S$ such that $\forall_{\alpha_0<\delta}\exists_{\alpha\in(\alpha_0,\delta)}\exists_{\beta<\delta}\ \alpha,\beta\in A^\delta_\gamma$ and $Z\cap(\kappa\times\alpha)=X_\beta$.

For as big $\xi<\kappa$ as possible we define be transfinite induction a pair (B_ξ,C_ξ) of subsets of κ^+, with C_ξ being closed unbounded, such that for every $\delta\in C_\xi\cap S$ and for

$$V^\delta_\xi=\{(\alpha,\beta)\in A^\delta_\gamma\times A^\delta_\gamma:\forall_{\eta<\xi}B_\eta\cap\alpha=(X_\beta)_\eta\}$$

either $sup\{\alpha:\exists_{\beta<\kappa}(\alpha,\beta)\in V^\delta_\xi\}<\delta$ or else $V^\delta_\xi\supsetneq V^\delta_{\xi+1}$.

The proof

Key lemma: $\exists_{\gamma<cf(\kappa)}\exists_{(X_\alpha)_{\alpha<\kappa^+}:X_\alpha\subseteq\kappa\times\kappa^+}\forall_{Z\subseteq\kappa\times\kappa^+}$ there are stationarily many $\delta\in S$ such that $\forall_{\alpha_0<\delta}\exists_{\alpha\in(\alpha_0,\delta)}\exists_{\beta<\delta}\ \alpha,\beta\in A^\delta_\gamma$ and $Z\cap(\kappa\times\alpha)=X_\beta$.

For as big $\xi<\kappa$ as possible we define be transfinite induction a pair (B_ξ,C_ξ) of subsets of κ^+, with C_ξ being closed unbounded, such that for every $\delta\in C_\xi\cap S$ and for

$$V^\delta_\xi=\{(\alpha,\beta)\in A^\delta_\gamma\times A^\delta_\gamma:\forall_{\eta<\xi}B_\eta\cap\alpha=(X_\beta)_\eta\}$$

either $sup\{\alpha:\exists_{\beta<\kappa}(\alpha,\beta)\in V^\delta_\xi\}<\delta$ or else $V^\delta_\xi\supsetneq V^\delta_{\xi+1}$.

Observe that this construction must stop below κ. Indeed, because assume that we define (B_ξ,C_ξ) for every $\xi<\kappa$.

The proof

Key lemma: $\exists_{\gamma<cf(\kappa)}\exists_{(X_\alpha)_{\alpha<\kappa^+}:X_\alpha\subseteq\kappa\times\kappa^+}\forall_{Z\subseteq\kappa\times\kappa^+}$ there are stationarily many $\delta\in S$ such that $\forall_{\alpha_0<\delta}\exists_{\alpha\in(\alpha_0,\delta)}\exists_{\beta<\delta}\ \alpha,\beta\in A^\delta_\gamma$ and $Z\cap(\kappa\times\alpha)=X_\beta$.

For as big $\xi<\kappa$ as possible we define be transfinite induction a pair (B_ξ, C_ξ) of subsets of κ^+, with C_ξ being closed unbounded, such that for every $\delta\in C_\xi\cap S$ and for

$$V^\delta_\xi=\{(\alpha,\beta)\in A^\delta_\gamma\times A^\delta_\gamma:\forall_{\eta<\xi}B_\eta\cap\alpha=(X_\beta)_\eta\}$$

either $sup\{\alpha:\exists_{\beta<\kappa}(\alpha,\beta)\in V^\delta_\xi\}<\delta$ or else $V^\delta_\xi\supsetneq V^\delta_{\xi+1}$.

Observe that this construction must stop below κ. Indeed, because assume that we define (B_ξ, C_ξ) for every $\xi<\kappa$. Take $Z=\{(\xi,\eta):\eta\in B_\xi\}$ (so $B_\xi=(Z)_\xi$), denote $C=\bigcap_{\xi<\kappa}C_\xi$ and by key lemma take $\delta\in S\cap C$ working for Z, so there are $\alpha,\beta\in A^\delta_\gamma$ with α being arbitrarily large (in δ), such that $Z\cap(\kappa\times\alpha)=X_\beta$.

The proof

Key lemma: $\exists_{\gamma<cf(\kappa)}\exists_{(X_\alpha)_{\alpha<\kappa^+}:X_\alpha\subseteq\kappa\times\kappa^+}\forall_{Z\subseteq\kappa\times\kappa^+}$ there are stationarily many $\delta\in S$ such that $\forall_{\alpha_0<\delta}\exists_{\alpha\in(\alpha_0,\delta)}\exists_{\beta<\delta}\ \alpha,\beta\in A_\gamma^\delta$ and $Z\cap(\kappa\times\alpha)=X_\beta$.

For as big $\xi<\kappa$ as possible we define be transfinite induction a pair (B_ξ, C_ξ) of subsets of κ^+, with C_ξ being closed unbounded, such that for every $\delta\in C_\xi\cap S$ and for

$$V_\xi^\delta=\{(\alpha,\beta)\in A_\gamma^\delta\times A_\gamma^\delta:\forall_{\eta<\xi}B_\eta\cap\alpha=(X_\beta)_\eta\}$$

either $sup\{\alpha:\exists_{\beta<\kappa}(\alpha,\beta)\in V_\xi^\delta\}<\delta$ or else $V_\xi^\delta\supsetneq V_{\xi+1}^\delta$.

Observe that this construction must stop below κ. Indeed, because assume that we define (B_ξ, C_ξ) for every $\xi<\kappa$. Take $Z=\{(\xi,\eta):\eta\in B_\xi\}$ (so $B_\xi=(Z)_\xi$), denote $C=\bigcap_{\xi<\kappa}C_\xi$ and by key lemma take $\delta\in S\cap C$ working for Z, so there are $\alpha,\beta\in A_\gamma^\delta$ with α being arbitrarily large (in δ), such that $Z\cap(\kappa\times\alpha)=X_\beta$. So in particular for any $\eta<\kappa$ we see

$$(X_\beta)_\eta=(Z\cap(\kappa\times\alpha))_\eta=(Z)_\eta\cap\alpha=B_\eta\cap\alpha.$$

The proof

So $(V_\xi^\delta)_{\delta<\kappa}$ is strictly decreasing, which is impossible, since those are subsets of $A_\gamma^\delta \times A_\gamma^\delta$, which is of cardinality less than κ.

The proof

So $(V_\xi^\delta)_{\delta<\kappa}$ is strictly decreasing, which is impossible, since those are subsets of $A_\gamma^\delta \times A_\gamma^\delta$, which is of cardinality less than κ.
So let $\xi_0 < \kappa$ be such that we defined (B_ξ, C_ξ) for $\xi < \xi_0$ and B_{ξ_0}, C_{ξ_0} could not be defined. Let $C = \bigcap_{\xi<\xi_0} C_\xi$ We will show that $(S_\delta)_{\delta \in S \cap C}$ defined as

$$S_\delta = \bigcup \{ (X_\beta)_{\xi_0} : \exists_{\alpha<\kappa} (\alpha, \beta) \in V_{\xi_0}^\delta \}$$

is a $\diamondsuit_{\kappa^+}(S \cap C)-$sequence.

The proof

$$S_\delta = \bigcup\{(X_\beta)_{\xi_0} : \exists_{\alpha<\kappa}(\alpha, \beta) \in V^\delta_{\xi_0}\}$$

$$V^\delta_\xi = \{(\alpha, \beta) \in A^\delta_\gamma \times A^\delta_\gamma : \forall_{\eta<\xi} B_\eta \cap \alpha = (X_\beta)_\eta\}$$

The proof

$$S_\delta = \bigcup\{(X_\beta)_{\xi_0} : \exists_{\alpha<\kappa}(\alpha,\beta) \in V^\delta_{\xi_0}\}$$

$$V^\delta_\xi = \{(\alpha,\beta) \in A^\delta_\gamma \times A^\delta_\gamma : \forall_{\eta<\xi} B_\eta \cap \alpha = (X_\beta)_\eta\}$$

Suppose that for some $B \subseteq \kappa^+$ the set $A = \{\delta \in S \cap C : S_\delta = B \cap \delta\}$ is not stationary.

The proof

$$S_\delta = \bigcup\{(X_\beta)_{\xi_0} : \exists_{\alpha<\kappa}(\alpha,\beta) \in V^\delta_{\xi_0}\}$$

$$V^\delta_\xi = \{(\alpha,\beta) \in A^\delta_\gamma \times A^\delta_\gamma : \forall_{\eta<\xi} B_\eta \cap \alpha = (X_\beta)_\eta\}$$

Suppose that for some $B \subseteq \kappa^+$ the set $A = \{\delta \in S \cap C : S_\delta = B \cap \delta\}$ is not stationary. Let $B_{\xi_0} = B$ and $C_{\xi_0} \subseteq C$ be closed unbounded in κ^+ and disjoint with A.

The proof

$$S_\delta = \bigcup \{(X_\beta)_{\xi_0} : \exists_{\alpha<\kappa}(\alpha,\beta) \in V^\delta_{\xi_0}\}$$

$$V^\delta_\xi = \{(\alpha,\beta) \in A^\delta_\gamma \times A^\delta_\gamma : \forall_{\eta<\xi} B_\eta \cap \alpha = (X_\beta)_\eta\}$$

Suppose that for some $B \subseteq \kappa^+$ the set $A = \{\delta \in S \cap C : S_\delta = B \cap \delta\}$ is not stationary. Let $B_{\xi_0} = B$ and $C_{\xi_0} \subseteq C$ be closed unbounded in κ^+ and disjoint with A. By definition of ξ_0 we can assume that there is a stationary subset $S_0 \subseteq C_{\xi_0} \cap S$ such that for $\delta \in S_0$ we have $sup\{\alpha : \exists_{\beta<\kappa}(\alpha,\beta) \in V^\delta_{\xi_0}\} = \delta$ and $V^\delta_{\xi_0} = V^\delta_{\xi_0+1}$,

The proof

$$S_\delta = \bigcup \{ (X_\beta)_{\xi_0} : \exists_{\alpha<\kappa} (\alpha,\beta) \in V^\delta_{\xi_0} \}$$

$$V^\delta_\xi = \{ (\alpha,\beta) \in A^\delta_\gamma \times A^\delta_\gamma : \forall_{\eta<\xi} B_\eta \cap \alpha = (X_\beta)_\eta \}$$

Suppose that for some $B \subseteq \kappa^+$ the set $A = \{\delta \in S \cap C : S_\delta = B \cap \delta\}$ is not stationary. Let $B_{\xi_0} = B$ and $C_{\xi_0} \subseteq C$ be closed unbounded in κ^+ and disjoint with A. By definition of ξ_0 we can assume that there is a stationary subset $S_0 \subseteq C_{\xi_0} \cap S$ such that for $\delta \in S_0$ we have $sup\{\alpha : \exists_{\beta<\kappa} (\alpha,\beta) \in V^\delta_{\xi_0}\} = \delta$ and $V^\delta_{\xi_0} = V^\delta_{\xi_0+1}$, but then we see that

$$B_{\xi_0} \cap \delta = \bigcup \{ B_{\xi_0} \cap \alpha : \exists_{\beta<\kappa} (\alpha,\beta) \in V^\delta_{\xi_0} \} =$$

$$\bigcup \{ (X_\beta)_{\xi_0} : \exists_{\alpha<\kappa} (\alpha,\beta) \in V^\delta_{\xi_0} \} = S_\delta .$$

□