Rosenthal compacta and Lexicographic products

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This presentation is based on the paper by Antonio Avilés and Stevo Todorcevic, Lexicographic products as compact spaces of the first Baire class, Topology Appl. 267 (2019), 106871.

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Rosenthal Compacta

Definition

A real-valued function $f : X \to \mathbb{R}$ on a metrizable space X is of the first Baire class if f is a pointwise limit of a sequence of continuous functions on space X.

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Theorem (Baire)

If X is completely metrizable, the following conditions are equivalent for a function $f : X \to \mathbb{R}$
(1) f is of the first Baire class,
(2) $f^{-1}(U)$ is an F_σ-set in X for every open $U \subseteq \mathbb{R}$
(3) $f \restriction_K$ has a point of continuity for every non-empty closed $K \subseteq X$

Rosenthal Compacta

Definition

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Fact

A compact Hausdorff space can always be represented as a compact set of functions $f : X \to R$ on a certain set X in pointwise topology.

Rosenthal Compacta

Examples

The Helly space of all nondecreasing functions $f : [0,1] \to [0,1]$ (with topolgy induced from the product $[0,1]^{[0,1]}$) and split interval $([0,1] \times \{0,1\}, <_{lex})$ with order topology are Rosenthal compact spaces.

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Theorem (Odell, Rosenthal)

For a separable Banach space X, the unit ball $B_{X^{**}}$ (in the second dual space) equipped with the weak topology is a separable Rosenthal compactum if and only if X contains no subspace isomorphic to ℓ_1*

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- K belongs to the class $\mathcal{CD}$ if K is homeomorphic to a compact set of functions $f : X \to \mathbb{R}$ with countably many discontinuities on a compact metric space X

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We have that $\mathcal{CD} \subseteq \mathcal{RK} \subseteq \mathcal{R}$

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Theorem (Pol)

$\mathcal{RK} \subsetneq \mathcal{R}$

Rosenthal Compacta

Definition

Space K belongs to class $\mathcal{RK}_0$ if K can be represented as a compact set of functions $f : X \to \mathbb{R}$ of the first Baire class on a compact metric space X, which is the closure of a (countable) set of contionous functions on X.

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Fact

A compact space K embeds in an element of the class $\mathcal{RK}_0$ if and only if it is homeomorphic to weak compact subset of E^{**} for some separable Banach space space E that does not contain ℓ_1*

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Mappings onto metric spaces with small fibers

Definition

Given class $\mathfrak{C}$ of compact spaces, we say that a compact space K is a $\mathfrak{C}$-to-one preimage of a metric space if there exists a continuous function $f : K \to M$ onto a metric space M such that $f^{-1}(x) \in \mathfrak{C}$ for every $x \in M$.

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Definition

A compact space is Corson if it can be represented as a compact $K \subseteq \mathbb{R}^I$, such that for every $x, y \in K$ the set $\{i \in I : x_i \neq y_i\}$ is at most countable, equivalently if it can be represented as a compact K, such that for every $x \in K$ set $supp(x) = \{i \in I : x_i \neq 0\}$ is at most countable.

Mappings onto metric spaces with small fibers

Proposition

If K is a compact set of functions with countably many discontinuities on a Polish space, then K is a Corson-to-one preimage of a metric space.

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If K is a compact set of functions with countably many discontinuities on a Polish space, then K is a Corson-to-one preimage of a metric space.

Proof.

Let K be a pointwise compact set of functions $f : X \to \mathbb{R}$, with X a Polish space. Fix a countable dense set $D \subseteq X$. The restriction map $r : K \to \mathbb{R}^D$ gives a continuous map into a metric space. If $r(f) = r(g)$, then f and g coincide on the points of common continuity of f and g, since they coincide on D. Thus, all functions in a given fiber $F = r^{-1}(r(f))$ coincide with f in all but countably many points. This implies, that each fiber is a Corson compactum. □

Mappings onto metric spaces with small fibers

Lemma

Let L_1 and L_2 be two complete linear orders, and $L_1 \times L_2$ its lexicographic product, endowed with the order topology. If L_1 is uncountable and $f : L_1 \times L_2 \to M$ is a continuous function onto a metric space, then there exists $x \in M$ such that $f^{-1}(x)$ contains a copy of L_2.

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Proof.

Since M is a compact metric space, it has a countable basis of open F_σ sets. Taking preimages, there is a countable family $\mathcal{F}$ of open F_σ sets in $L_1 \times L_2$, such that if $f(x) \neq f(y)$, then x and y are separated by elements of $\mathcal{F}$. Since open intervals (a, b) form a basis for the topolgy of $L_1 \times L_2$, there is also a countable family $\{(a_n, b_n) : n \in \omega\}$ such that if $f(x) \neq f(y)$, then x and y are separated by these intervals. Since L_1 is uncountable, there exists $t \in L_1$ which is not the first coordinate of any a_n or b_n, then $f(t, s) = f(t, s')$ for all $s, s' \in L_2$. We have $L_2 \subseteq f^{-1}(f(t, s))$ for any $s \in L_2$. □

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Linearly ordered Corson compactum is metrizable.

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Corollary

Let L_1 and L_2 be two complete linear orders, and $L_1 \times L_2$ its lexicographic product, endowed with order topology. If L_1 is uncountable and L_2 is not metrizable, then $L_1 \times L_2$ is not a Corson-to-one preimage of a metric space.

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Let L_1 and L_2 be two complete linear orders, and $L_1 \times L_2$ its lexicographic product, endowed with order topology. If L_1 is uncountable and L_2 is not metrizable, then $L_1 \times L_2$ is not a Corson-to-one preimage of a metric space.

Proof.

Assume $L_1 \times L_2$ is a Corson-to-one preimage of a metric space M for $f : L_1 \times L_2 \to M$. By previous Lemma we would have an $x \in M$, such that $L_2 \subseteq f^{-1}(x)$. Subspace $f^{-1}(x)$ is metrizable while L_2 is not. Contradiction. □

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Take $L_1 = ([0,1], <)$ and $L_2 = ([0,1] \times \{0,1\}, <_{lex})$ □

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Space $([0,1]^2 \times \{0,1\}, \tau_{\prec})$ *is not Corson-to-one preimage of metric compacta.*

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Take $L_1 = ([0,1], <)$ and $L_2 = ([0,1] \times \{0,1\}, <_{lex})$ □

Now we will prove, that $([0,1]^2 \times \{0,1\}, \tau_{\prec}) \in \mathcal{RK}$