Self-distributive structures and reflections to set-theoretic solutions of the Yang–Baxter equation

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Seminar Algebra **– University of Warsaw**

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This talk is essentially based on

A. Albano, M. Mazzotta, P.S.:
Reflections to set-theoretic solutions of the Yang–Baxter equation,
arXiv:2405.19105.

whose study is strongly motivated by findings in

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Quandles as pre-Lie skew braces, set-theoretic Hopf algebras & universal $\mathcal{R}$-matrices, J. Phys. A: Math. Theor. **57**, 405203 (2024).

Outline:

- Overview on *set-theoretic solutions of the YBE*.
- Focus on the structure of the *shelf/rack* and *derived solutions* of the YBE.
- Overview on *reflections (RE)* for set-theoretic solutions of the YBE.
- Description of some classes of reflections in terms of shelves/racks.

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The quantum Yang–Baxter equation

The *quantum Yang–Baxter equation* has roots in statistical mechanics and takes its name after two independent works,

- C.N. Yang: *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Phys. Rev. Lett. **19** (1967) 1312–1315.
- R.J. Baxter: *Partition function of the eight-vertex lattice model*, Ann. Physics **70** (1972) 193–228.

where a particular one-dimensional quantum mechanical many body problem was studied. The equation depends on the idea that, in some scattering situations, particles may preserve their momentum while changing their quantum internal states.

Set-theoretic solutions of the Yang–Baxter equation

G. Drinfel'd: *On some unsolved problems in quantum group theory, in: Quantum Groups*, Leningrad, 1990, in: Lecture Notes in Math. vol. **1510** (2) Springer, Berlin, (1992), 1–8.

If X is a set, a map $r : X \times X \to X \times X$ satisfying the *braid relation*

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)$$

is said to be a *set-theoretic solution*, or briefly *solution*, of the YBE.

If we consider two maps $\lambda_a, \rho_b : X \to X$ and write r as

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

for all $a, b \in X$, then r is said to be

- *left non-degenerate* if λ_a is bijective, for every $a \in X$;
- *right non-degenerate* if ρ_b is bijective, for every $b \in X$;
- *non-degenerate* if r is both left and right non-degenerate.
- *involutive* if $r^2 = \mathrm{id}_{X \times X}$.
- *idempotent* if $r^2 = r$.

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Lyubashenko solutions

If X is a set, the map $r : X \times X \to X \times X$ given by

$$r(a, b) = (f(b), g(a))$$

is a solution, where f, g are maps from X to X such that $fg = gf$.

In particular,

- r is *left (resp. right) non-degenerate* if and only if f is bijective (resp. g is bijective);
- r is *involutive* if and only if f, g are bijective and $g = f^{-1}$;
- If we fix $k \in X$ and consider $f : X \to X$ defined by $f(a) = k$, for every $a \in X$, and $g = f$, then $r(a, b) = (k, k)$ is a *degenerate* and *idempotent* solution.

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If X is a set, the map $r : X \times X \to X \times X$ given by

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Skew braces solutions

[Rump (2007) - Cedó, Jespers, Okniński (2014) - Guarnieri, Vendramin (2017)]

A triple $(B, +, \circ)$ is said to be a *skew brace* if $(B, +)$ and $(B, \circ)$ are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds, for all $a, b, c \in B$. If $(B, +)$ is abelian then B is a *brace*.

Any Jacobson radical ring is a brace. Indeed, if $(R, +, \cdot)$ is a Jacobson radical ring, then $(R, +, \circ)$ is a brace with $\circ$ is the *adjoint operation*, i.e., $\mathbf{a} \circ \mathbf{b} := \mathbf{a} + \mathbf{b} + \mathbf{a} \cdot \mathbf{b}$, for all $a, b \in R$.

Theorem [Ru07 - GuVe17]

If B is a skew brace, then the map $r_B : B \times B \to B \times B$ defined by

$$r_B(a, b) := \left(-a + a \circ b,\ (-a + a \circ b)^- \circ a \circ b\right)$$

is a non-degenerate bijective solution (with a^- the inverse of a with respect to $\circ$, for every $a \in B$). In particular,

$$r_B \text{ is involutive} \iff (B, +, \circ) \text{ is a brace.}$$

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Shelf structures

Definition

Let X be a non-empty set and $\triangleright$ a binary operation on X. Then, $(X, \triangleright)$ is said to be a *left shelf* if $\triangleright$ is *left self-distributive*, i.e., the identity

$$\forall\ a, b, c \in X \quad a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$$

holds. Moreover, a left shelf $(X, \triangleright)$ is called

1. a *left spindle* if $a \triangleright a = a$, for all $a \in X$;
2. a *left rack* if $(X, \triangleright)$ is a *left quasigroup*, i.e., the map $L_a : X \to X$ defined by
$$L_a(b) := a \triangleright b,$$
for all $b \in X$, are bijective, for all $a \in X$;
3. a *quandle* if $(X, \triangleright)$ is both a left spindle and a left rack.

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Examples

1. If X is a set and $f : X \to X$ is an idempotent map, then $(X, \triangleright)$ is a left shelf where $a \triangleright b := f(a)$. The case with $f = \mathrm{id}_X$ is called the *trivial spindle*.

2. If X is a set and $f \in Sym_X$, then $(X, \triangleright)$ is a left rack where $a \triangleright b := f(b)$. The case with $f = \mathrm{id}_X$ is called the *trivial rack*.

3. *Conjugation quandle*: If $(X, +)$ is a group, then $(X, \triangleright)$ is a left quandle where $a \triangleright b := -a + b + a$.

4. *Dihedral quandle*: Let $n \in \mathbb{N}_0$ and $X = \mathbb{Z}_n$. Then, the structure $(X, \triangleright)$ where $a \triangleright b := 2a - b (mod\ n)$, for all $a, b \in X$, is a left quandle.

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Left non-degenerate solutions and left shelves

Proposition

If $(X, \triangleright)$ is a left shelf, then the map $r_\triangleright : X \times X \to X \times X$ defined by

$$r_\triangleright(a, b) = (b,\ b \triangleright a)$$

is a *left non-degenerate* solution of *derived type*.
Conversely, if (X, r) is a left non-degenerate solution, then the structure $(X, \triangleright_r)$ is a left shelf where $\triangleright_r$ is the binary operation on X given by

$$a \triangleright_r b := \lambda_a \rho_{\lambda_b^{-1}(a)}(b).$$

- If $(X, \triangleright)$ is a *left shelf*, we call $r_\triangleright$ the *solution associated to* $(X, \triangleright)$.
- If (X, r) is a *left non-degenerate solution*, we call $(X, \triangleright_r)$ the *left shelf associated to* r. Moreover, the solution associated to $(X, \triangleright_r)$ is called the *left derived solution of* (X, r).

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Drinfel'd homomorphisms

Definition – cf. [Soloviev (2000) - Doikou (2021)]

If (X, r) and (Y, s) are solutions, we say that a map $\Phi : X \times X \to Y \times Y$ is a *Drinfel'd homomorphism* or, in short, *D-homomorphism* if

$$\Phi\, r = s\, \Phi.$$

If Φ is a bijection, we call Φ a *D-isomorphism* and we say that (X, r) and (Y, s) are *D-isomorphic (via Φ)*, and we denote it by $r \cong_D s$.

Important fact: Let (X, r) be a left non-degenerate solution, then

$$r \cong_D r_{\triangleright}$$

via the map $\Phi : X \times X \to X \times X, (a, b) \mapsto (a, \lambda_a(b))$.

Drinfel'd homomorphisms

Definition – cf. [Soloviev (2000) - Doikou (2021)]

If (X, r) and (Y, s) are solutions, we say that a map $\Phi : X \times X \to Y \times Y$ is a *Drinfel'd homomorphism* or, in short, *D-homomorphism* if

$$\Phi\, r = s\, \Phi.$$

If Φ is a bijection, we call Φ a *D-isomorphism* and we say that (X, r) and (Y, s) are *D-isomorphic (via Φ)*, and we denote it by $r \cong_D s$.

Important fact: Let (X, r) be a left non-degenerate solution, then

$$\boldsymbol{r} \cong_D \boldsymbol{r}_{\triangleright}$$

via the map $\Phi : X \times X \to X \times X, (a, b) \mapsto (a, \lambda_a(b))$.

A description of all left non-degenerate solutions

Let $(X, \triangleright)$ be a left shelf. We say that $\lambda : X \to \mathrm{Aut}(X, \triangleright), a \mapsto \lambda_a$ is a *twist* if

$$\forall\, a, b \in X \quad \lambda_a \lambda_b = \lambda_{\lambda_a(b)} \lambda_{\lambda^{-1}_{\lambda_a(b)}(\lambda_a(b) \triangleright a)}.$$

Theorem [DRS (2024)]

Let $(X, \triangleright)$ be a left shelf and $\lambda : X \to \mathrm{Sym}_X, a \mapsto \lambda_a$. Then, the map $r_\lambda : X \times X \to X \times X$ defined by

$$\forall\, a, b \in X \quad r_\lambda(a, b) = \left(\lambda_a(b),\, \lambda^{-1}_{\lambda_a(b)}(\lambda_a(b) \triangleright a)\right),$$

is a solution if and only if λ is a twist. Moreover, any left non-degenerate solution can be obtained that way.

- *idempotent l.n. solutions* $\rightsquigarrow$ $a \triangleright b = a$, for all $a, b \in X$.
- *involutive l.n. solutions* $\rightsquigarrow$ $a \triangleright b = b$, for all $a, b \in X$.
- *skew braces solutions* $\rightsquigarrow$ $a \triangleright b = -a + b + a$, for all $a, b \in X$, with $(X, +)$ an arbitrary group.

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Some consequences and observations

Corollaries

Let (X, r) be a left non-degenerate solution. Then, the following hold:

1. (X, r) is bijective if and only if $(X, \triangleright_r)$ is a rack;
2. If X is a finite set, (X, r) is right non-degenerate and bijective if and only if $(X, \triangleright_r)$ is a rack.

Remark: Let $(X, \triangleright)$ and $(Y, \blacktriangleright)$ be shelves and λ, ψ twists of $(X, \triangleright)$ and $(Y, \blacktriangleright)$, respectively. If $(X, \triangleright) \cong (Y, \blacktriangleright)$ then $(X, r_\lambda) \cong_D (Y, r_\psi)$.

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The reflection equation

Similarly to the YBE, the *reflection equation* serves as a significant tool in the theory of quantum groups and integrable systems.

It was first studied to encode the reflection on the boundary of particles in quantum field theory

I.V. Cherednik: *Factorizing particles on a half line, and root systems*, Teoret. Mat. Fiz. **61** (1) (1984) 35–44.

and to prove the integrability of quantum models with boundaries

E.K. Sklyanin: *Boundary conditions for integrable quantum systems*, J. Phys. A **21** (10) (1988) 2375–2389.

Reflections for set-theoretic solutions

The set-theoretic version of this equation was formulated in

V. Caudrelier, Q.C. Zhang: *Yang–Baxter and reflection maps from vector solitons with a boundary, Nonlinearity* **27** (6) (2014) 1081–1103.

Set-theoretic RE jointly with the YBE ensures the factorization property of the interactions of N-soliton solutions on the half-line. The interplay between solutions to the YBE and RE was deepened in

V. Caudrelier, N. Crampé, Q.C. Zhang: *Set-theoretical reflection equation: classification of reflection maps*, J. Phys. A **46** (9) (2013) 095203, 12.

Definition

Let (X, r) be a solution. A map $\kappa : X \to X$ is a *set-theoretic solution of the reflection equation*, or briefly a *reflection*, for (X, r) if it holds the identity

$$r\,(\mathrm{id}_X \times \kappa)\,r\,(\mathrm{id}_X \times \kappa) = (\mathrm{id}_X \times \kappa)\,r\,(\mathrm{id}_X \times \kappa)\,r.$$

Question: Let (X, r) be a solution. What are *all* the reflections for (X, r)?

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Some easy but essential properties

Let (X, r) be a solution and $\mathcal{K}(X, r)$ the set of all reflections for (X, r).

- If (Y, s) is a solution *equivalent* to (X, r) via a bijection $\alpha : X \to Y$, i.e, $(\alpha \times \alpha)r = s(\alpha \times \alpha)$, and $\kappa \in \mathcal{K}(X, r)$, then $\kappa_\alpha := \alpha\kappa\alpha^{-1} \in \mathcal{K}(Y, s)$.
- If (X, r) is bijective and $\kappa \in \mathcal{K}_{\text{bij}}(X, r)$, then $\kappa^{-1} \in \mathcal{K}_{\text{bij}}(X, r^{-1})$.
- $\kappa, \varphi \in \mathcal{K}(X, r)$, then, in general, $\kappa\varphi \notin \mathcal{K}(X, r)$.

Example

Let X be a set, $f, g \in \text{Sym}_X$ s.t. $fg = gf$, and consider the *Lyubashenko's solution* $r(x, y) = (f(y), g(x))$. Then, if $\kappa : X \to X$ is a map,

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Brief recap

- [De Commer (2019)] introduced a notion of braided action of a group with braiding and showed that it provides reflections.
- [Katsamaktsis (2019)] investigated solutions to the reflection equation with braces.
- [Smoktunowicz, Vendramin, Weston (2020)] proposed a first more systematic approach that makes use of ring-theoretic methods, and more generally methods coming from brace theory, to produce families of reflections in the involutive case.
- [Doikou, Smoktunowicz (2021)] investigated connections between set-theoretic Yang–Baxter and reflection equations and quantum integrable systems.
- [Lebed, Vendramin (2022)] focused on reflections for involutive non-degenerate solutions.

RE for involutive solutions

[Smoktunowicz, Vendramin, Weston (2020)] and [Lebed, Vendramin (2022)] provided reflections for *involutive solutions* which lie into two specific classes.

Definition

Let (X, r) be a solution. We say that a map $\kappa : X \to X$ is

- *λ-centralizing* if $\kappa\lambda_a = \lambda_a\kappa$, for every $a \in X$.
- *ρ-invariant* if $\rho_{\kappa(a)} = \rho_a$, for every $a \in X$.

Proposition

Let (X, r) be an involutive solution. Then,

1. [SmVeWe20] If (X, r) is left non-degenerate, any λ-centralizing map is a reflection for (X, r).
2. [LeVe22] If (X, r) is right non-degenerate, any ρ-invariant map is a reflection for (X, r).

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How to describe RE for bijective non-degenerate solutions?

Bearing in mind that a left non-degenerate solution (X, r) can be written as

$$\forall\ a, b \in X \quad r(a, b) = \left(\lambda_a(b),\ \lambda^{-1}_{\lambda_a(b)} L_{\lambda_a(b)}(a)\right),$$

with $\lambda_a \in \operatorname{Aut}(X, \triangleright_r)$, for all $a \in X$, and $L_a(b) = a \triangleright_r b$, for all $a, b \in X$, we can extend the previous results directly involving the structure $(X, \triangleright_r)$.

- This naturally suggests splitting the study of reflections into their behaviour with respect to the maps $\lambda_a \in \operatorname{Aut}(X, \triangleright_r)$ and left multiplications L_a. $\rightsquigarrow$ In the involutive case, $L_a = \mathrm{id}_X$, for every $a \in X$.

- We initially focus on reflections for solutions that are only left or right non-degenerate since they turn out to be different from each other. Indeed, a map $\kappa : X \to X$ is a reflection for (X, r) if and only if

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λ-invariant reflections for l.n.-d. solutions

Theorem

Let (X, r) be a left non-degenerate solution, $(X, \triangleright_r)$ its associated shelf, and $\kappa : X \to X$ a λ-centralizing map. Then, $\kappa \in \mathcal{K}(X, r)$ if and only if, for all $a, b \in X$, the following hold:

1. $\kappa L_{L_a(b)}(a) = L_{\kappa L_a(b)}\kappa(a)$,
2. $\kappa L_a = \kappa L_{\kappa(a)}$.

Remark

If $(X, \triangleright_r)$ is a rack, by 1. in the previous theorem, any λ-centralizing reflection κ of a left non-degenerate solution (X, r) is an endomorphism of $(X, \triangleright_r)$. However, in the general case, if $\kappa \in \mathcal{K}(X, r)$, then $\kappa \notin \mathrm{End}\,(X, \triangleright_r)$.

λ-invariant reflections for l.n.-d. solutions

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The left multiplication group of $(X, \triangleright)$ and RE

Let $(X, \triangleright)$ be a left rack. The *left multiplication group* of $(X, \triangleright)$ is the normal subgroup of $\mathrm{Aut}(X, \triangleright)$ defined by

$$\mathrm{LMlt}(X, \triangleright) := \langle L_a \mid a \in X \rangle .$$

Theorem [AMS (2024)]

Let (X, r) be a bijective left non-degenerate solution and $\kappa : X \to X$ a λ-centralizing map. Then the following hold:

1. If $\kappa \in C_{\mathrm{End}(X, \triangleright_r)}(\mathrm{LMlt}(X, \triangleright_r))$, then $\kappa \in \mathcal{K}(X, r)$;
2. If $\kappa \in \mathcal{K}_{\mathrm{inj}}(X, r)$, then $\kappa \in C_{\mathrm{End}(X, \triangleright_r)}(\mathrm{LMlt}(X, \triangleright_r))$;
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Theorem [AMS (2024)]

If (X, r) is a bijective non-degenerate solution and κ a λ-centralizing map, then

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ρ-invariant reflections for bijective r.n.-d. solutions

If (X, r) is a right non-degenerate solution, one can consider the right rack $(X, \triangleleft_r)$ where $a \triangleleft_r b := \rho_a \lambda_{\rho_b^{-1}(a)}(b)$, for all $a, b \in X$. Moreover, $(X, r_\triangleleft)$ is a solution called the *right derived solution* associated to (X, r).

Considering the *right multiplication group* of $(X, \triangleleft)$

$$\mathrm{RMlt}(X, \triangleleft) := \langle R_a \mid a \in X \rangle$$

we have the following result.

Corollary [AMS (2024)]

Let (X, r) be a bijective right non-degenerate solution, $(X, \triangleleft_r)$ its associated right rack, and $\kappa : X \to X$ a ρ-invariant map. Then, the following hold:

1. $\kappa \in \mathcal{K}(X, r) \iff \kappa \in C_{\mathrm{Map}_X}(\mathrm{RMlt}(X, \triangleleft_r))$.
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Theorem [AMS (2024)]

If (X, r) is a bijective non-degenerate solution and κ a ρ-invariant map, then

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where $R_a(b) = b \triangleleft_r a$, for all $a, b \in X$.

Remark

If $\kappa, \omega : X \to X$ are ρ-invariant reflections for a right non-degenerate solution (X, r), then $\kappa\omega \in \mathcal{K}(X, r)$ and $\kappa\omega$ is ρ-invariant.

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① The study of shelf/rack endomorphisms

The study of rack automorphisms is addressed by many authors and it can be also helpful in the study of reflections.

For instance, in [Elhamdadi, Macquarrie, Restrepo (2012)], the authors proved that $Aut(X)$ of the dihedral quandle $(X, \triangleright)$ coincides with the group $Aff(\mathbb{Z}_n)$ of affine transformations of $\mathbb{Z}_n$.

With analogous computations, we have that

$$End(X) = \{f_{b,a} : \mathbb{Z}_n \to \mathbb{Z}_n \mid f_{b,a}(x) = b + ax,\ x, a, b \in \mathbb{Z}_n\}.$$

and, in light of results obtained,

$$\kappa = f_{b,a} \in \mathcal{K}(X, r_\triangleright) \quad \Longleftrightarrow \quad \begin{cases} 2ab = 0 & \pmod{n}, \\ 2a(a-1) = 0 & \pmod{n}. \end{cases}$$

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② Other reflections for bijective n.-d. solutions

Let (X, r) be a bijective non-degenerate solution and let $\kappa : X \to X$ be a ρ-centralizing and ρ-invariant map. Then, the following holds

$$\kappa \in \mathcal{K}(X, r) \iff \kappa \text{ is } \lambda\text{-centralizing}.$$

Theorem [AMS (2024)]

Let (X, r) be a solution, $\kappa \in \mathcal{K}(X, r)$, and $\varphi, \psi : X \to X$ maps that are λ, ρ-centralizing and λ, ρ-invariant. Then,

$$\omega := \varphi\kappa\psi \in \mathcal{K}(X, r).$$

If (X, r) is an *involutive solution*, then, for the maps φ, ψ, the assumptions of being λ-invariant and ρ-centralizing are redundant. Hence, the result contained in [Lebed, Vendramin (2022)] is a special case of the previous theorem.

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③ Other reflections for bijective n.-d. solutions: Examples

1. There exists a skew brace $(B, +, \circ)$ with $(B, +) \simeq D_8$ and $(B, \circ) \simeq C_8$ such that the associated solution (B, r) has **288** reflections. Among these:
   - **256** are only λ-centralizing;
   - **16** are only ρ-invariant;
   - **16** are of both type.

2. There exists a skew brace $(B, +, \circ)$ with $(B, +) \simeq D_8$ and $(B, \circ) \simeq C_8$ such that the associated solution (B, r) has **128** reflections. Among these, we have:
   - $\kappa_1 = 21222211$ is a λ-centralizing reflection which is not ρ-invariant;
   - $\kappa_2 = 11346578$ is ρ-invariant reflection which is not λ-centralizing;
   - $\kappa_3 = 21436578$ is a λ-centralizing and ρ-invariant reflection;
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④ Other aspects of the study: the parametric case

Let X, Y be non-empty sets, $Y \subseteq X$, and $z_{i,j} \in Y$ with $i, j \in \mathbb{Z}^+$.

A pair $(X, R^{z_{ij}})$ is a *solution of the parametric set-theoretic YBE* if

$$R_{12}^{z_{12}}\, R_{13}^{z_{13}}\, R_{23}^{z_{23}} = R_{23}^{z_{23}}\, R_{13}^{z_{13}}\, R_{12}^{z_{12}}$$

where, in the notation, z_{ij} denotes the dependence on (z_i, z_j).

Definition [Doikou (2024)]

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Some results already obtained on reflections allows for starting a systematic study of them for the parametric case.

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A. Doikou, M. Mazzotta, P.S.: *Parametric reflection maps: an algebraic approach*, arXiv:2412.15839.

Thank you!

Dziękuję!

Grazie!

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