# Reachability
# in
# Petri Nets

Wojciech Czerwiński

# Plan

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- basic notions and problem

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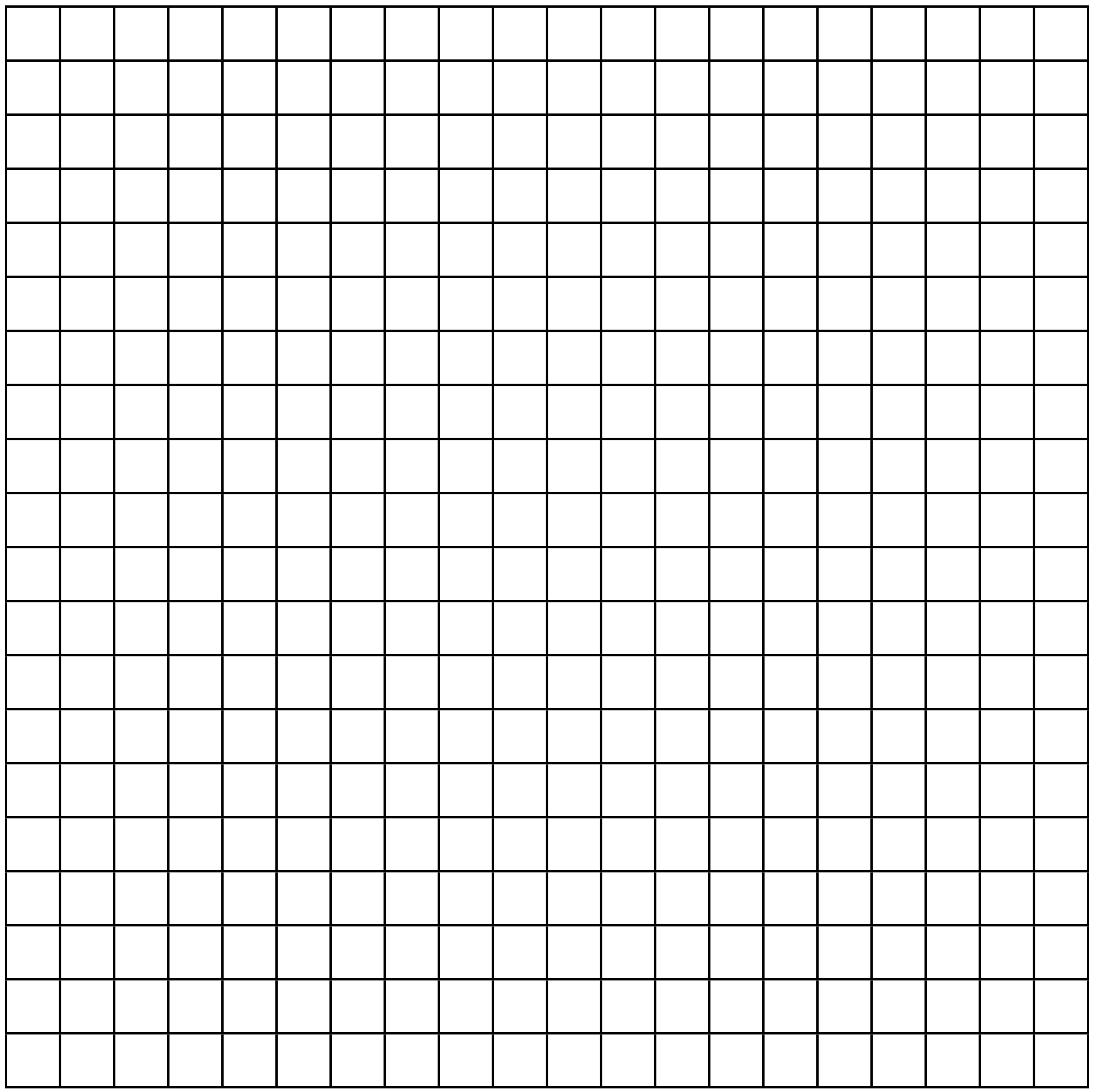

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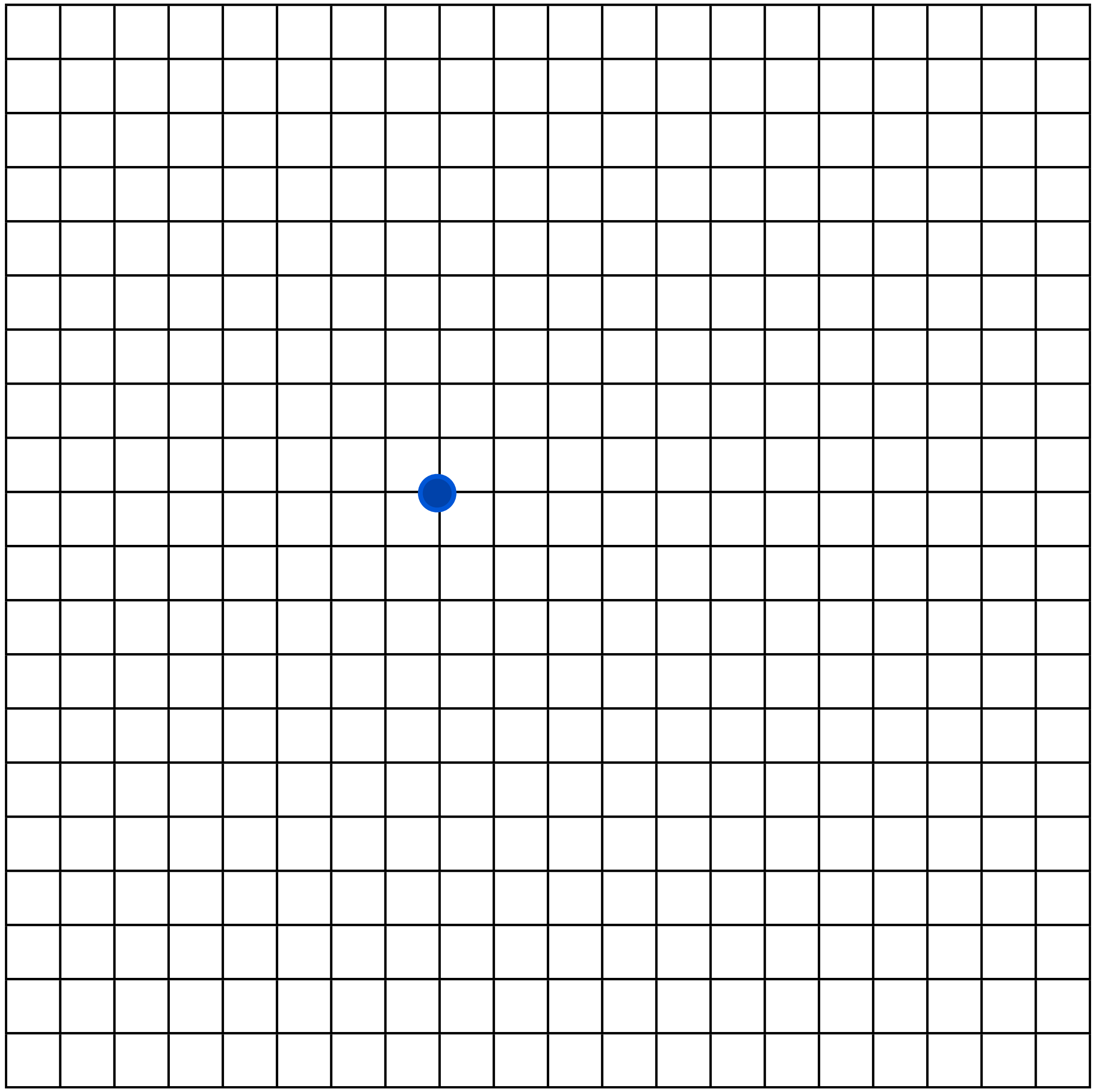

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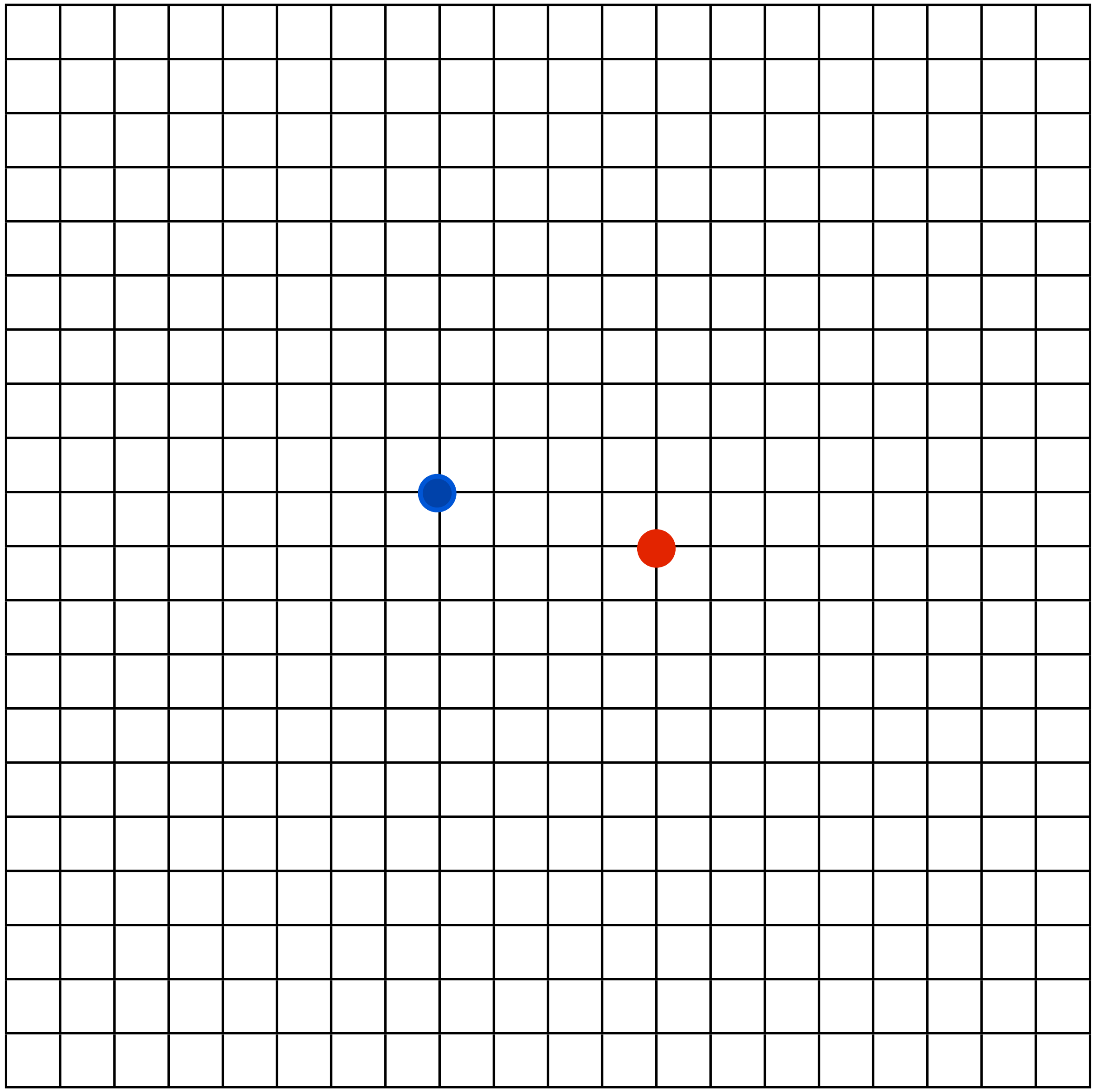

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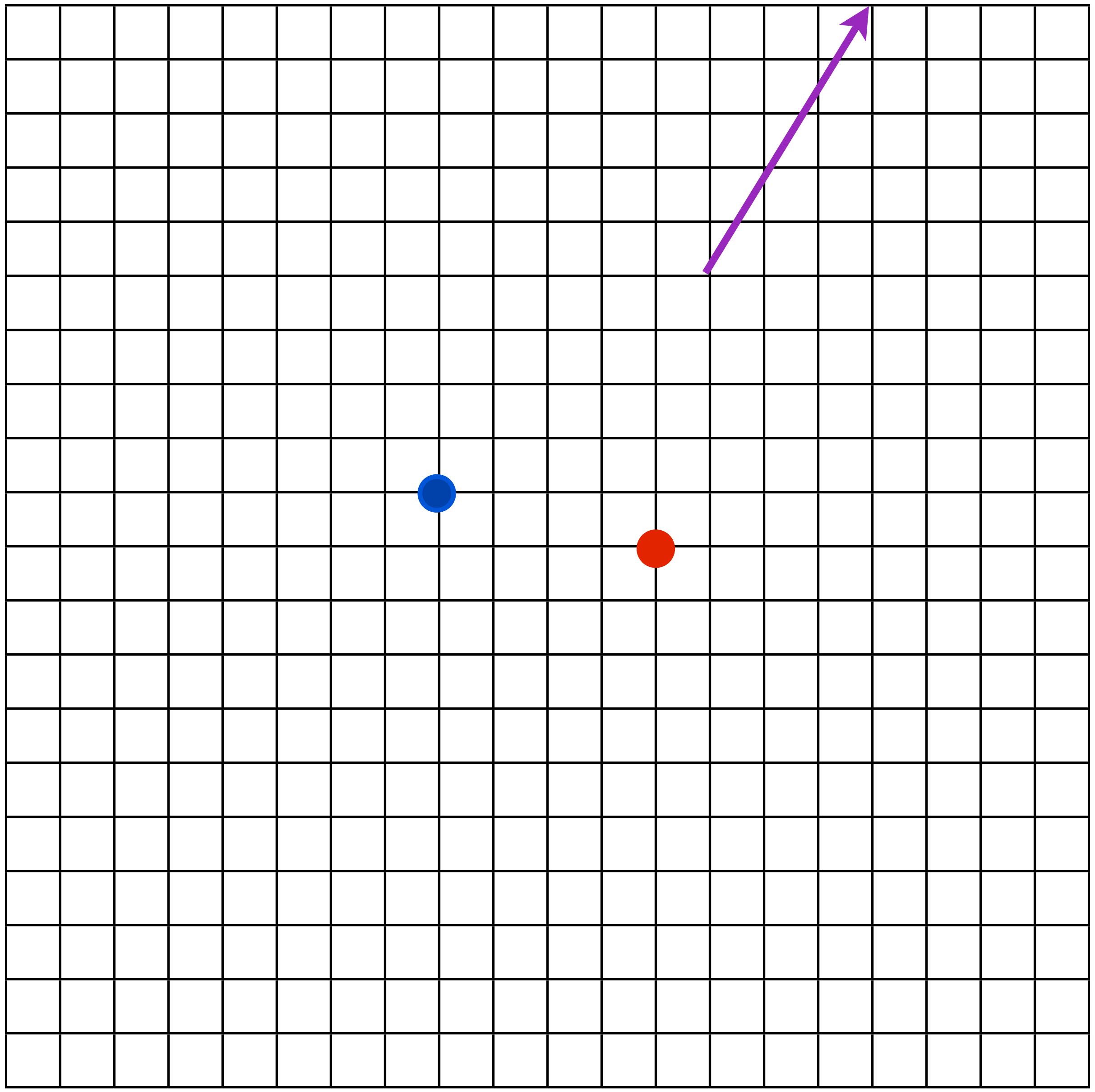

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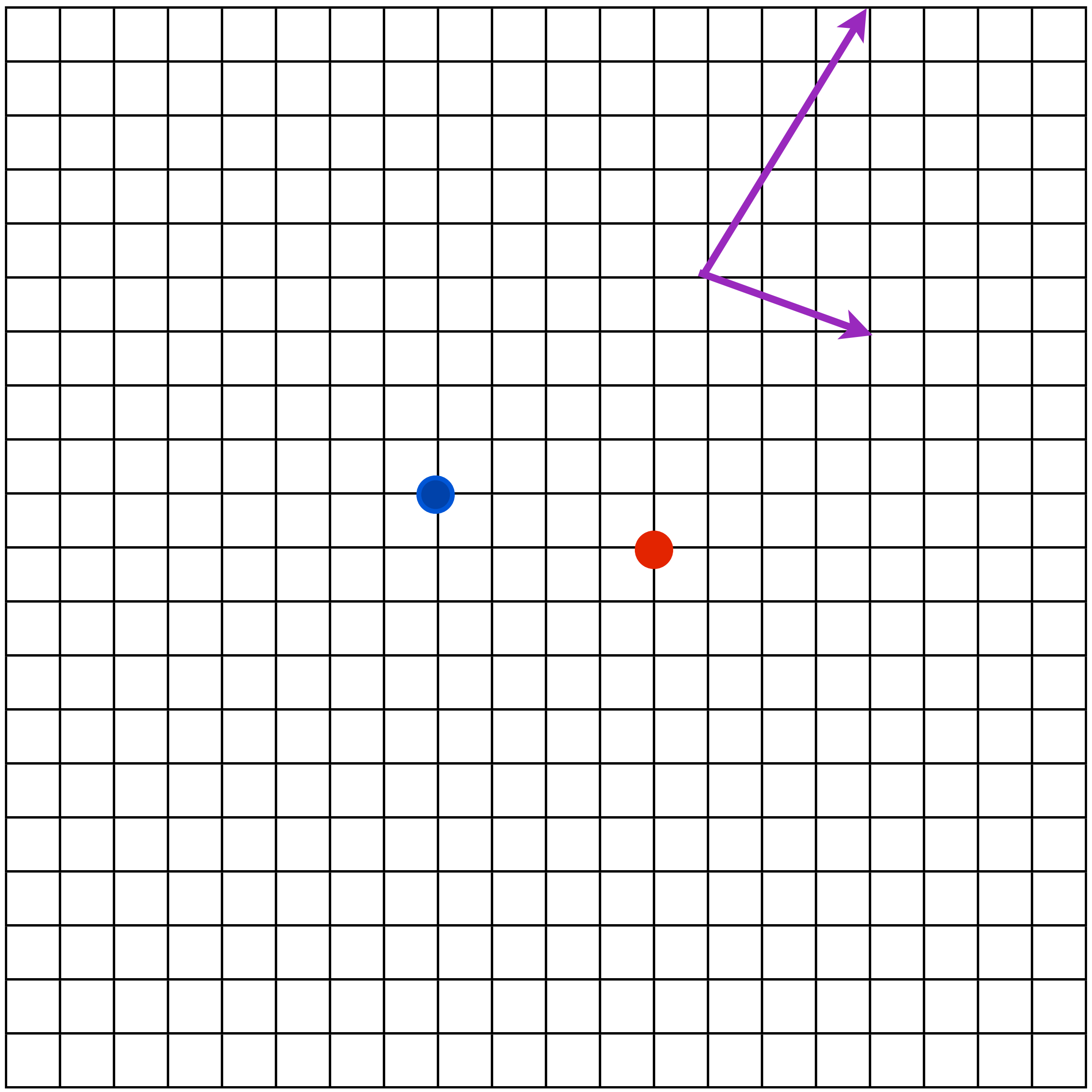

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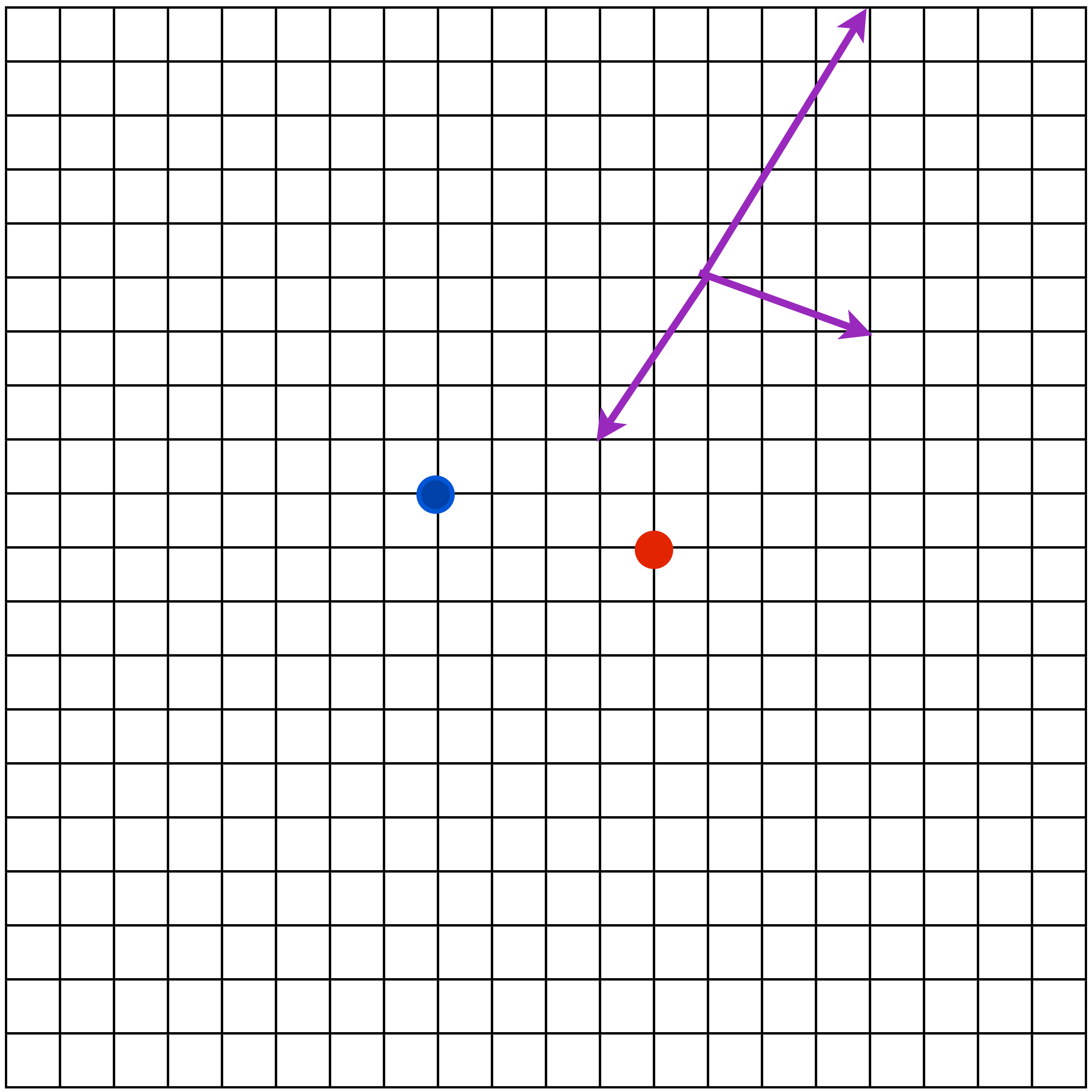

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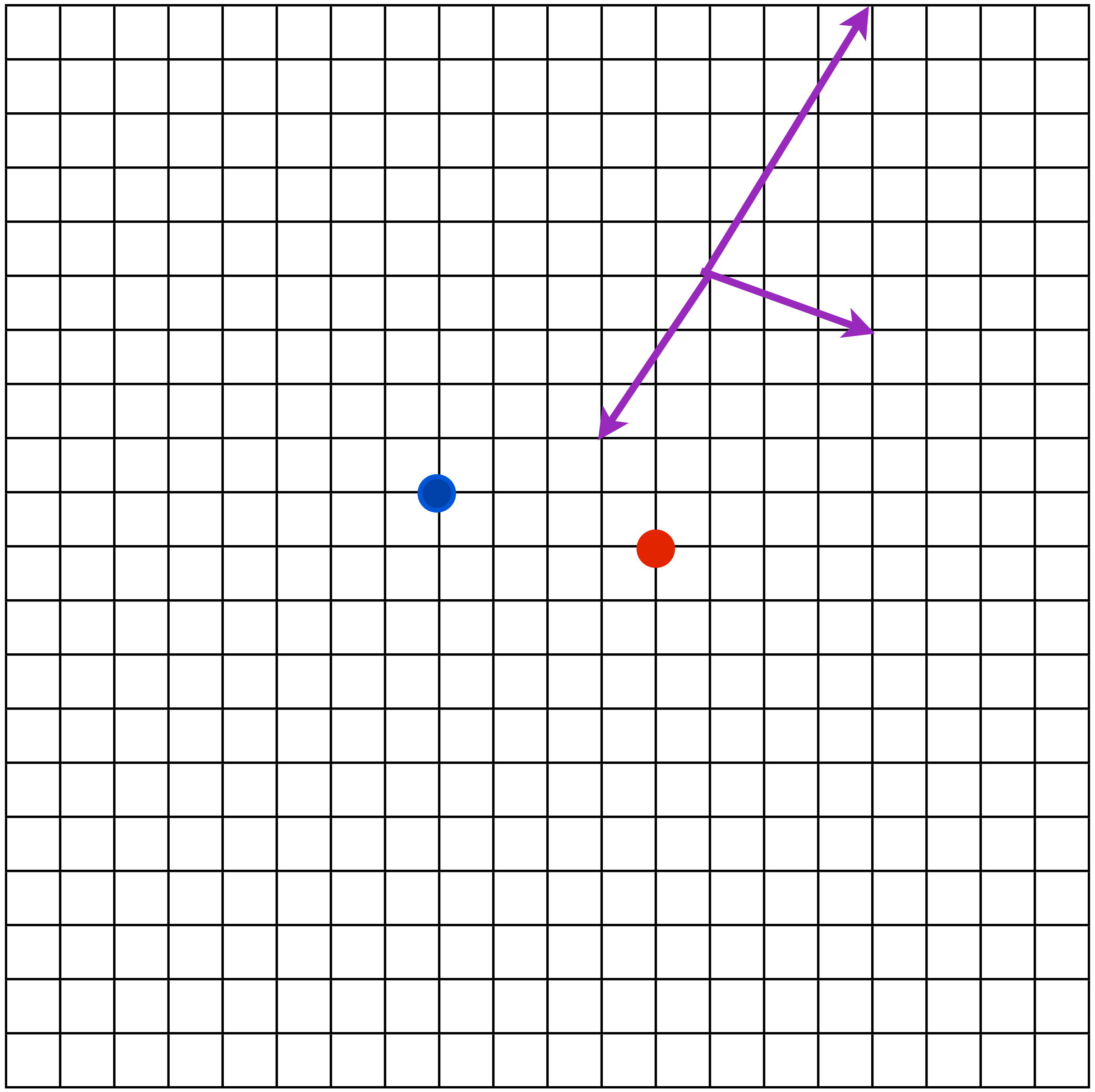

Can I reach red from blue using violet vectors?

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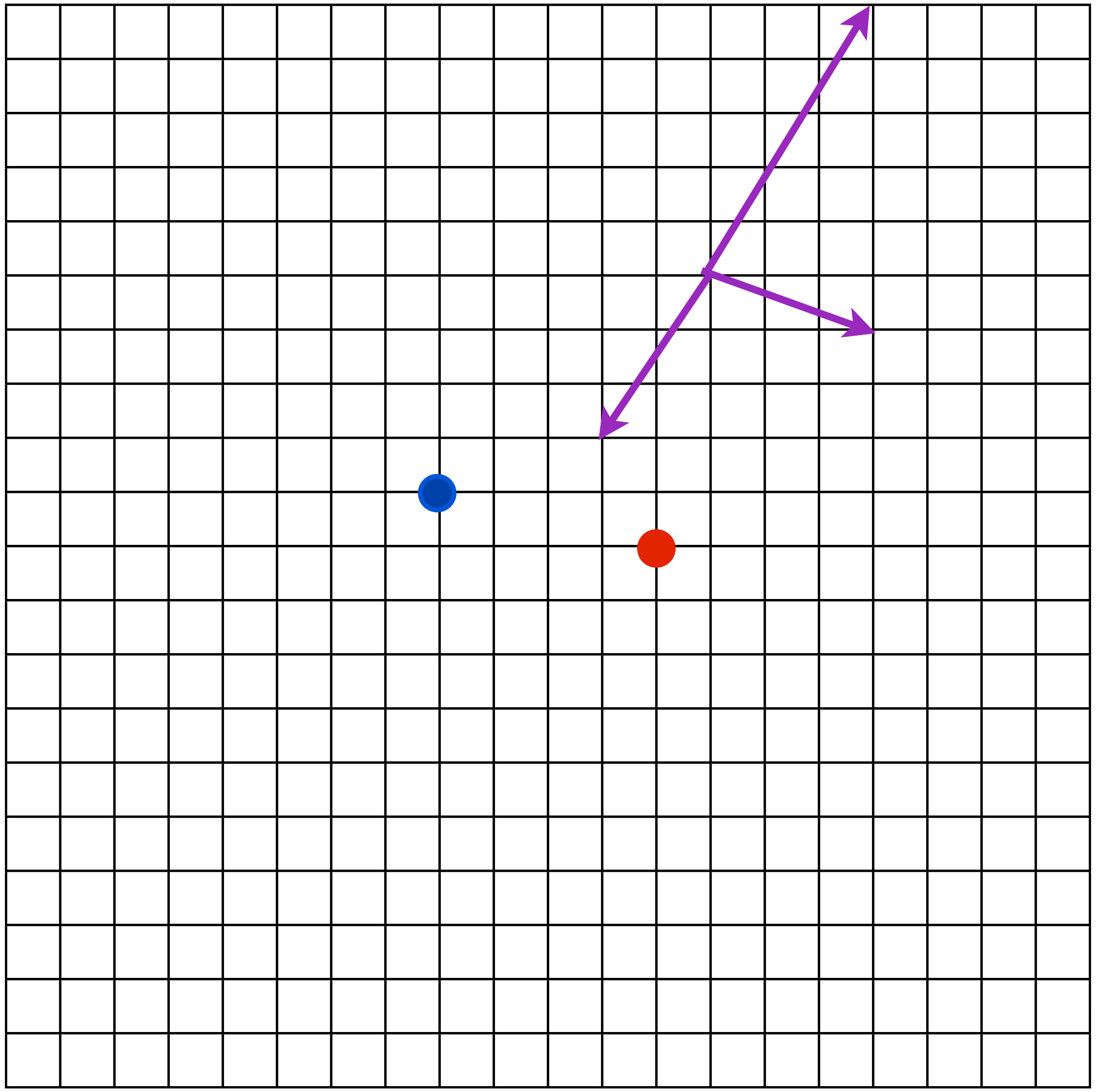

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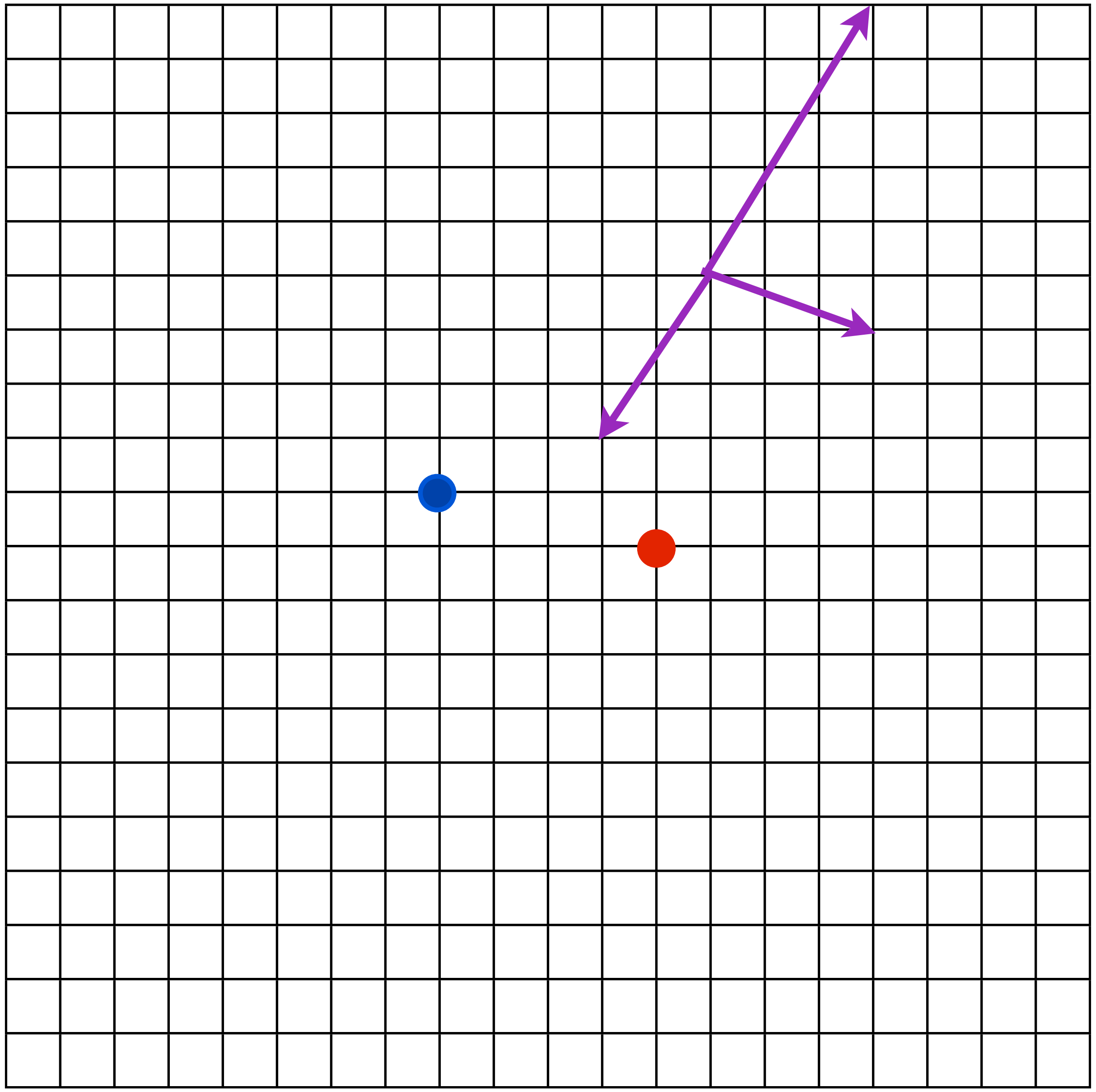

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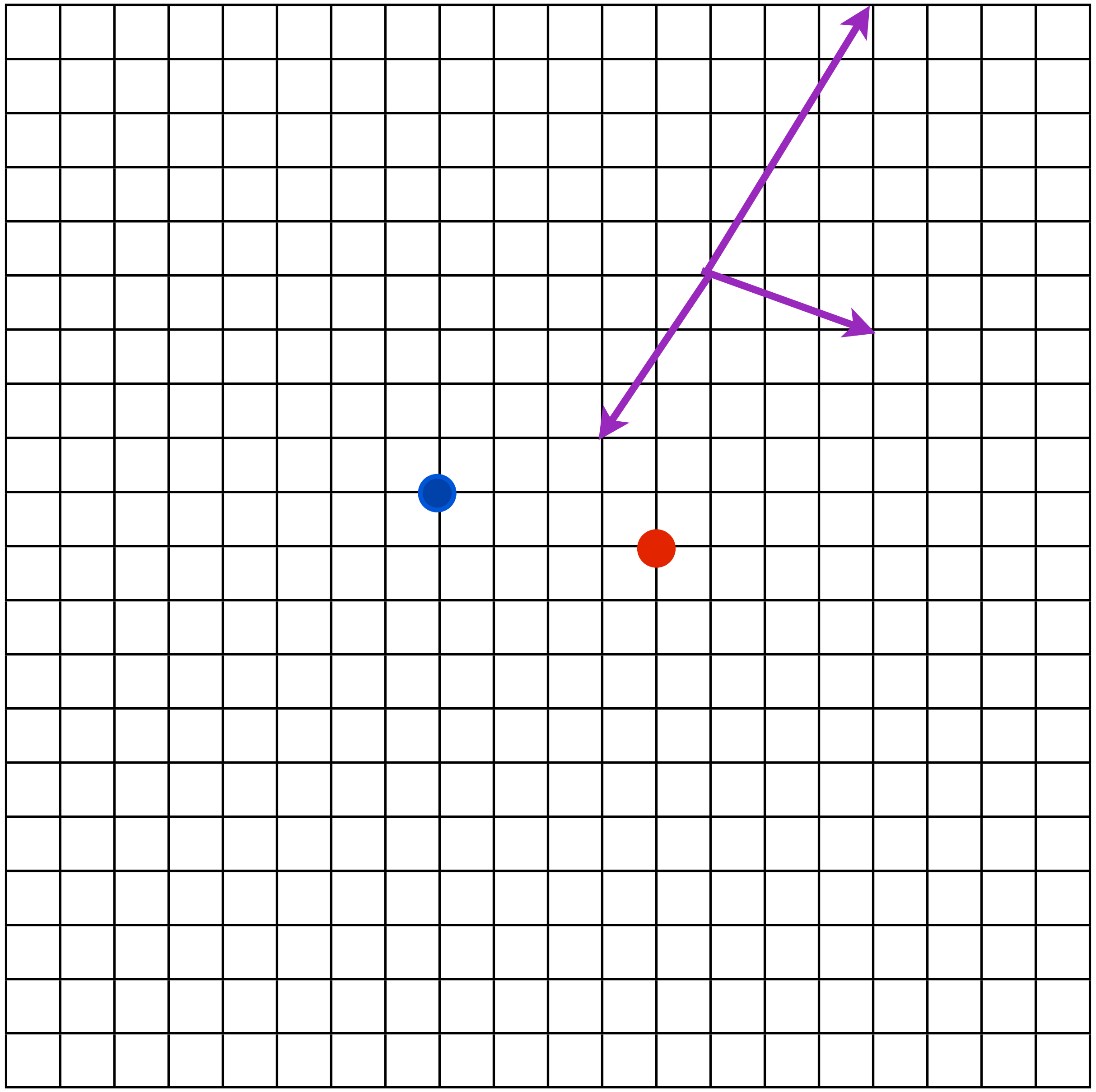

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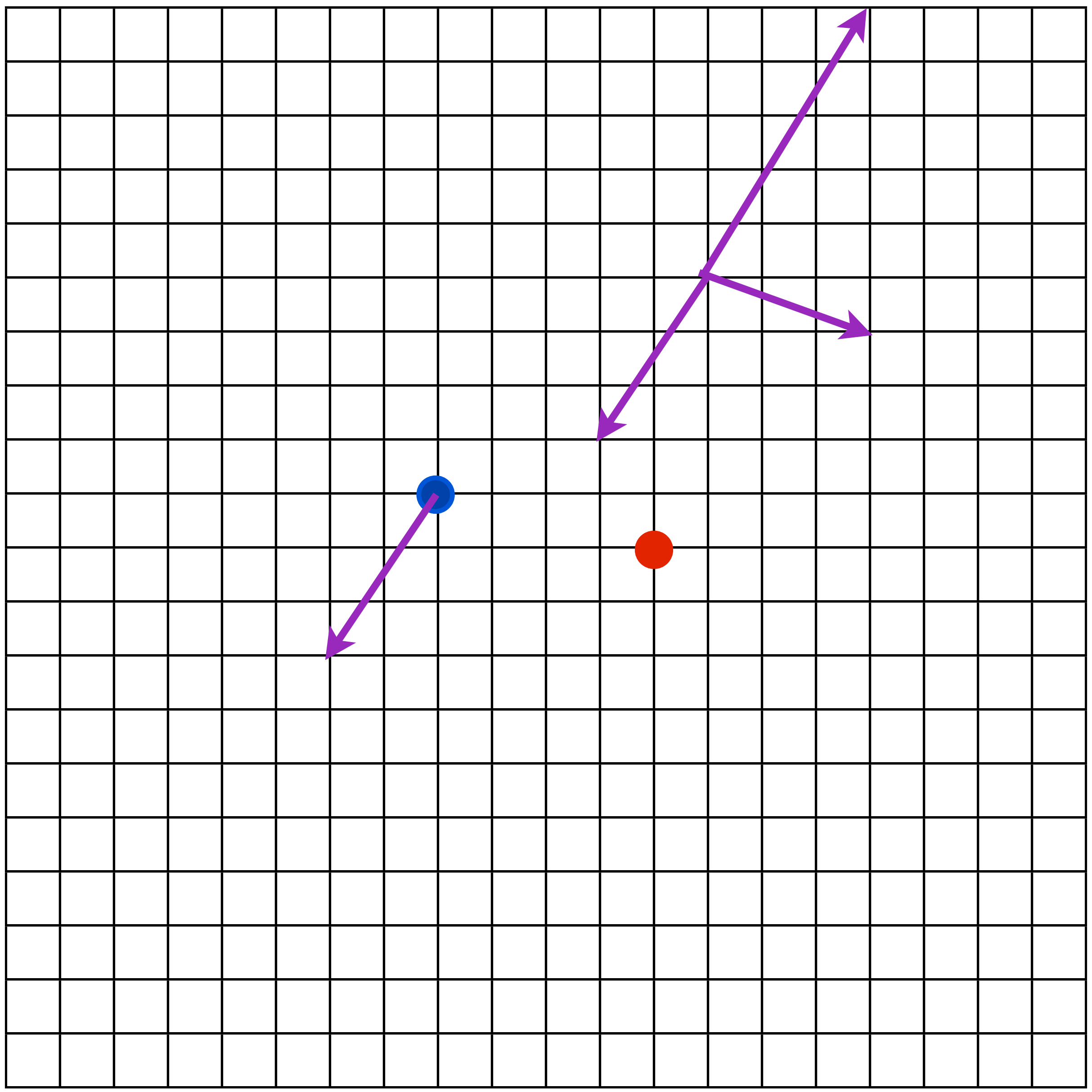

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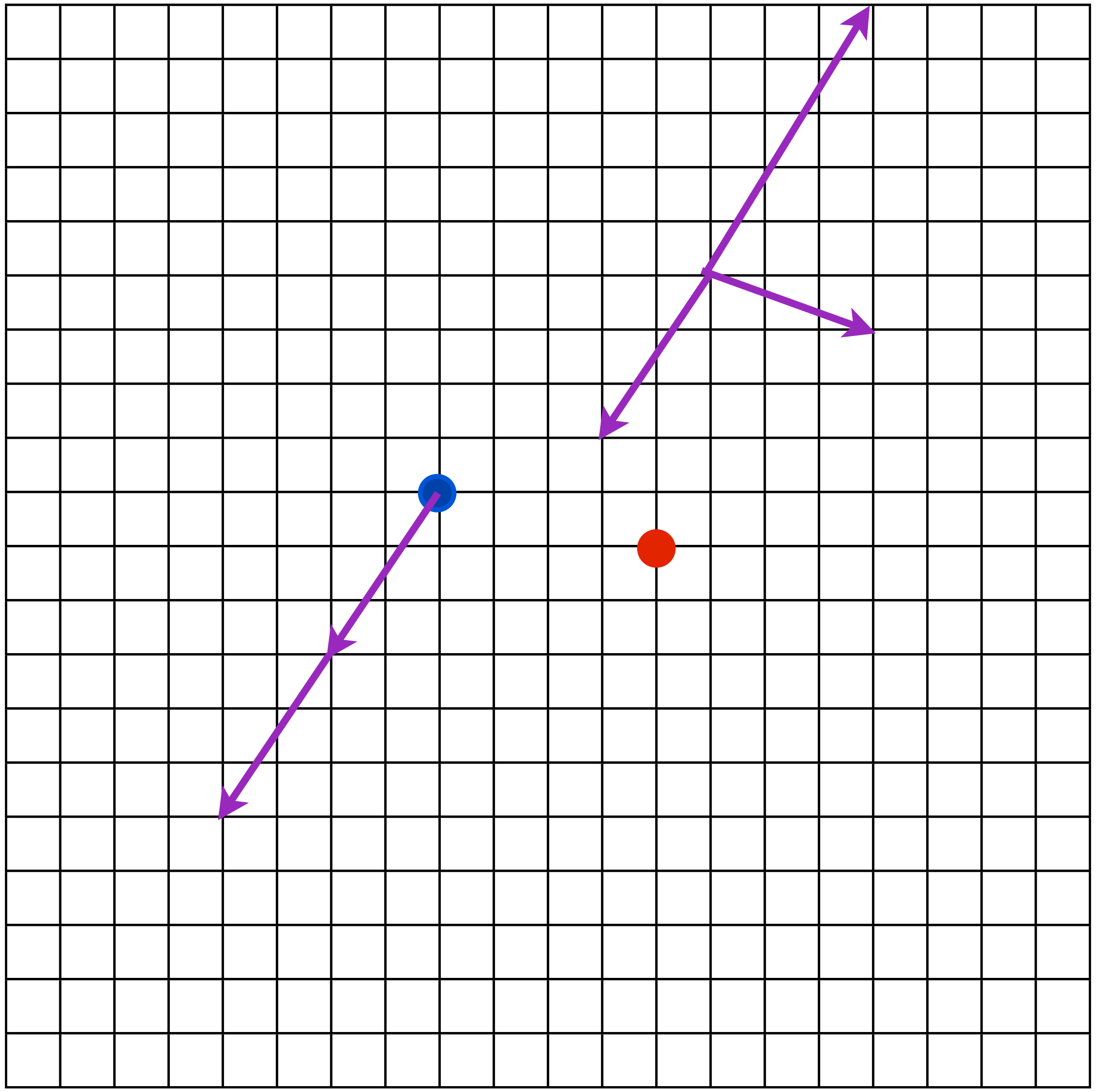

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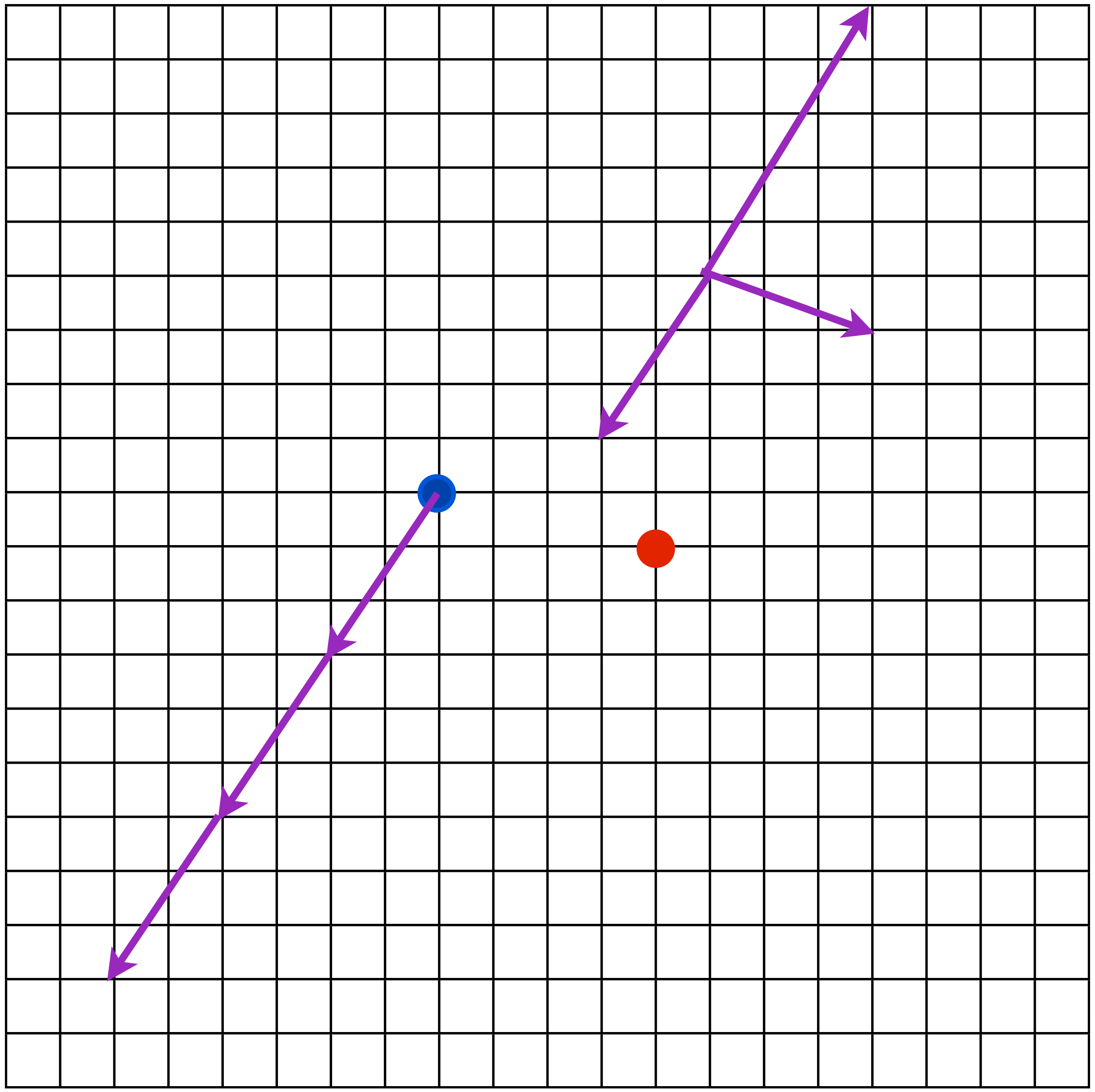

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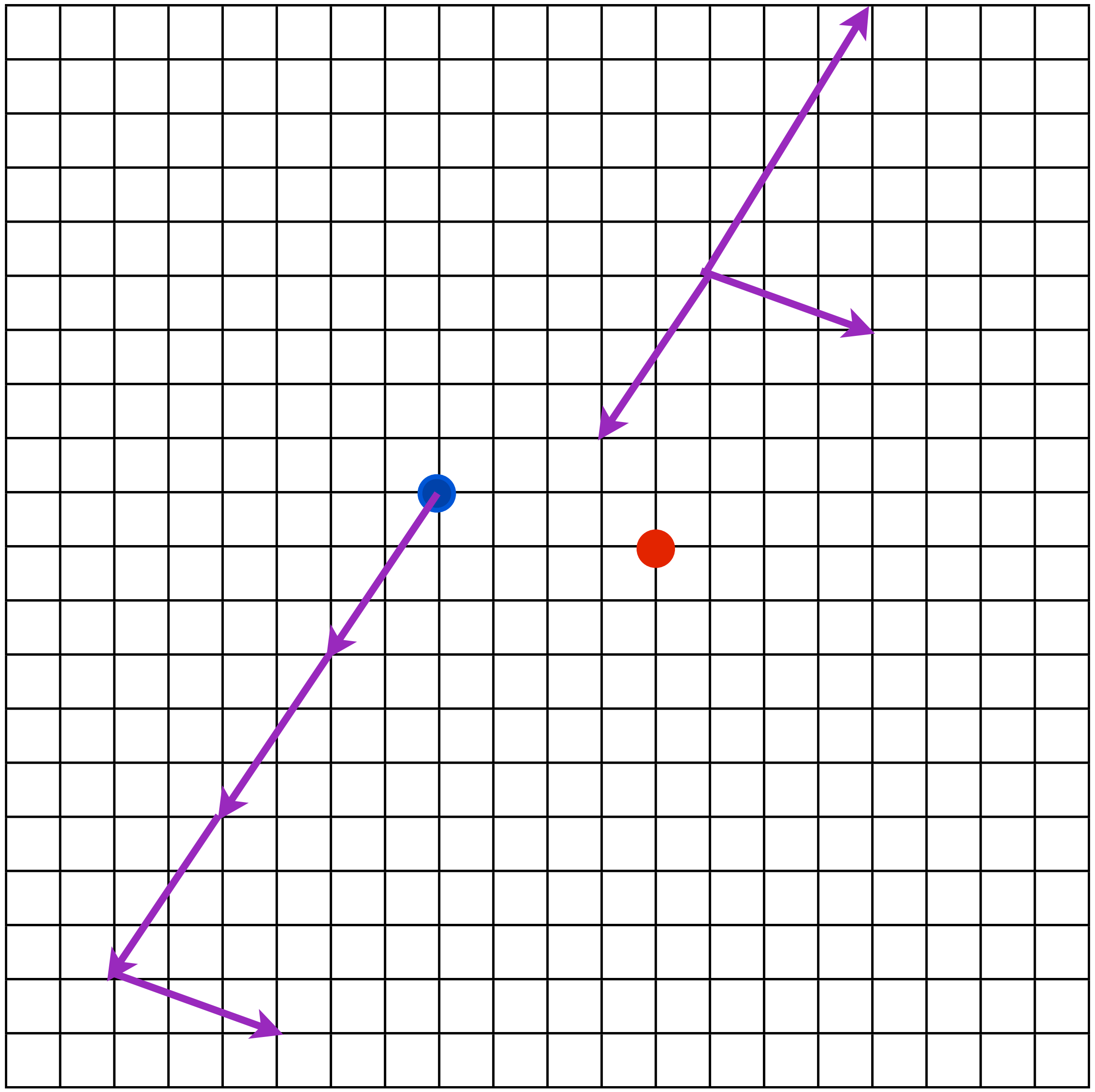

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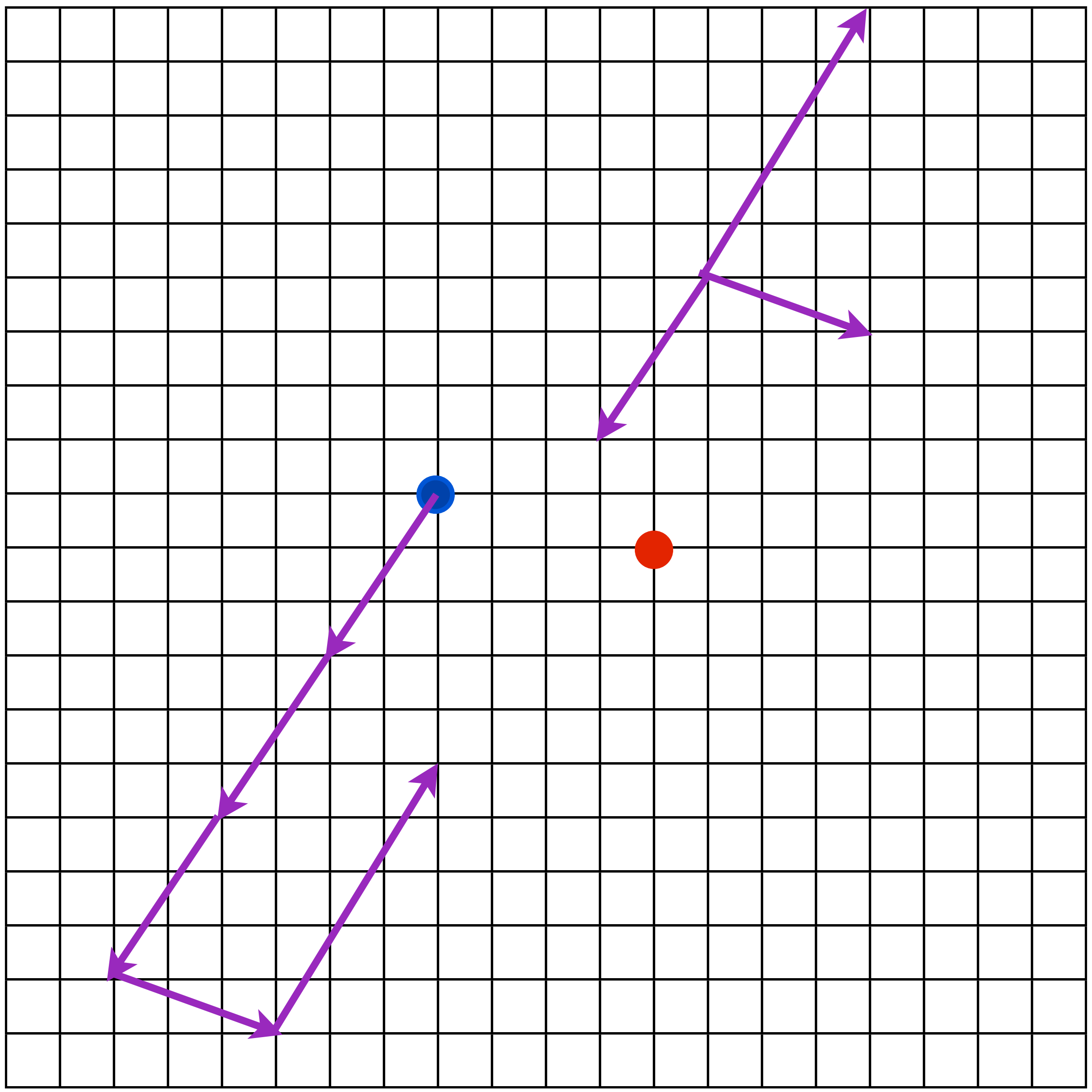

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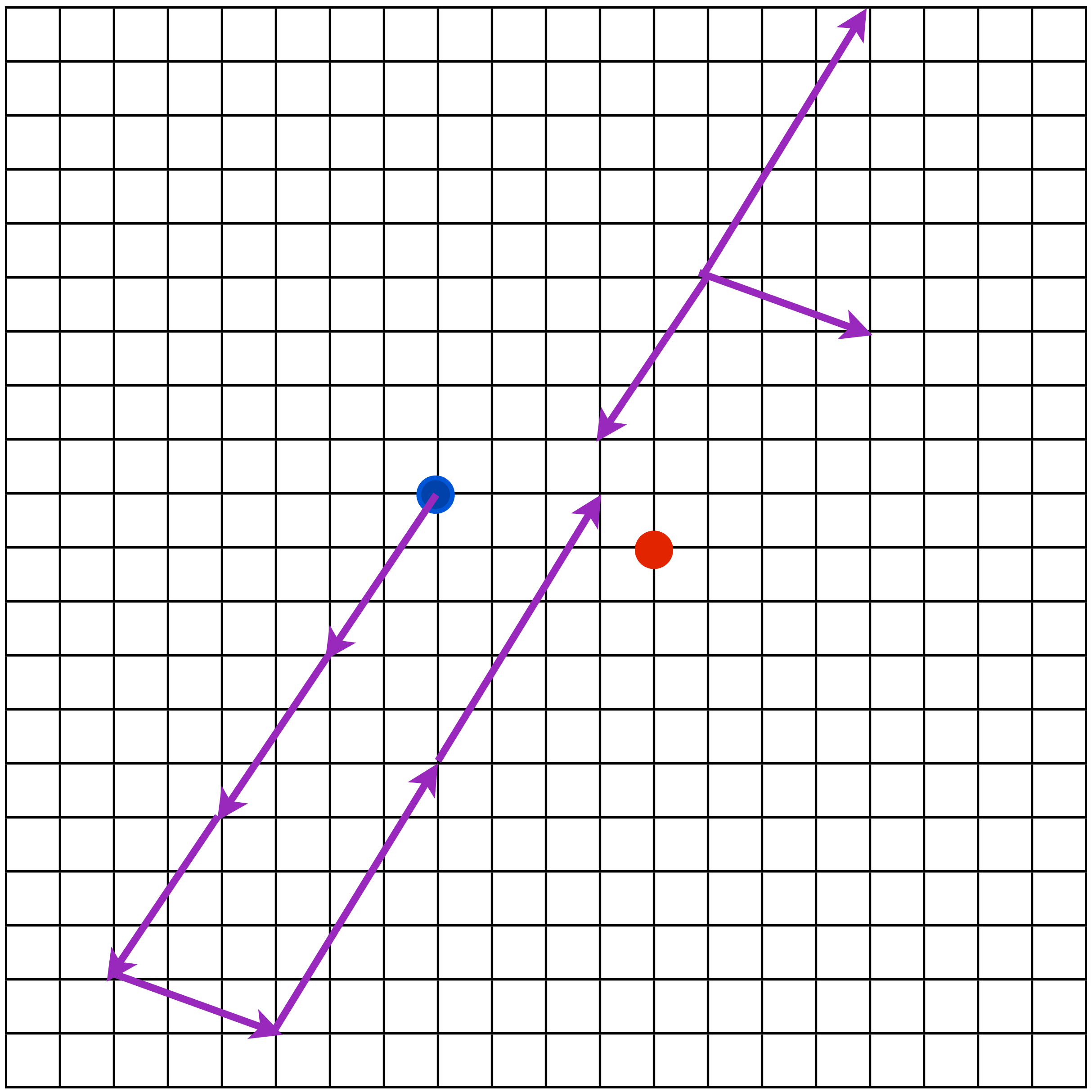

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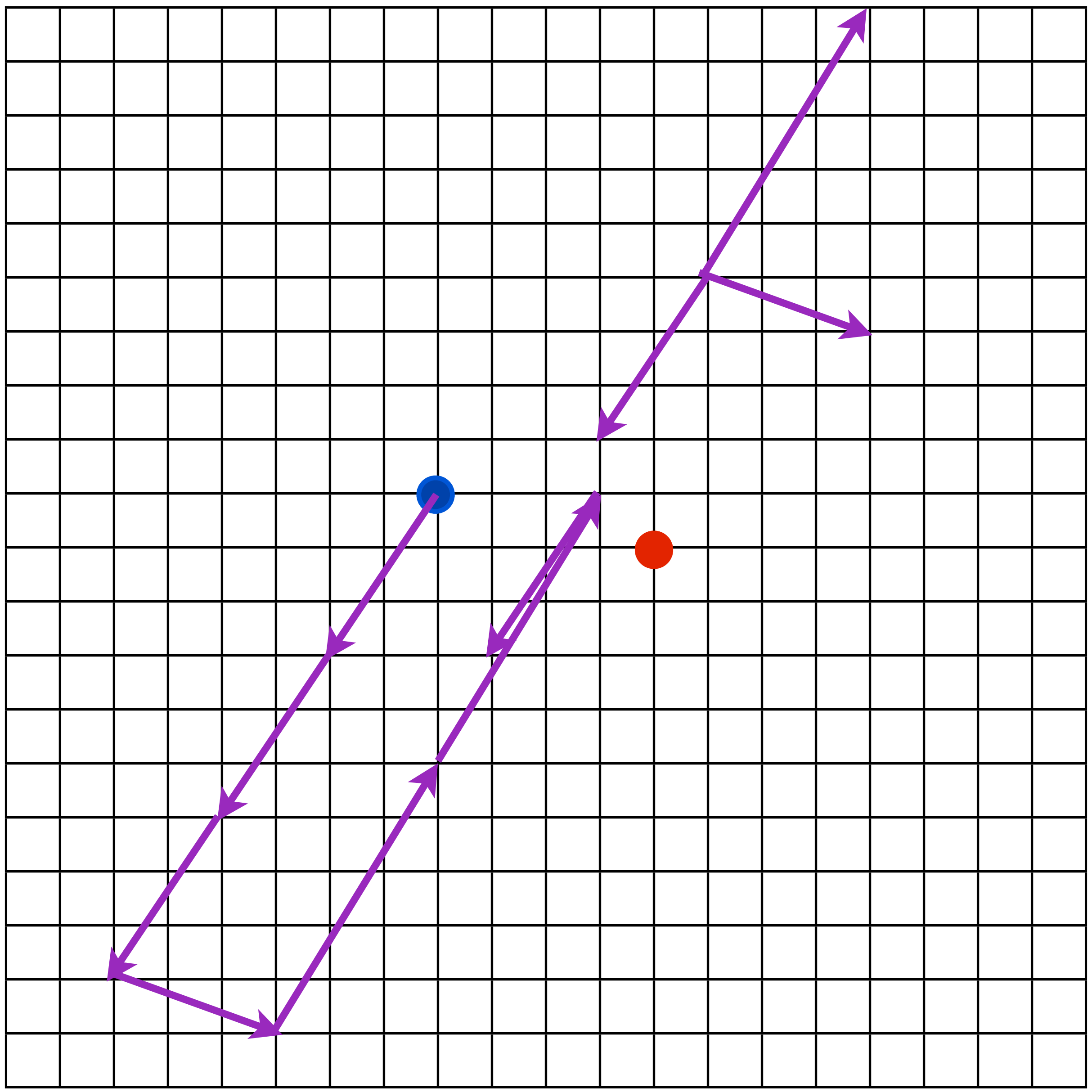

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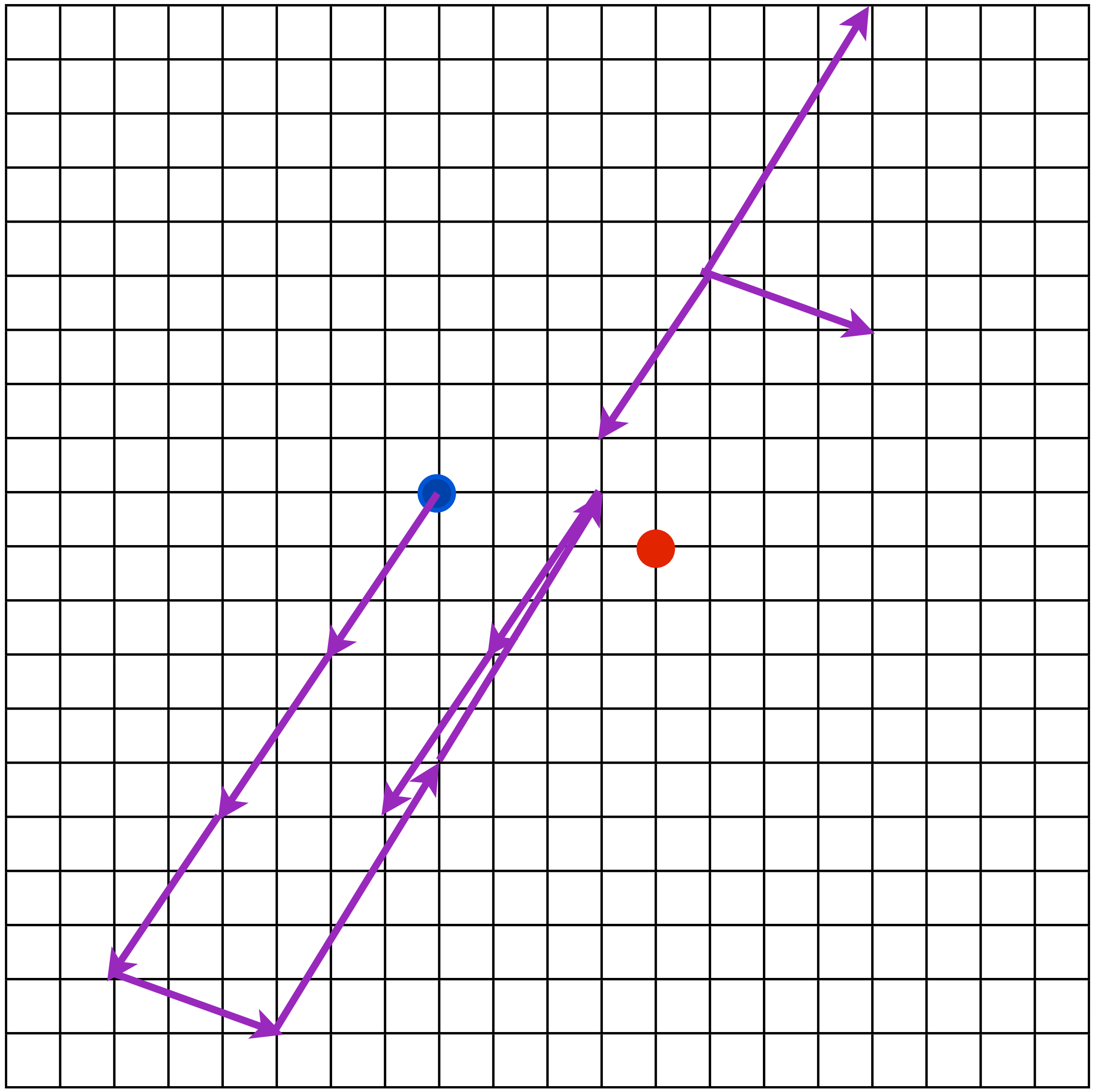

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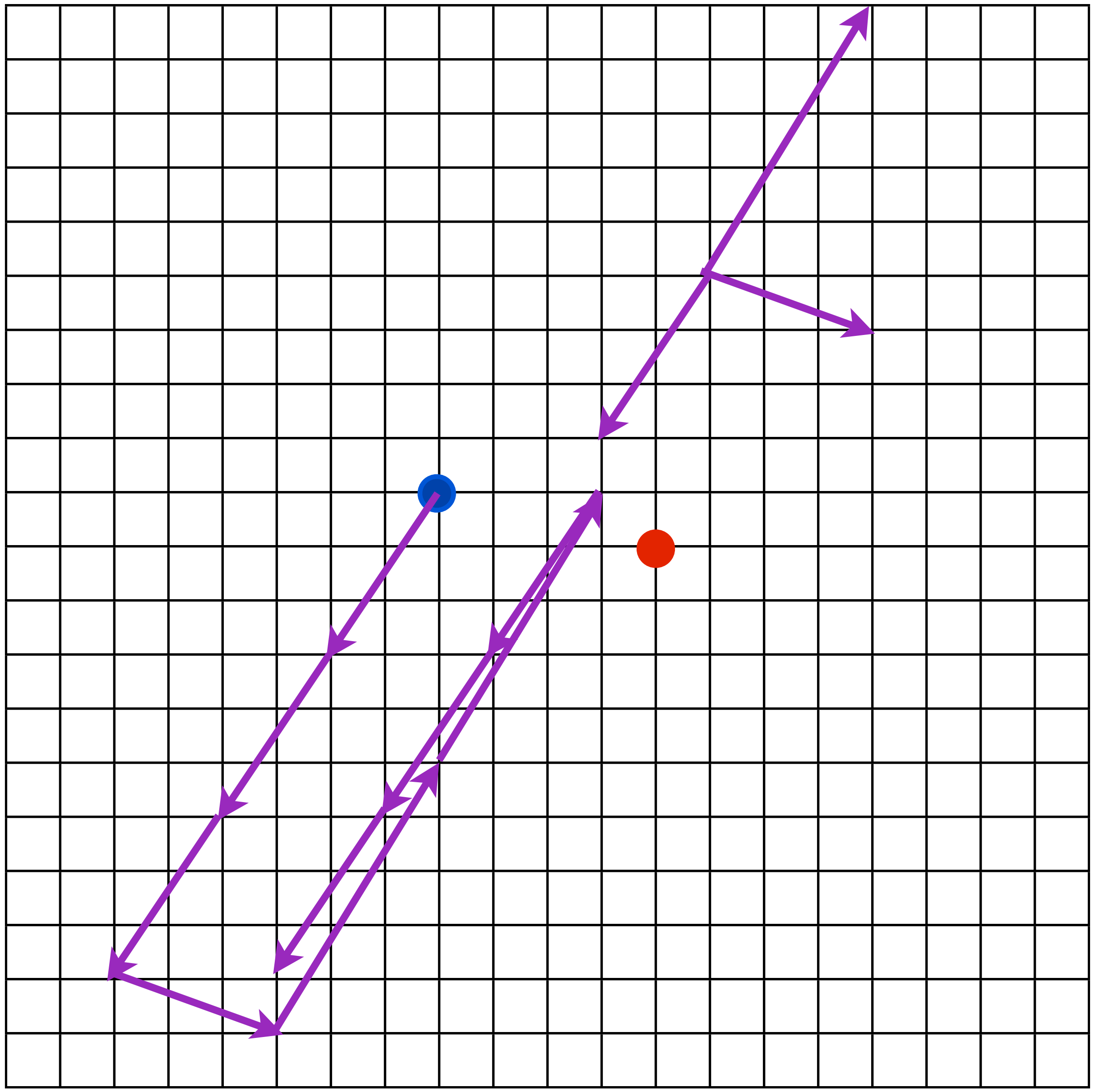

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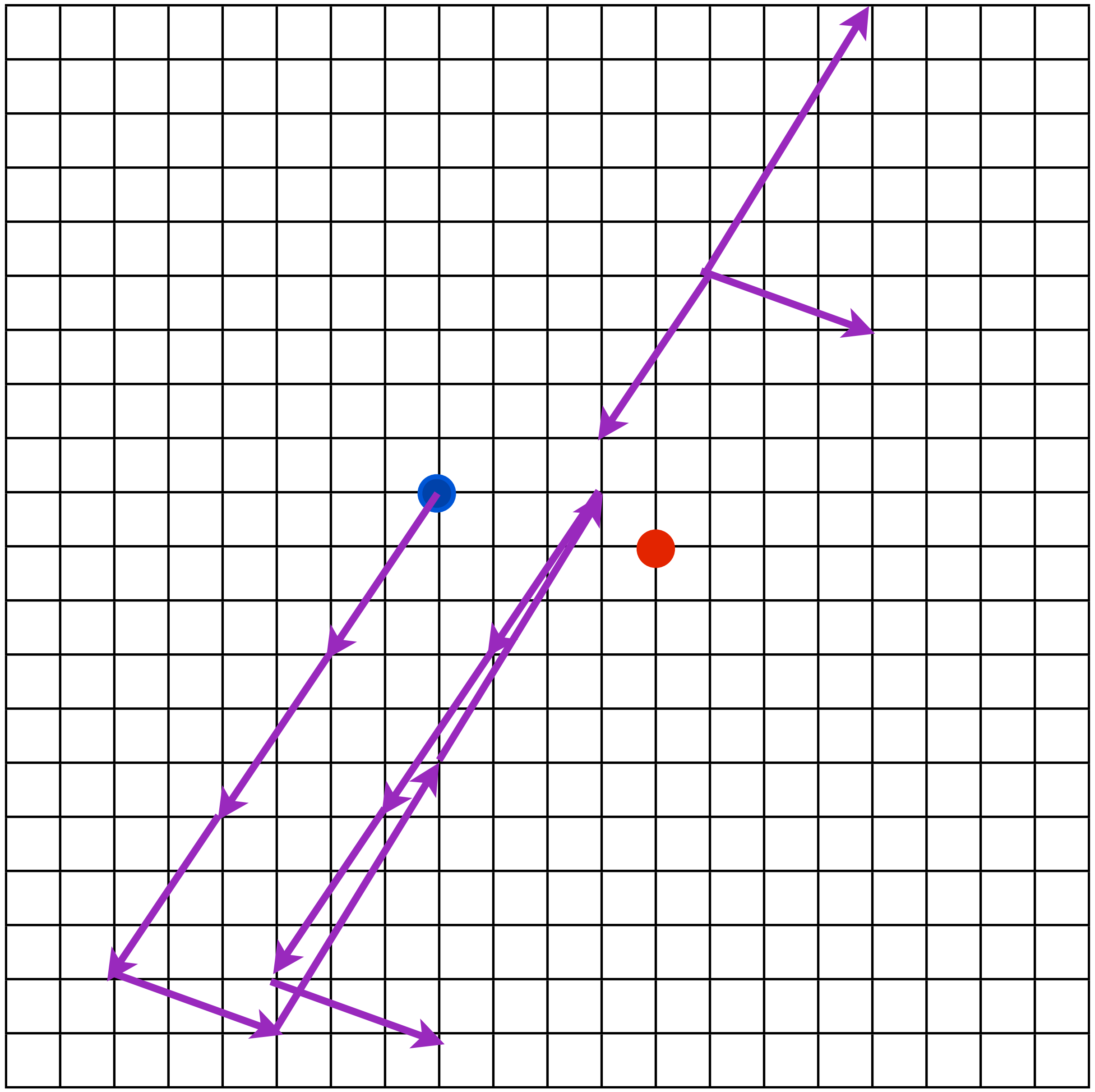

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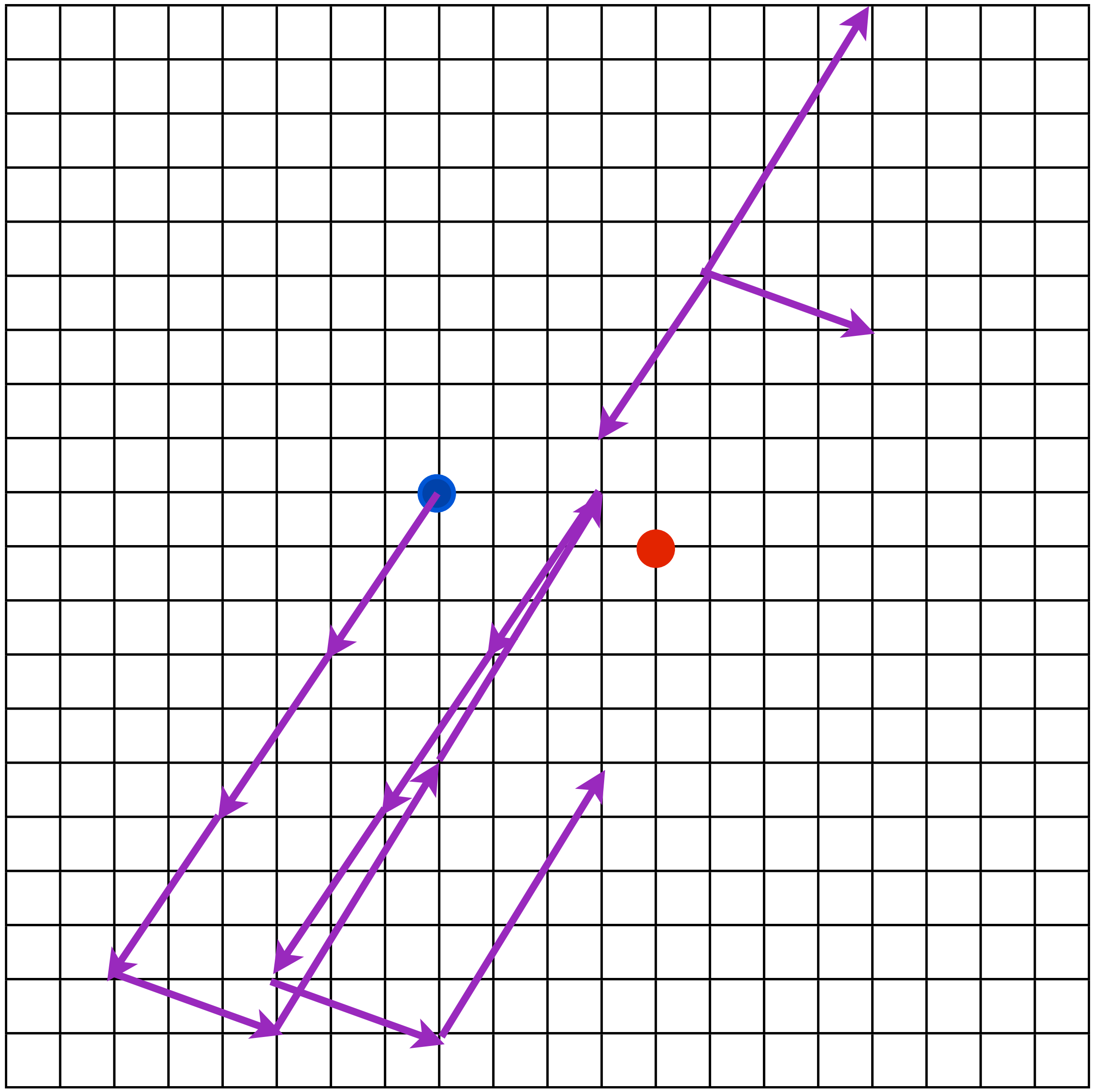

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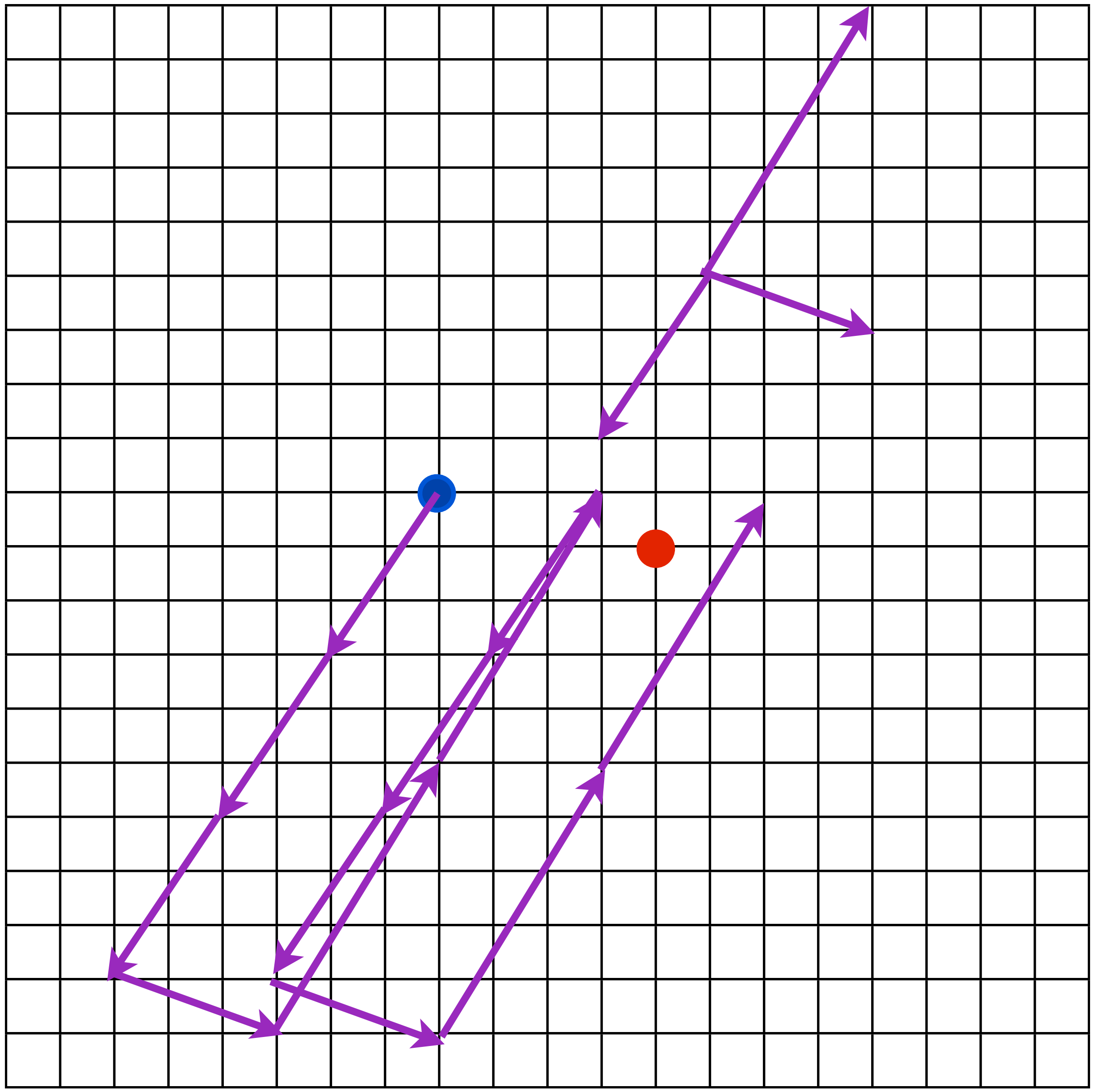

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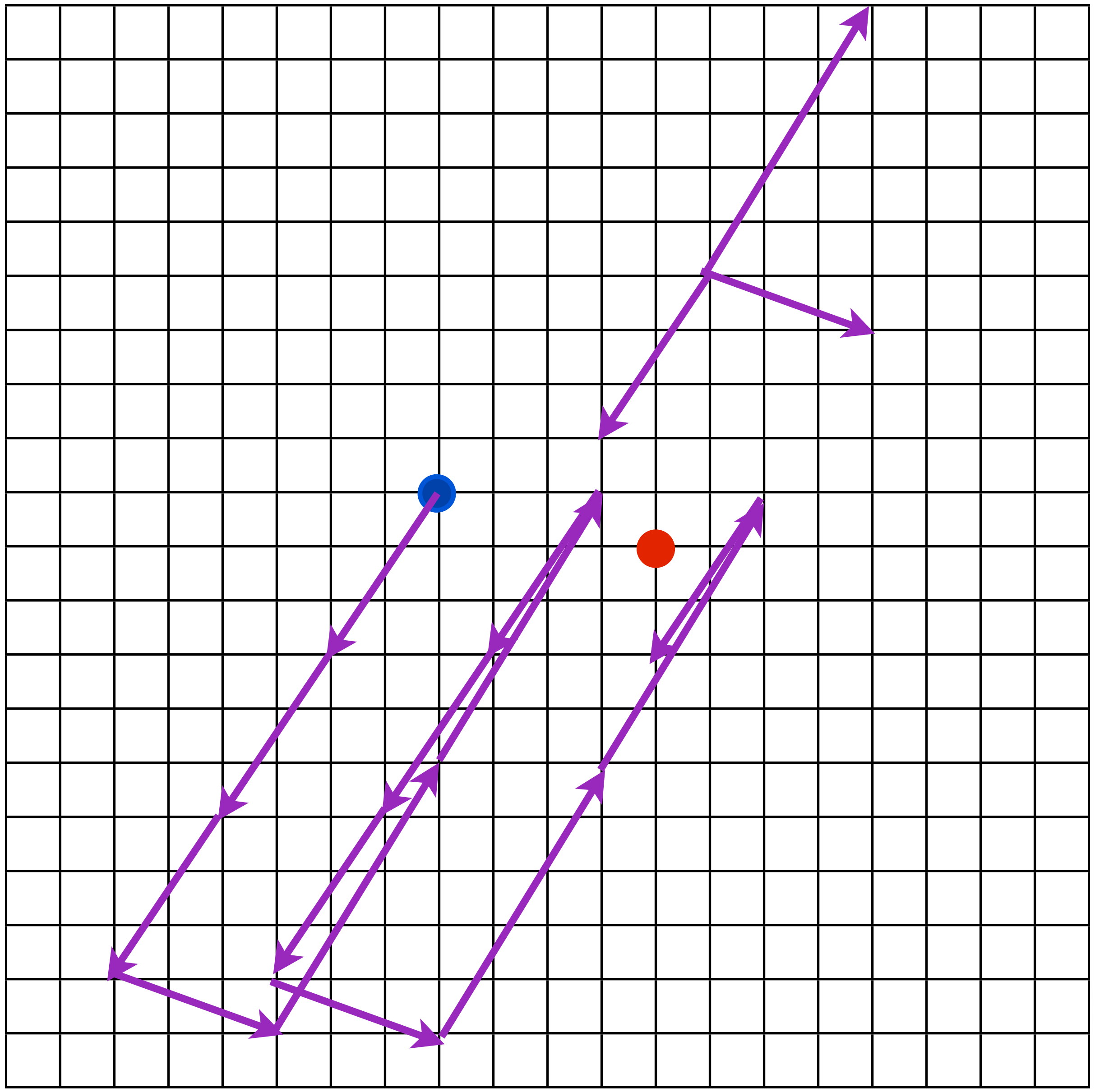

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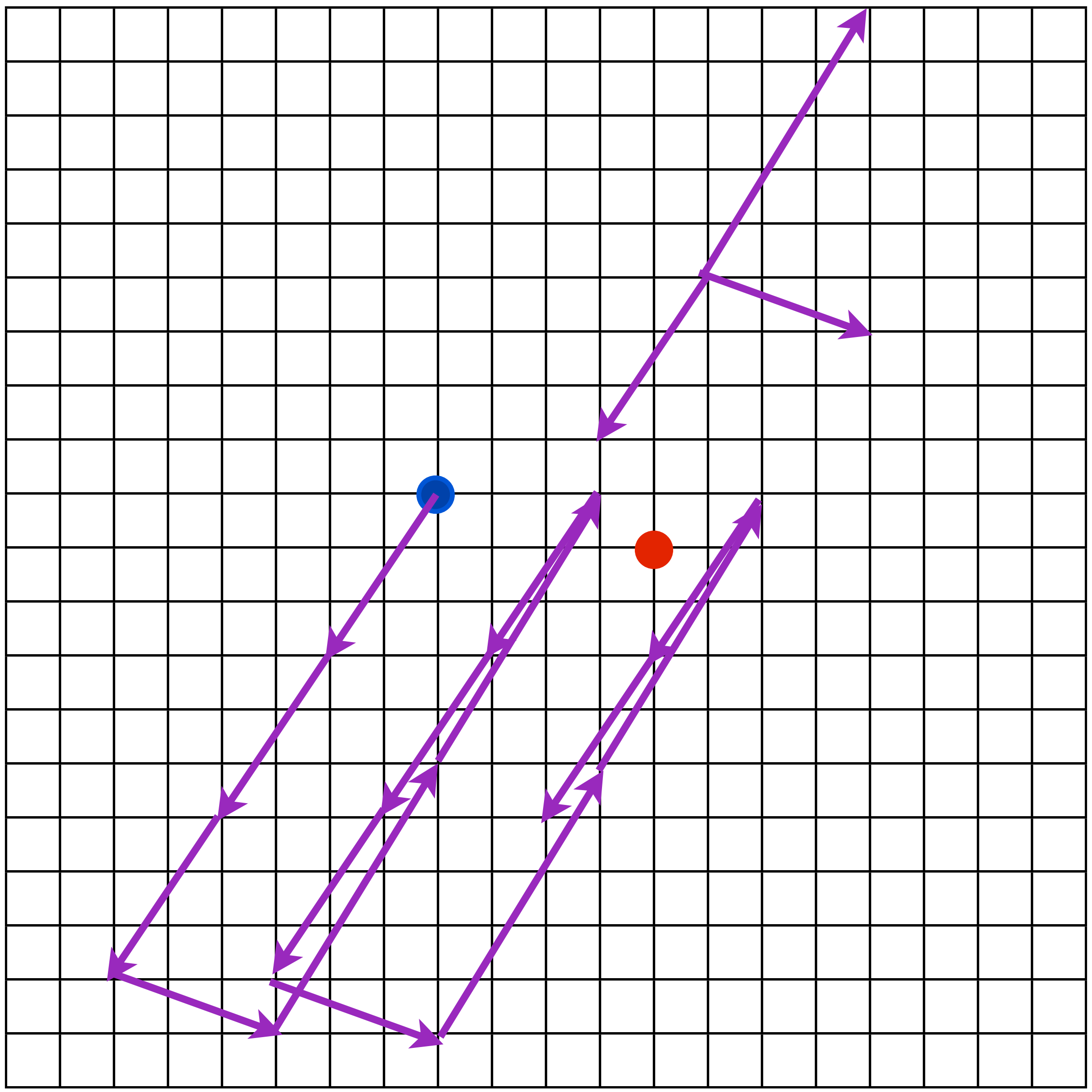

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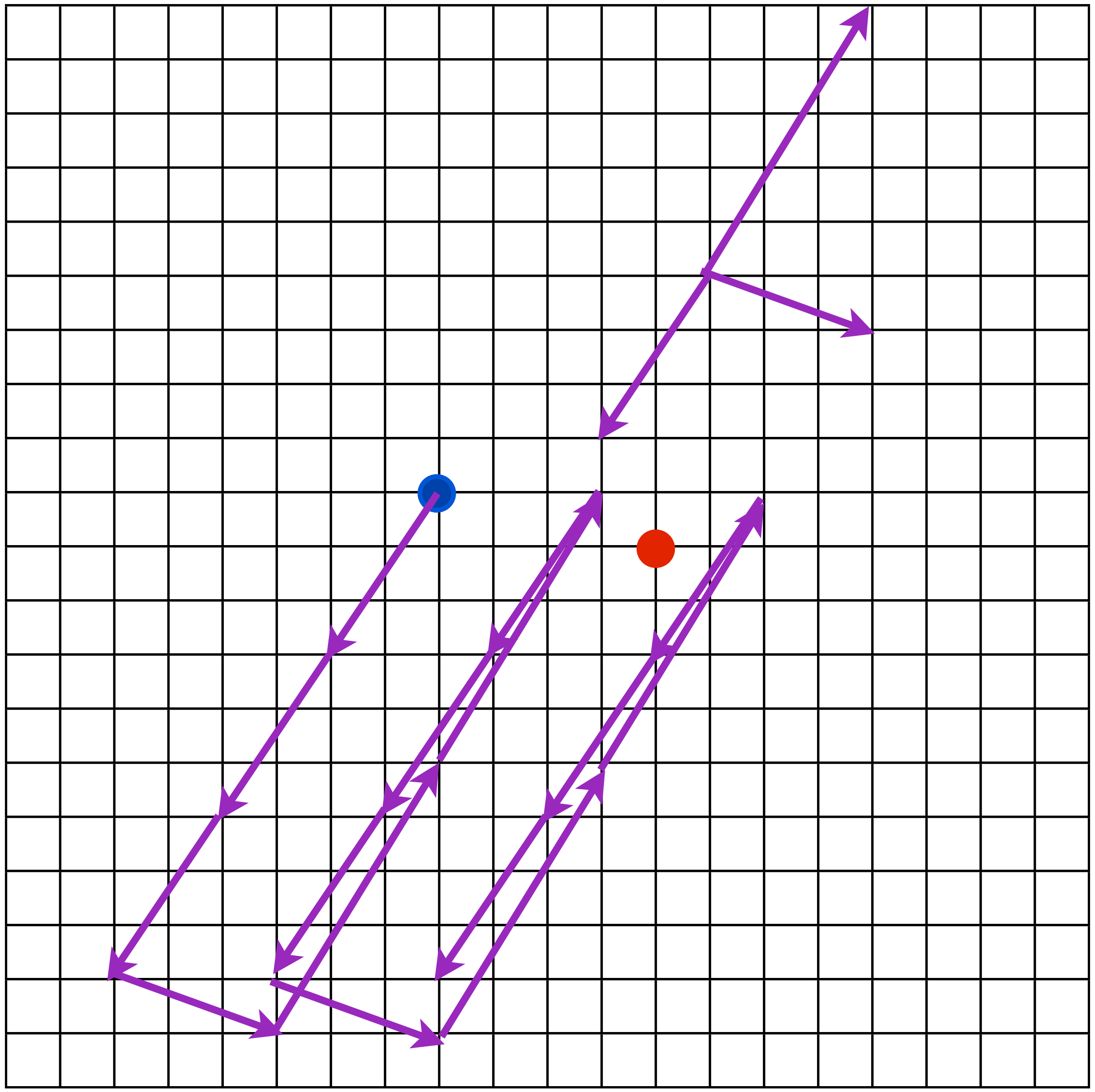

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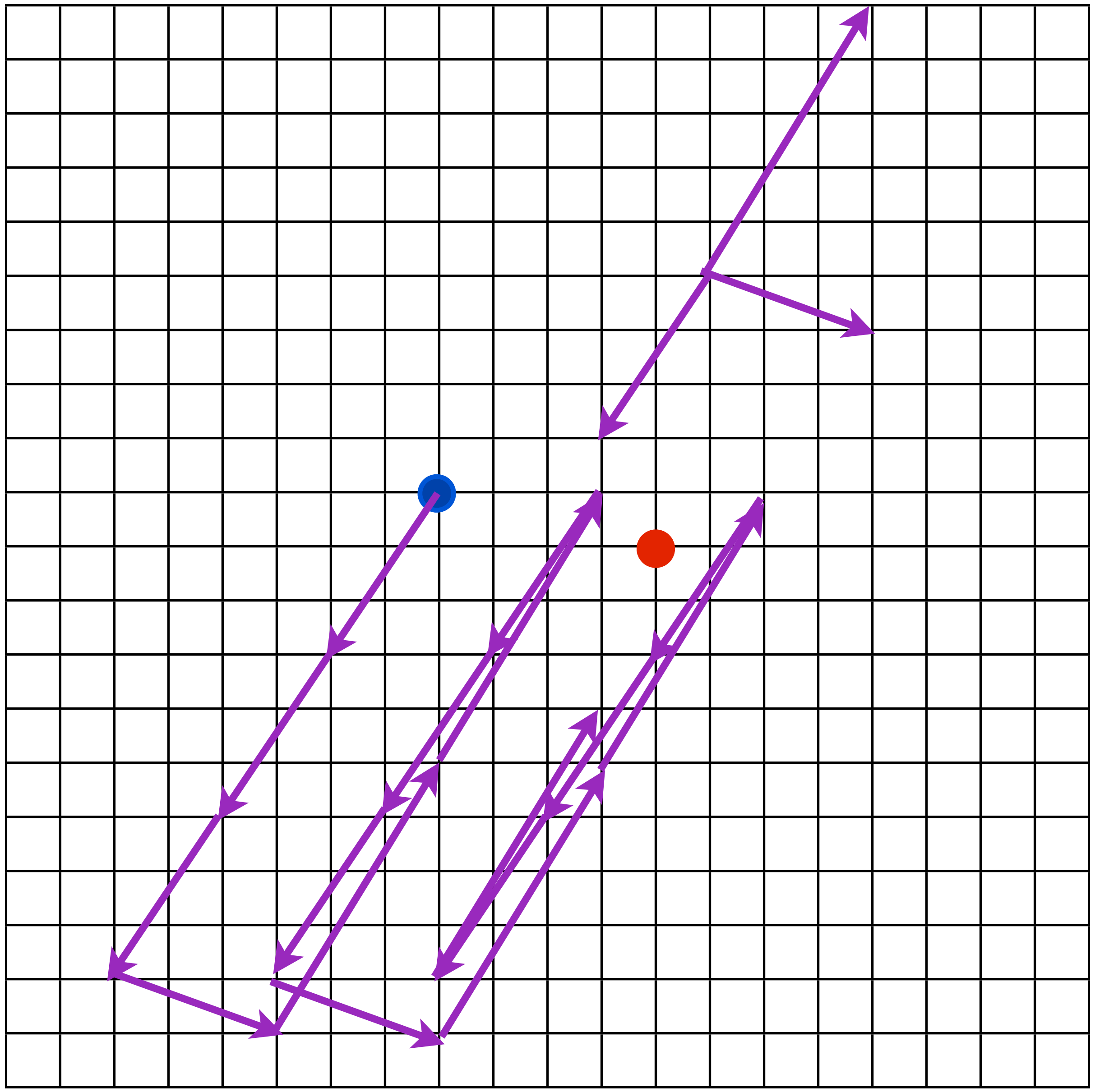

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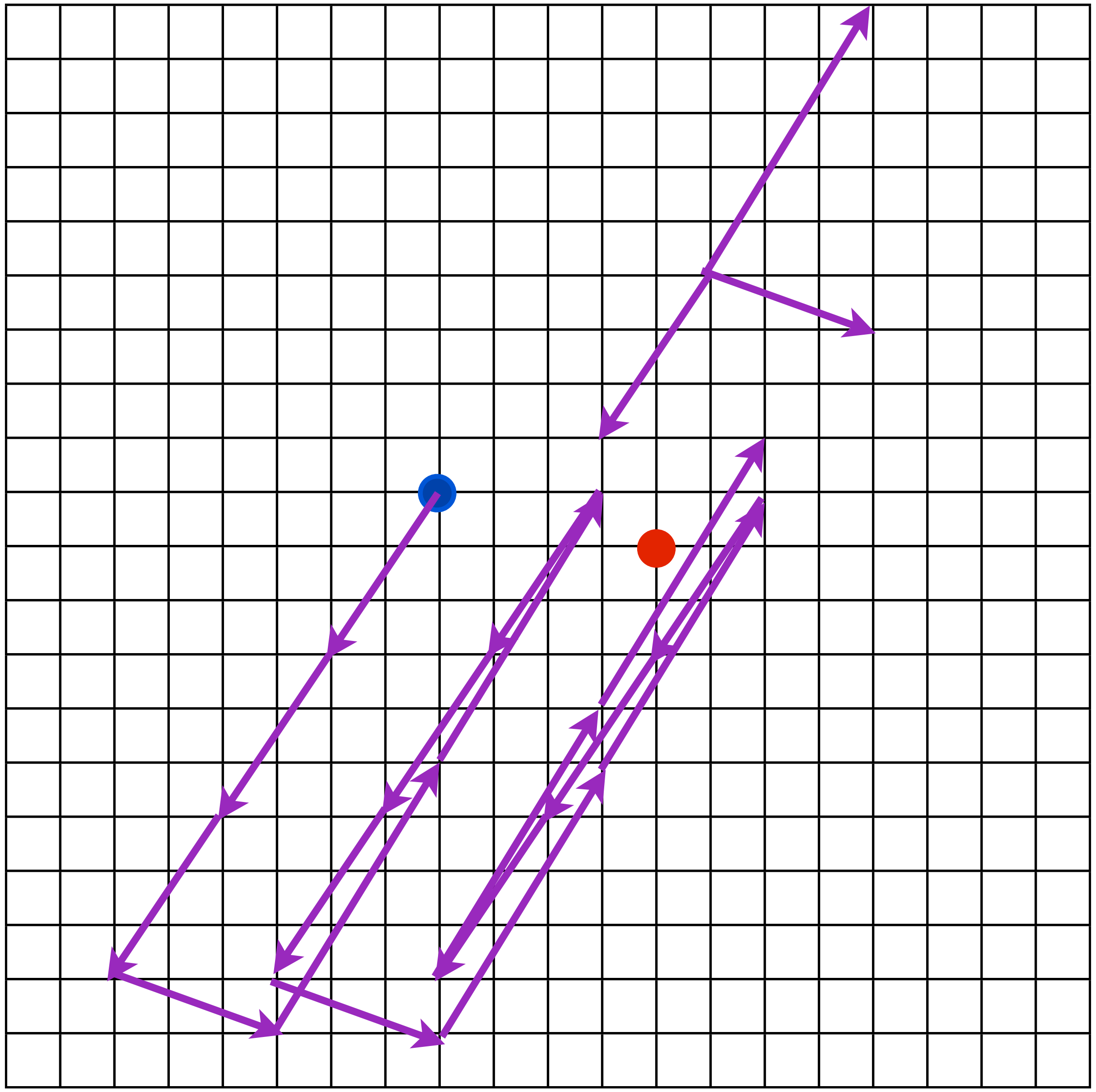

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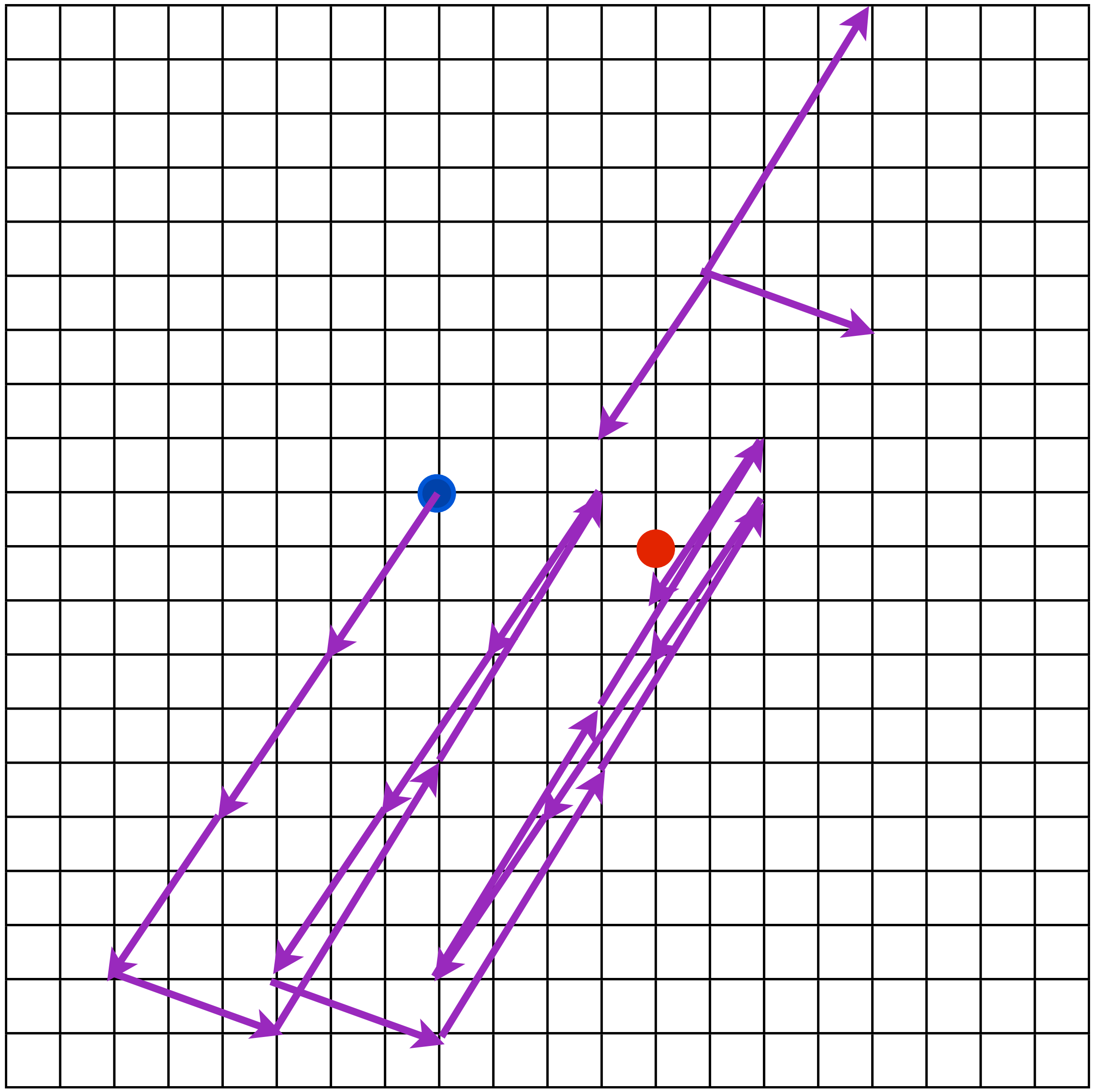

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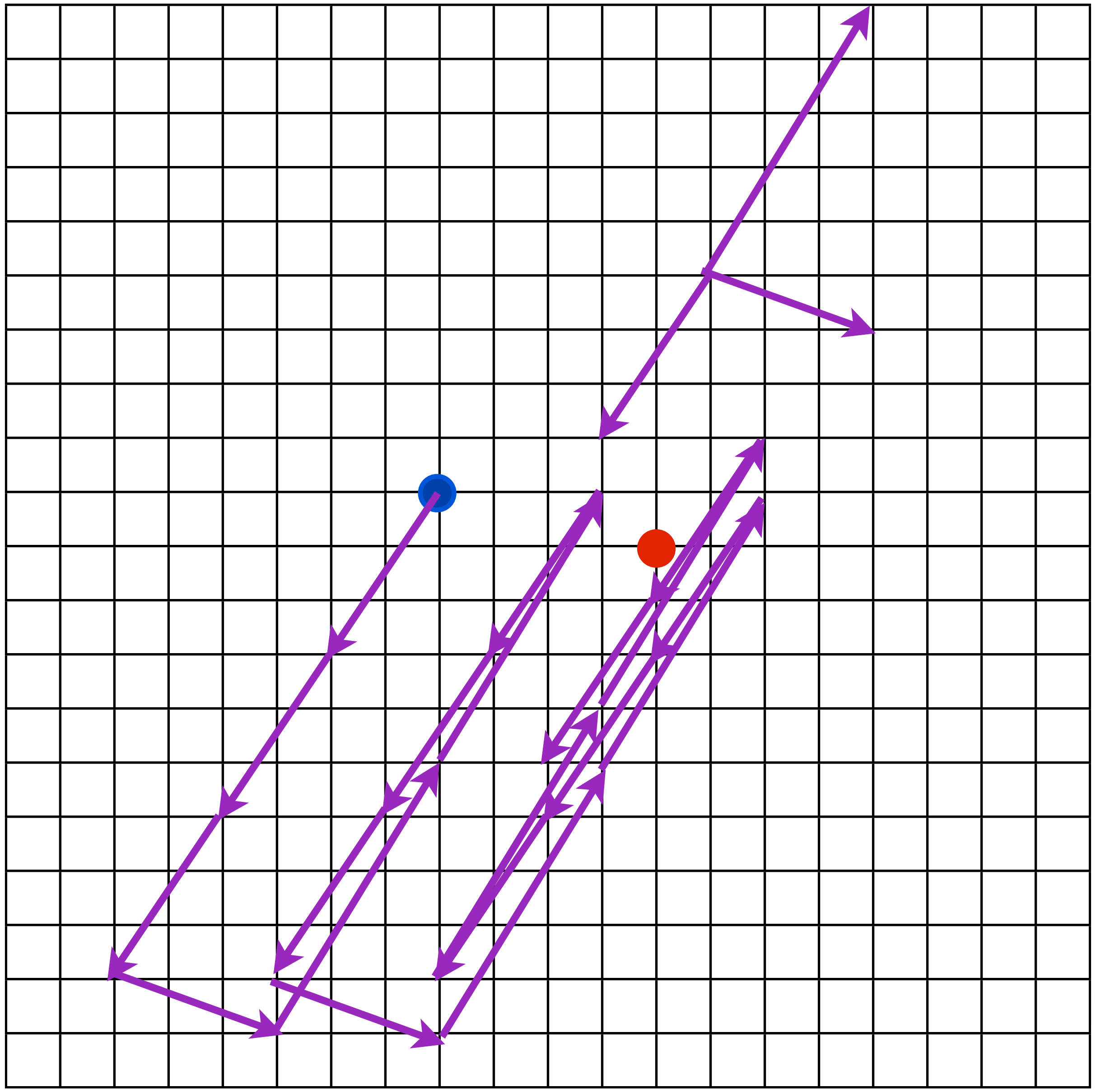

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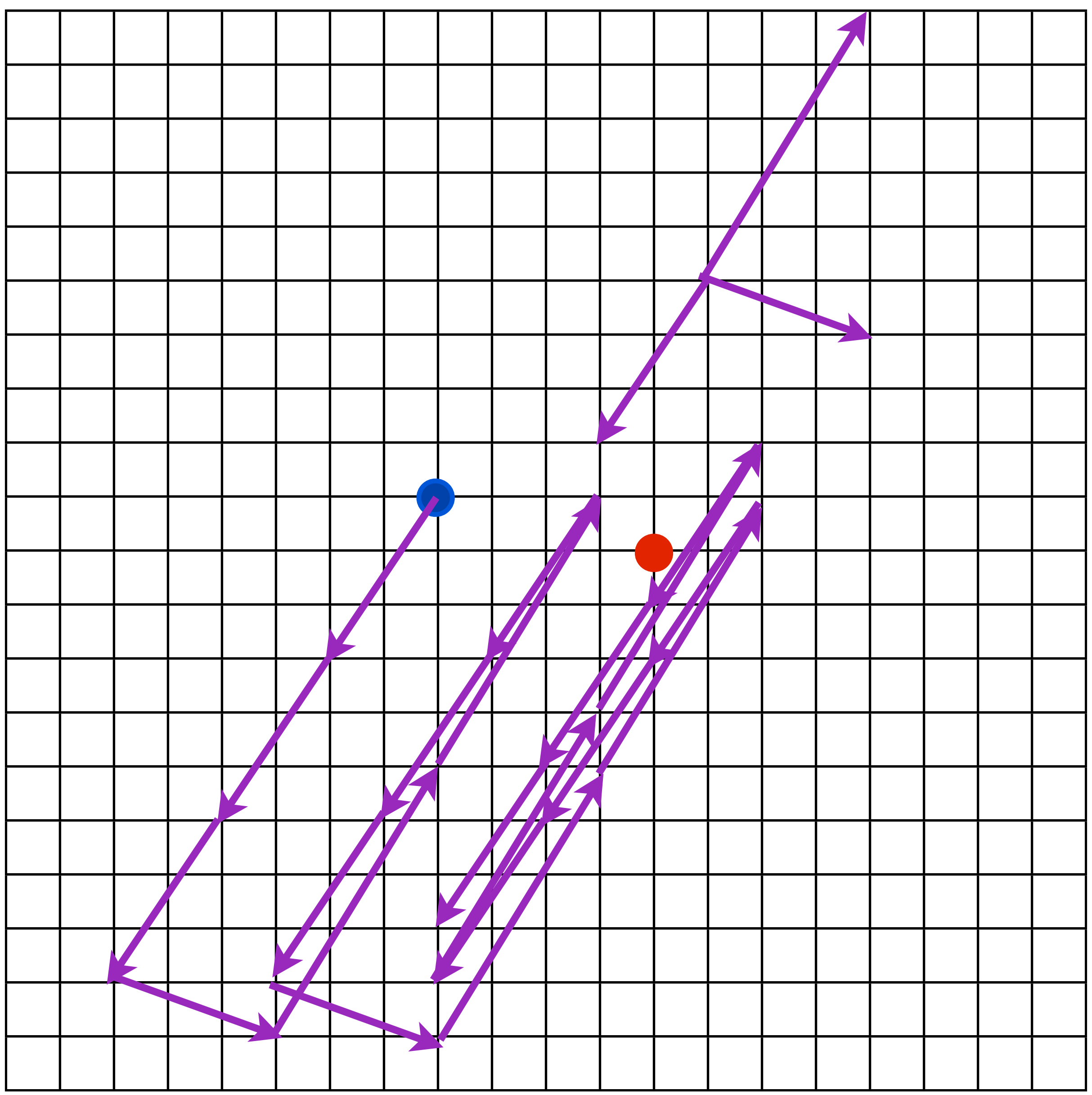

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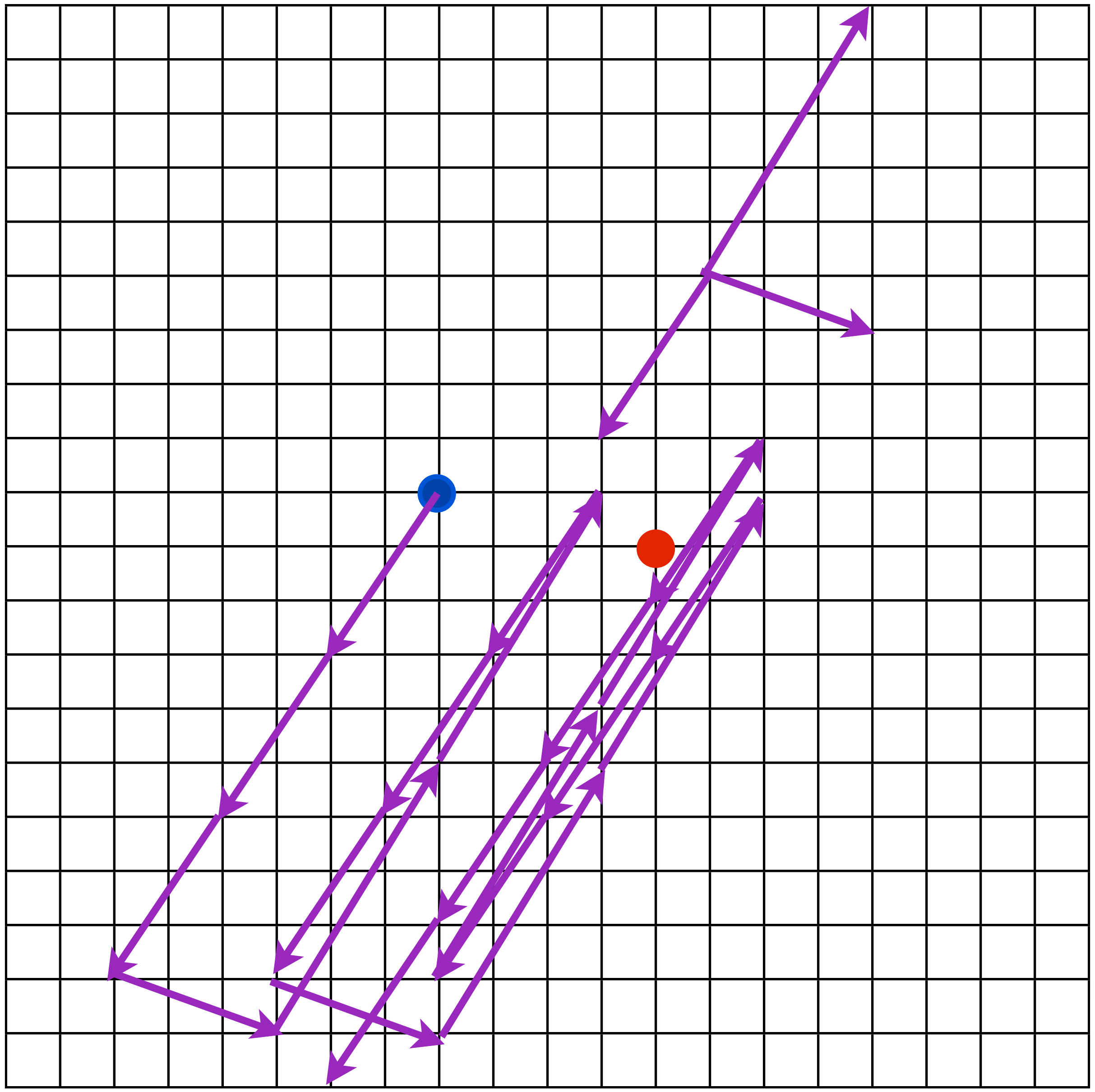

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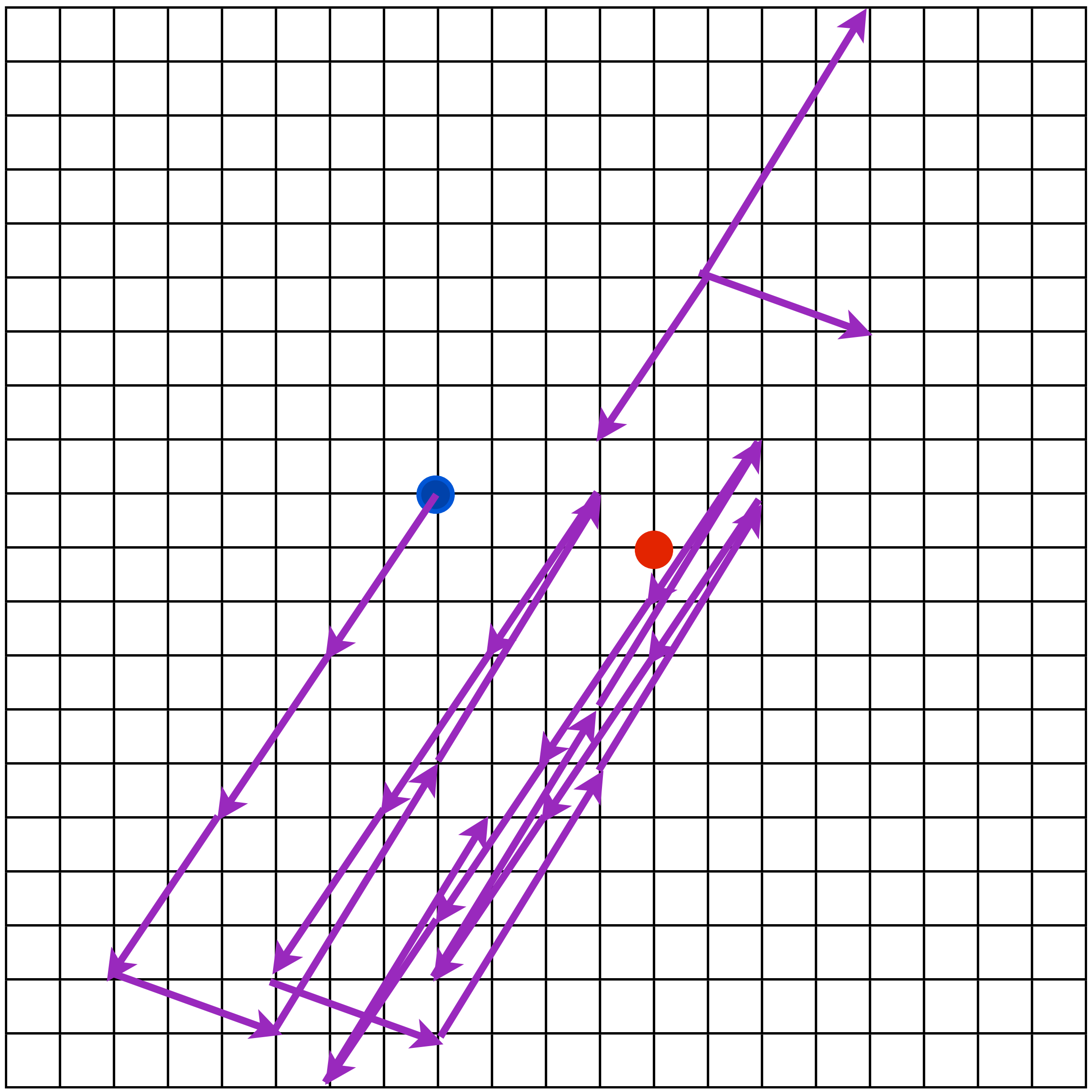

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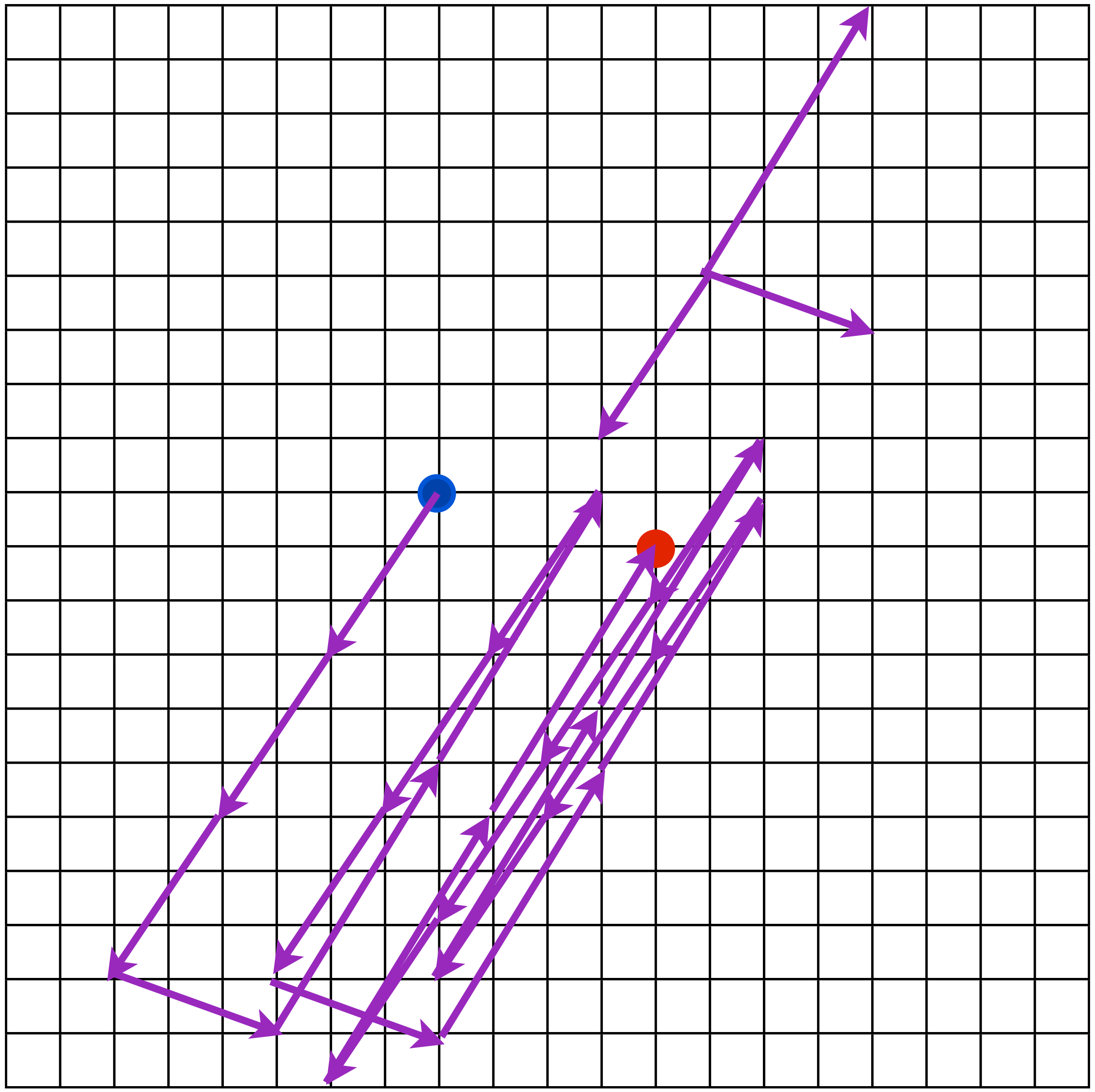

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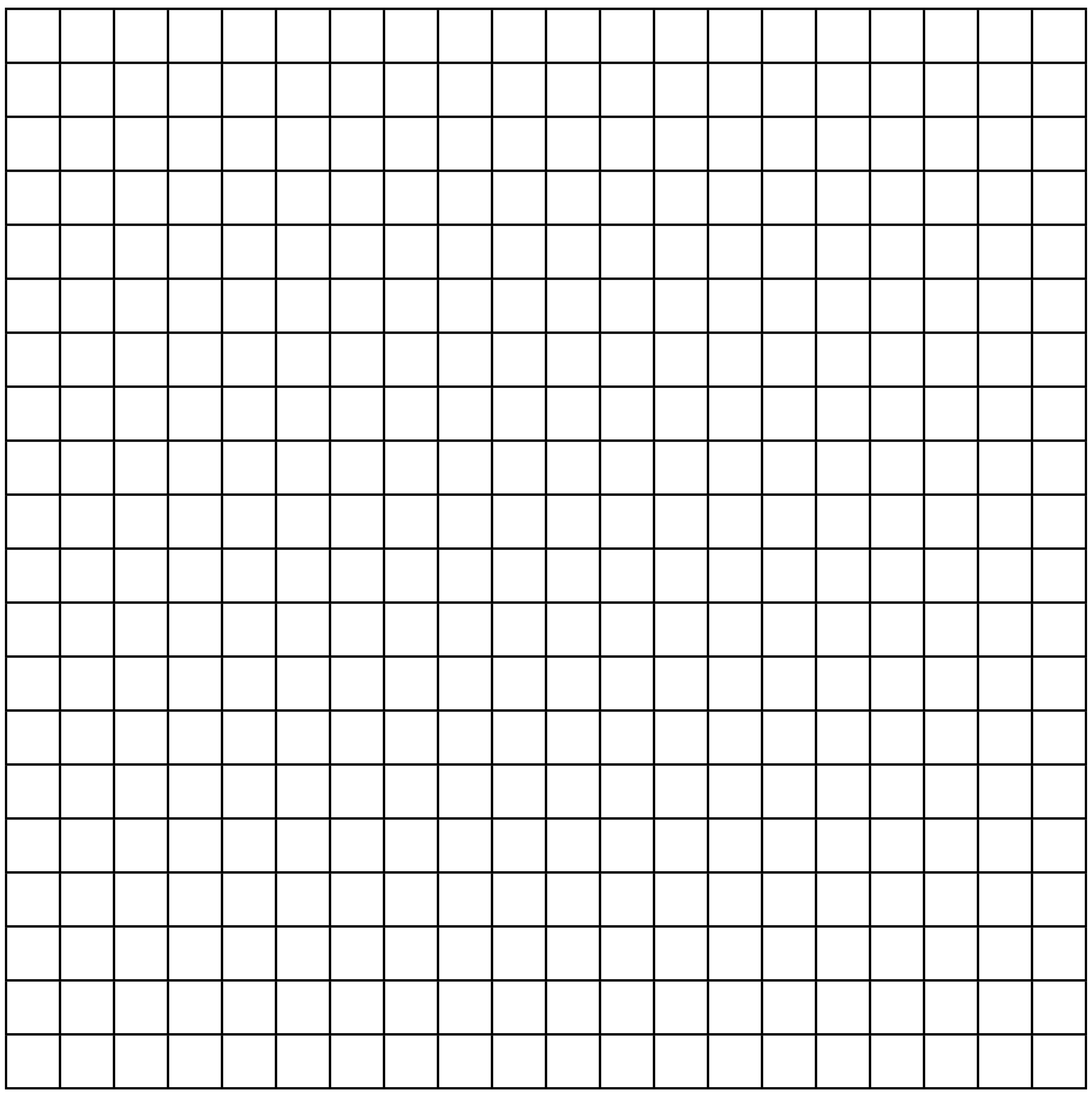

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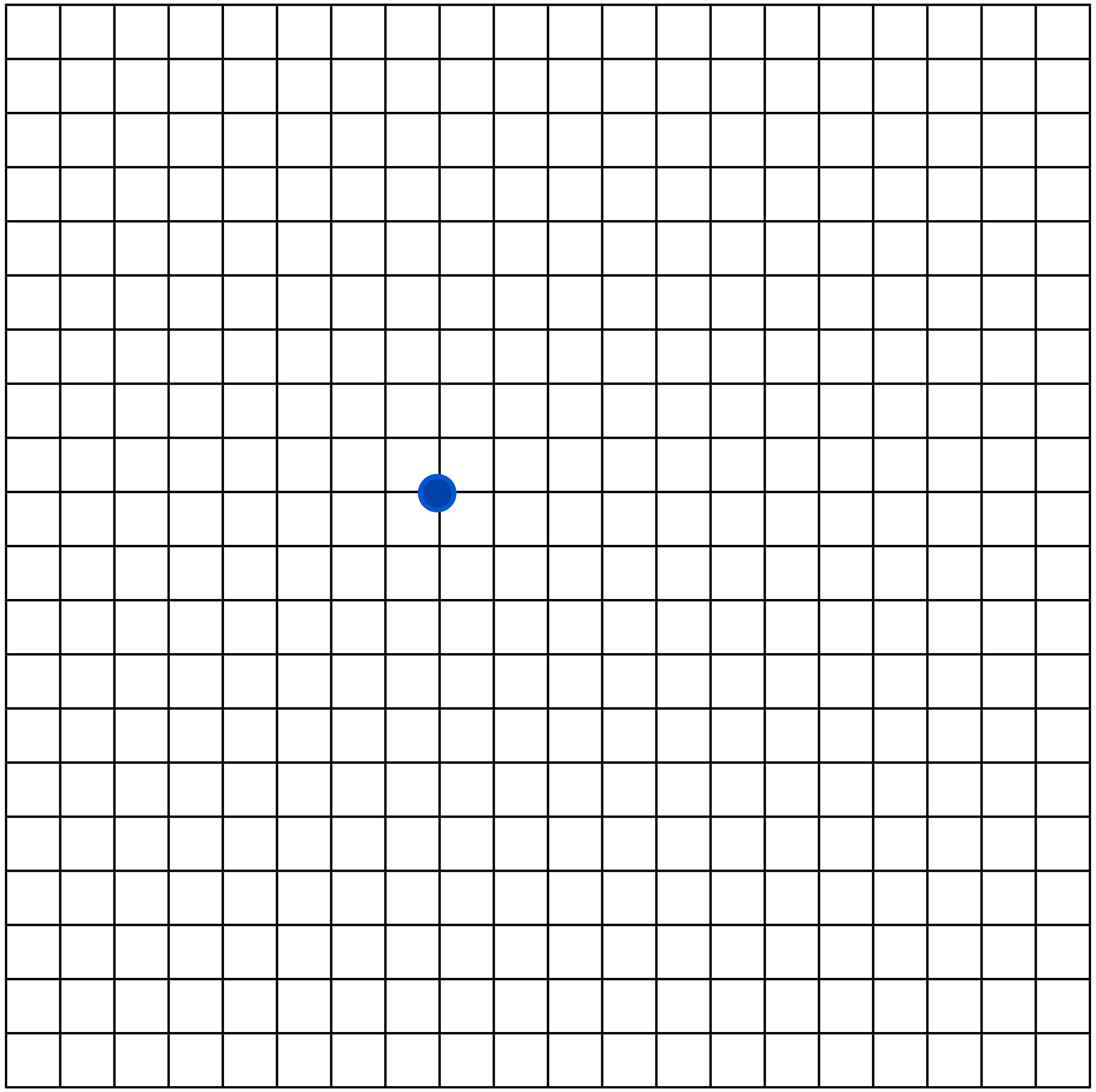

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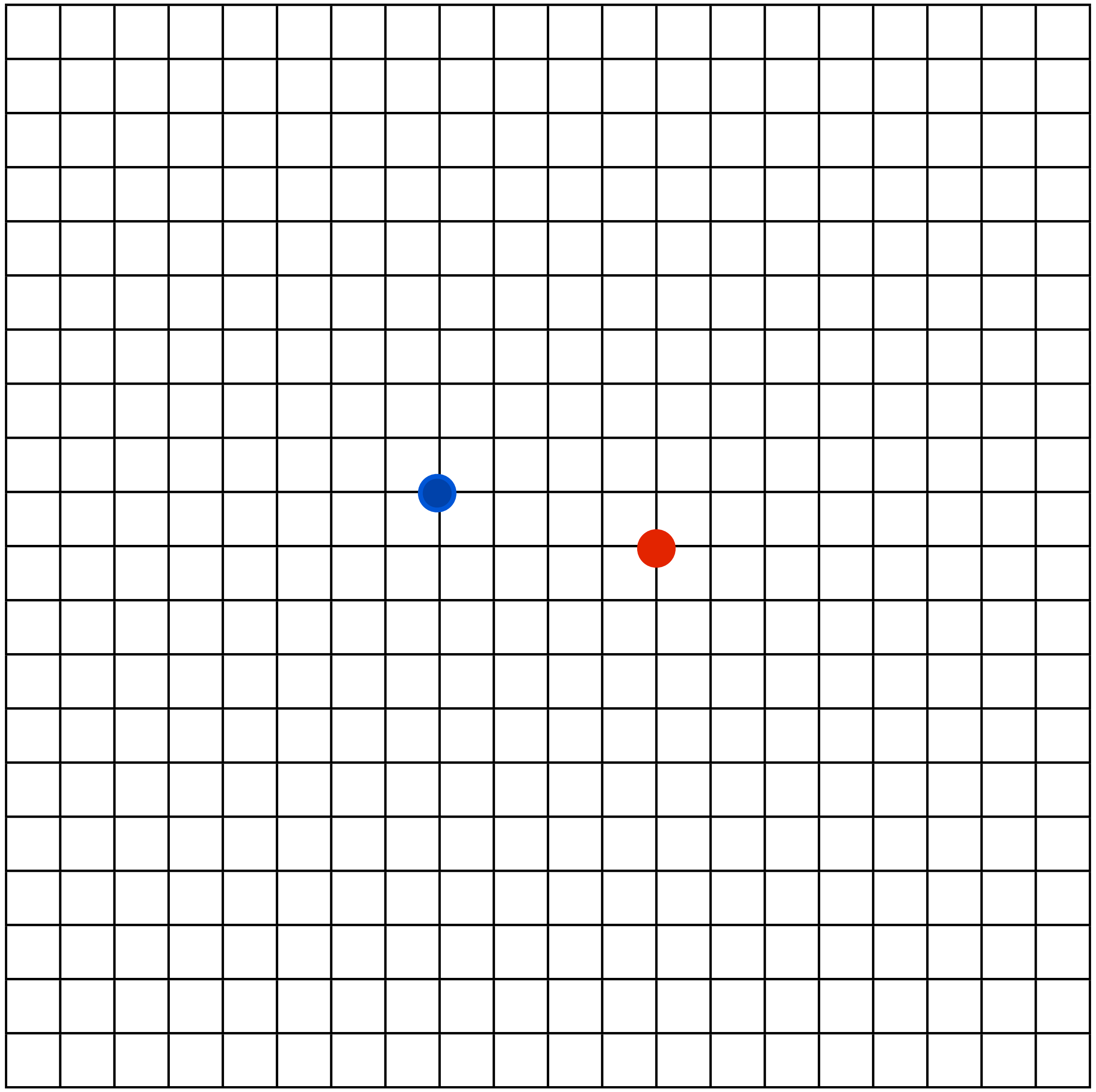

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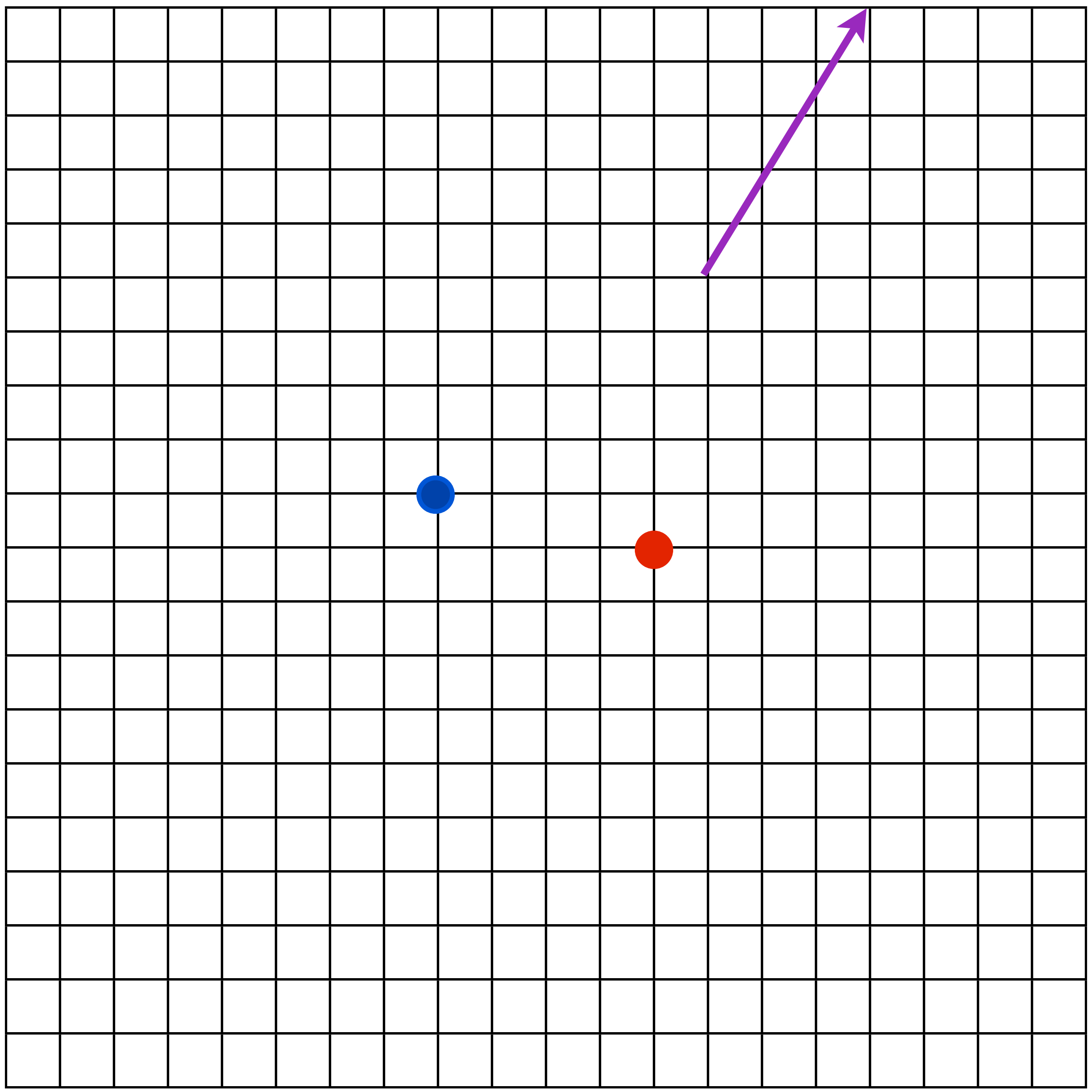

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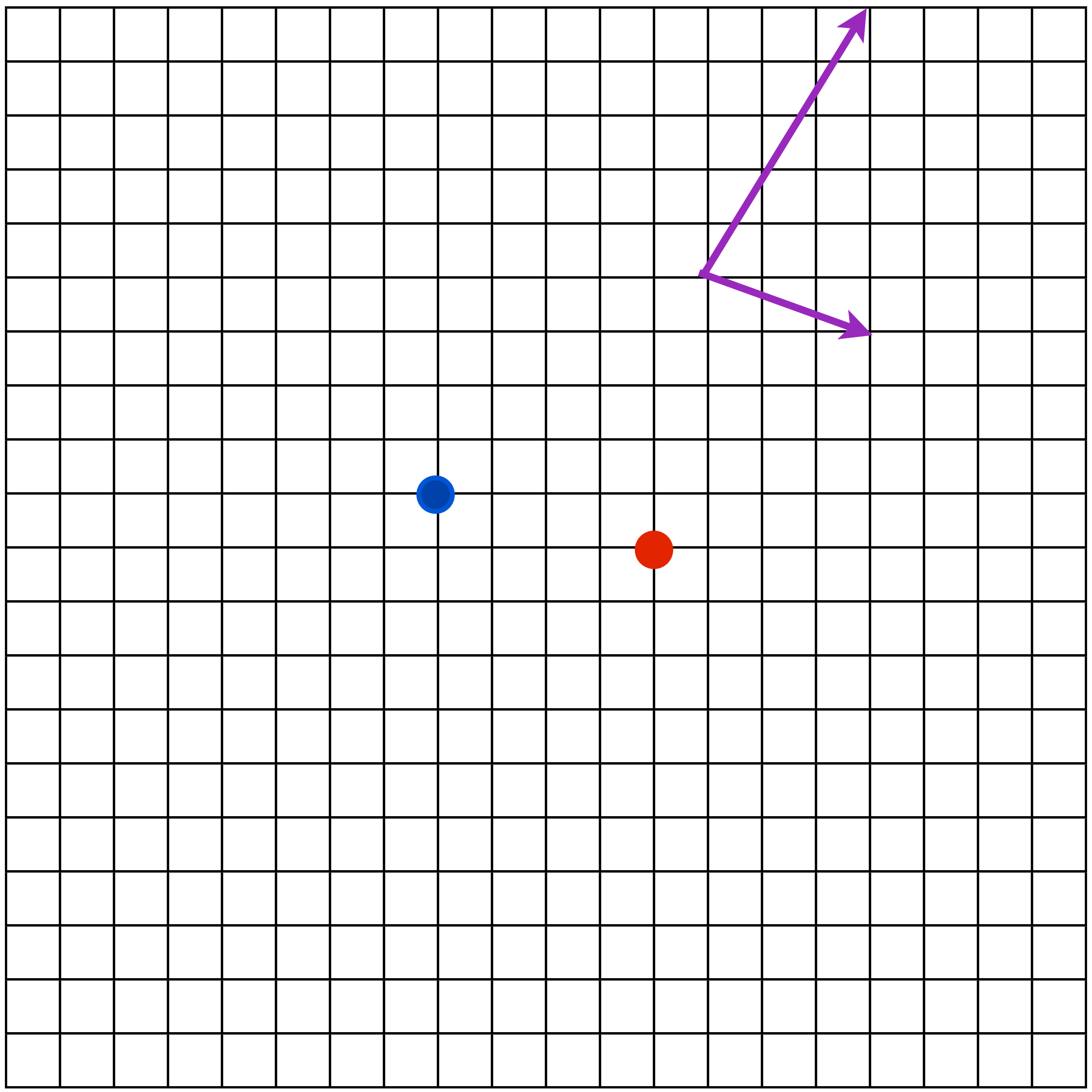

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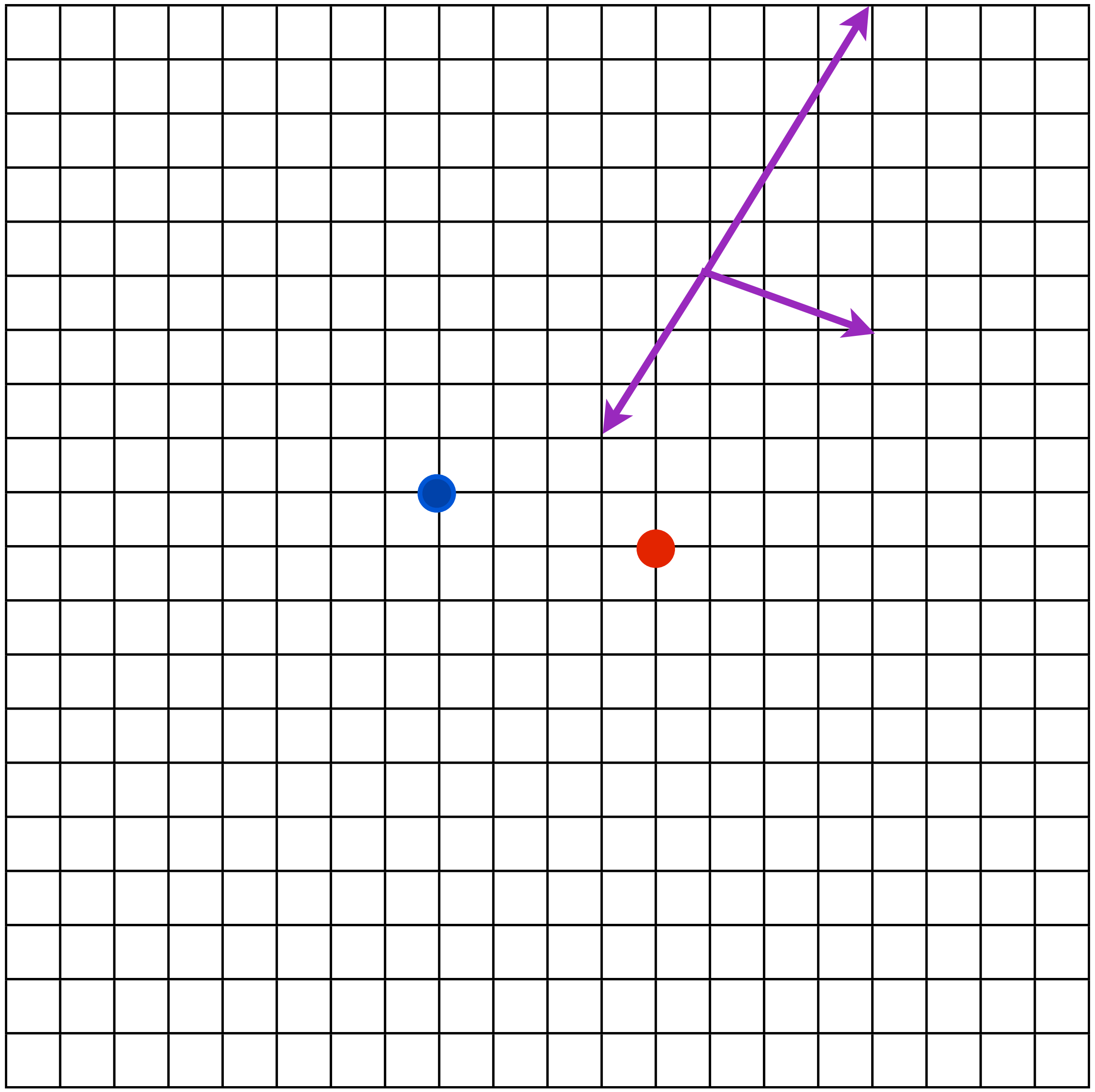

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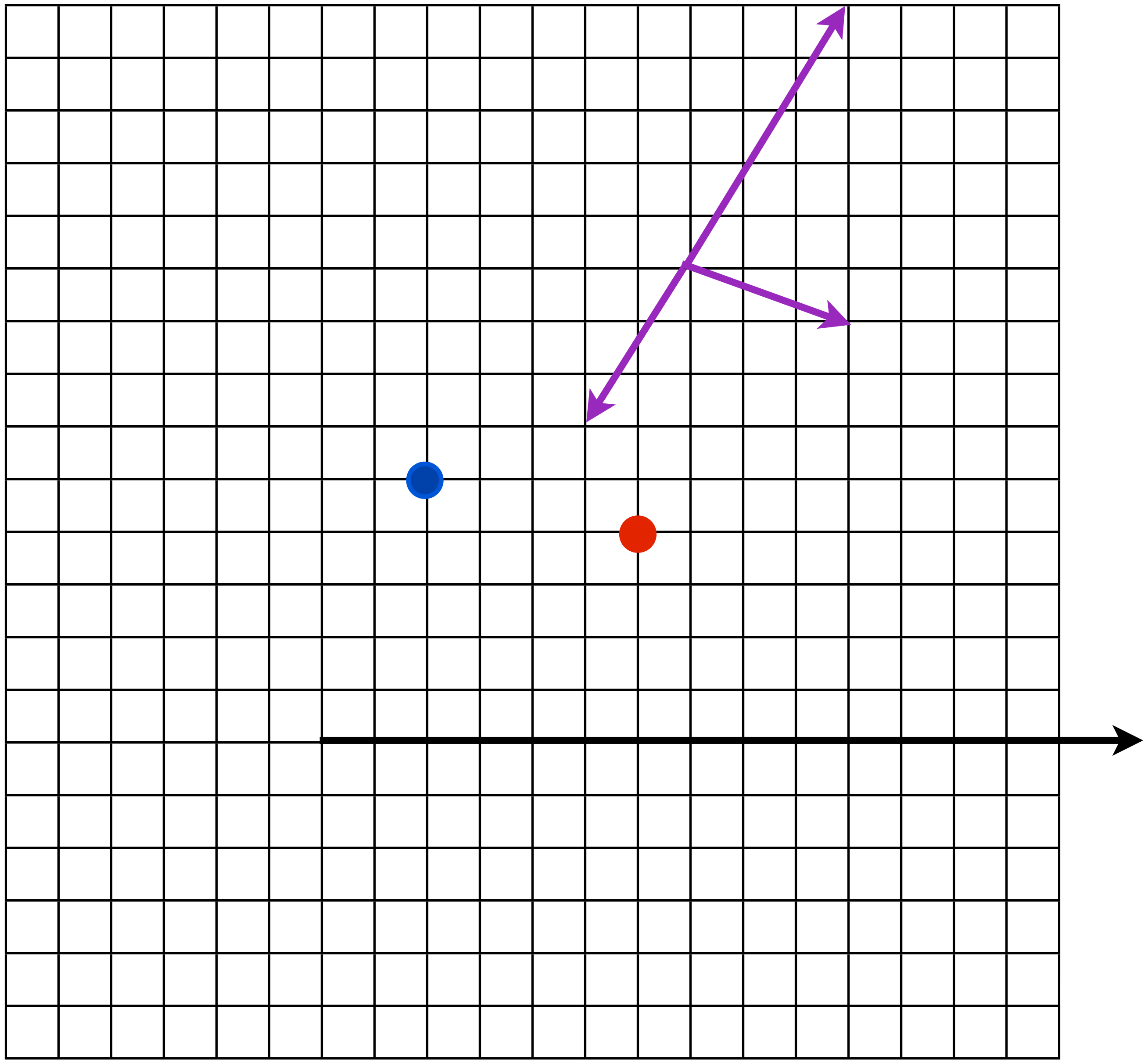

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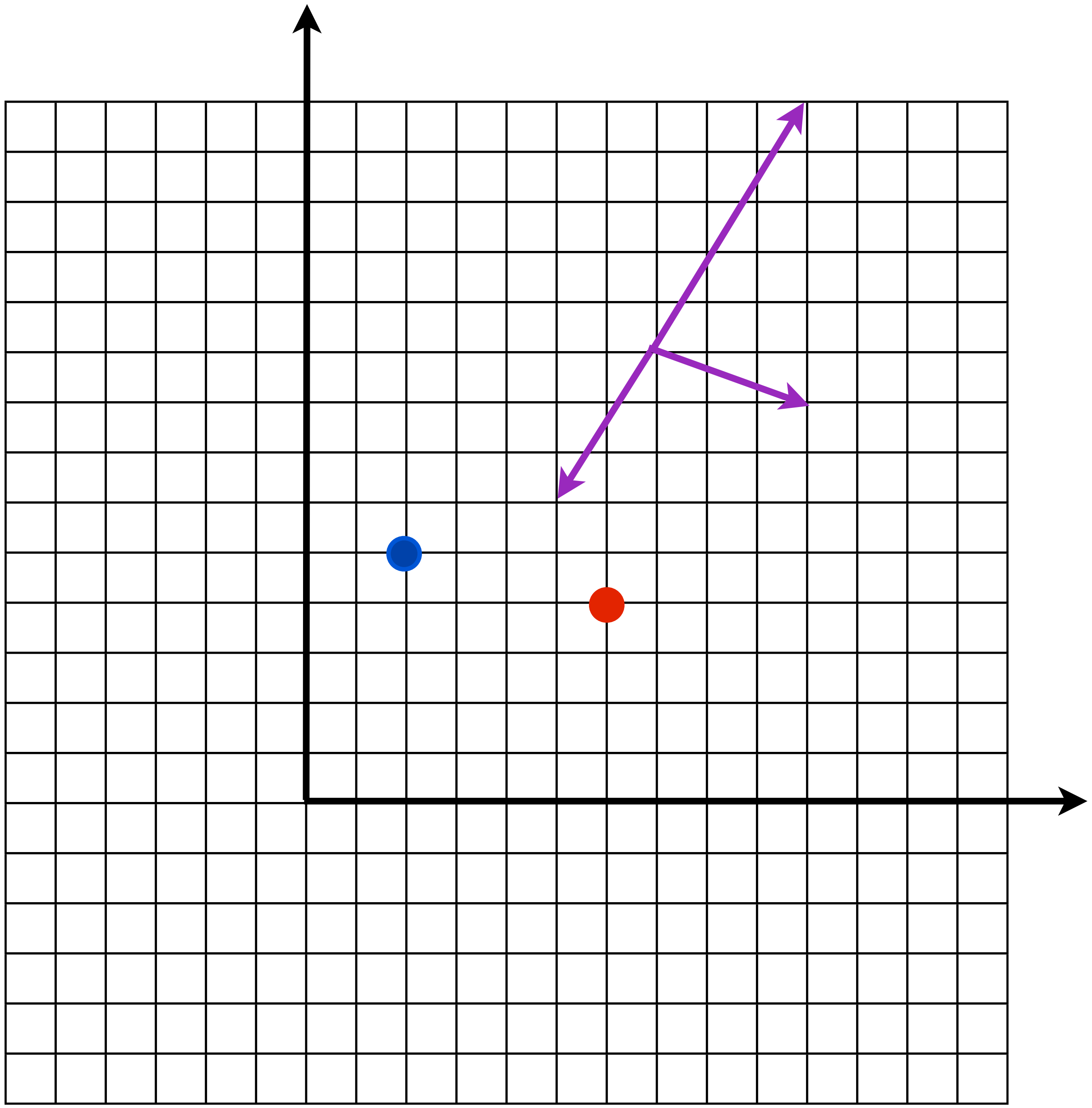

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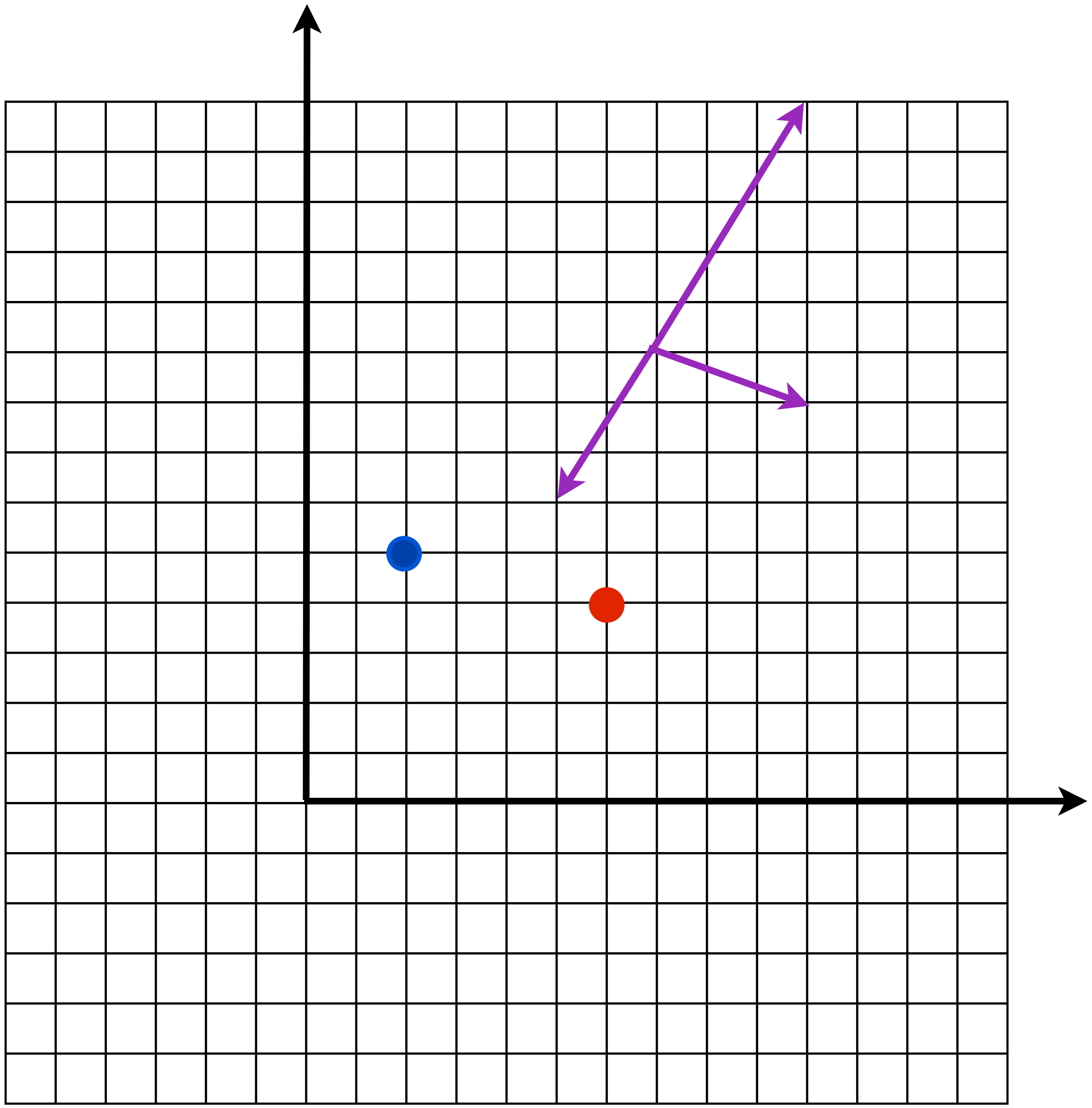

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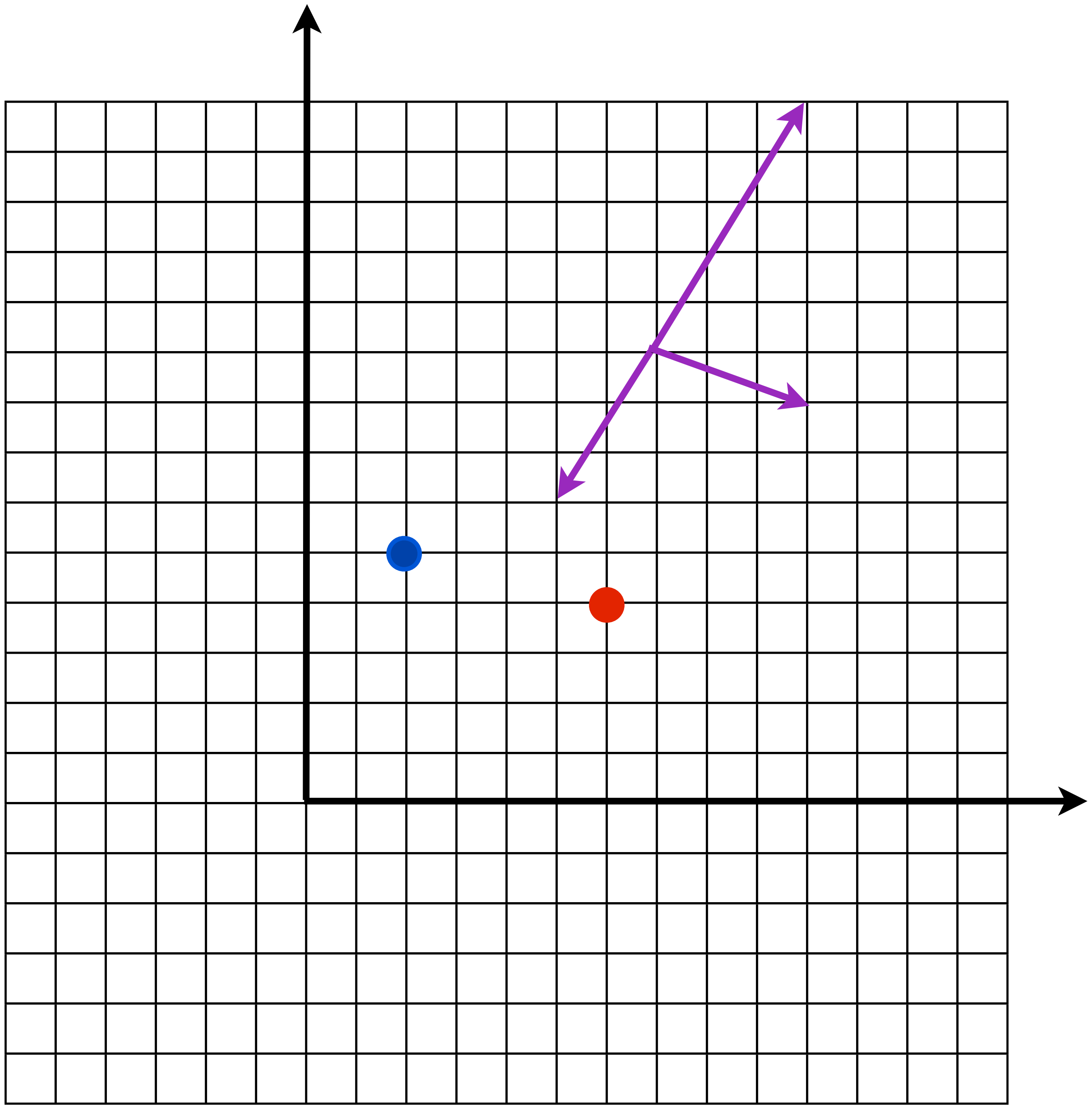

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Complicated!

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source s, target t in $N^d$

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Input: finite set of transitions T in $Z^d$
source s, target t in $N^d$

Question: can one reach t starting in s
by finitely many transitions from T
inside the positive quadrant?

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**Huge complexity gap**

Common feeling: possibly in exponential space

# Our result

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**Thm [C., Lasota, Lazic, Leroux, Mazowiecki]**

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The Reachability Problem for Vector Addition Systems is not elementary.

Not in time: $2^{2^{2^{\cdot^{\cdot^{\cdot^{2^{n}}}}}}}$

fixed height

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- hope: always a double-exponential path
- this is the case for similar coverability problem
- coverability: can I go from s above t?
- conjecture: reachability and coverability should behave similarly

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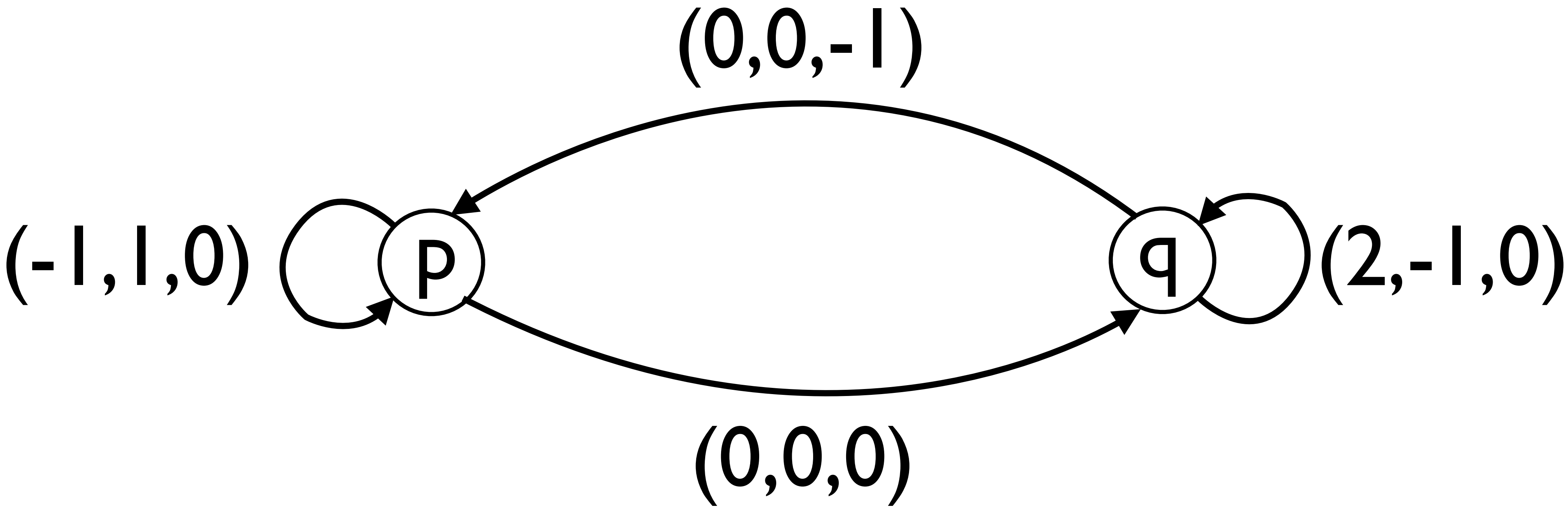

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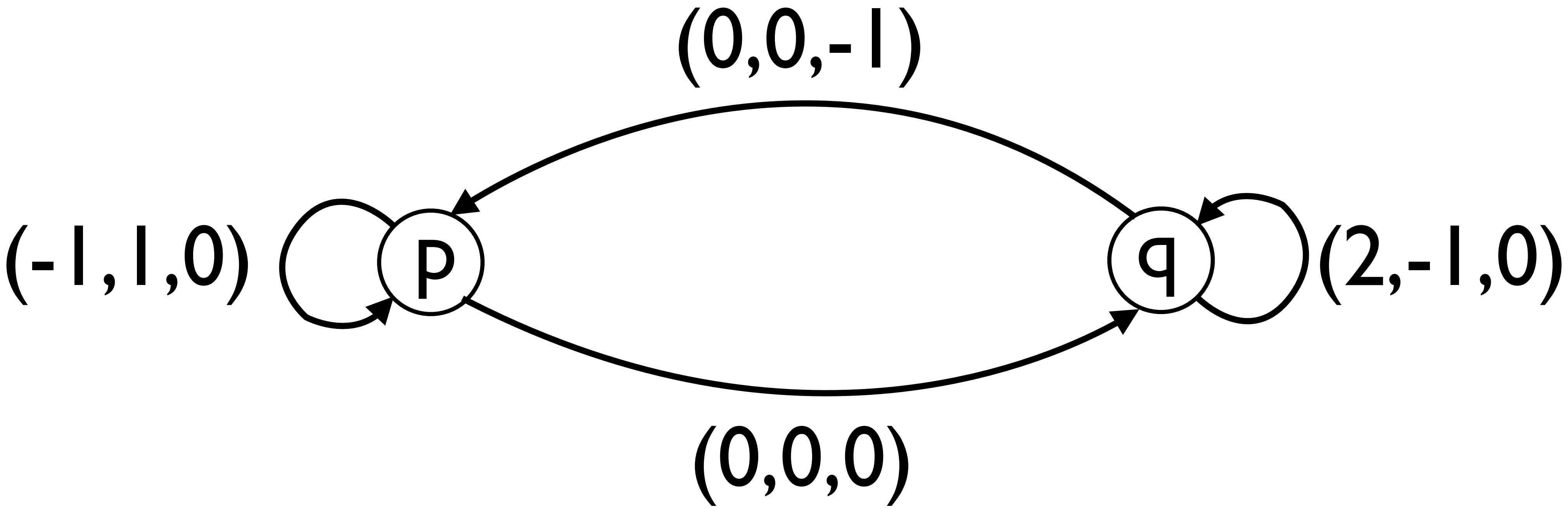


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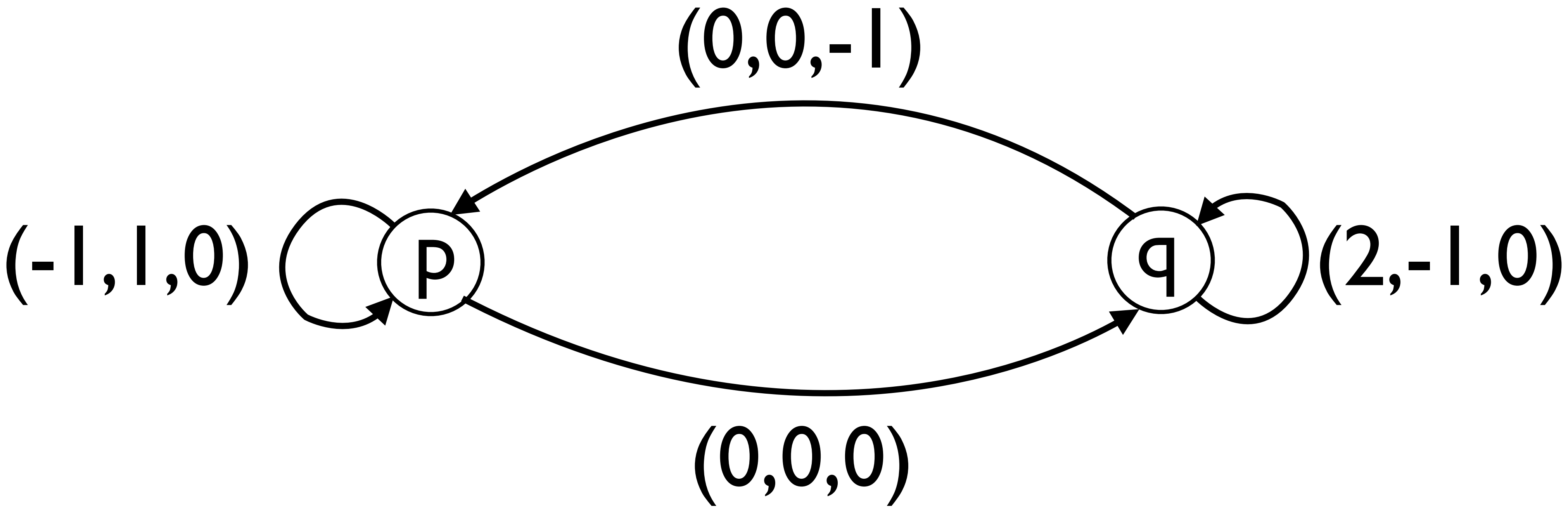


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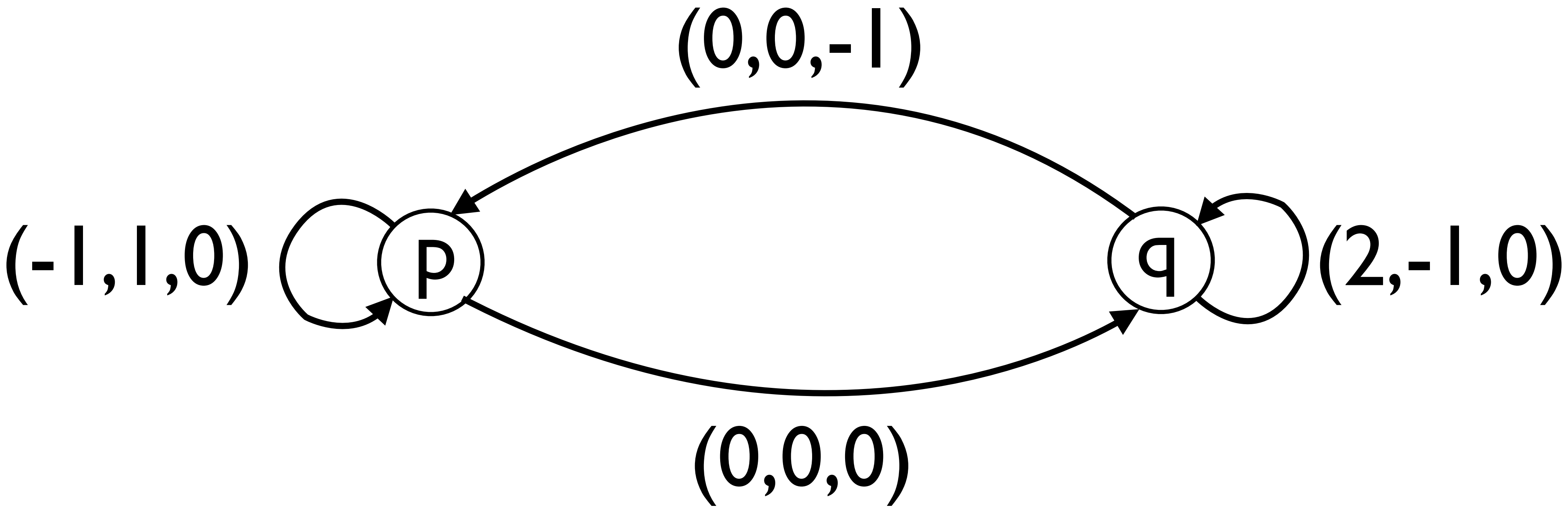


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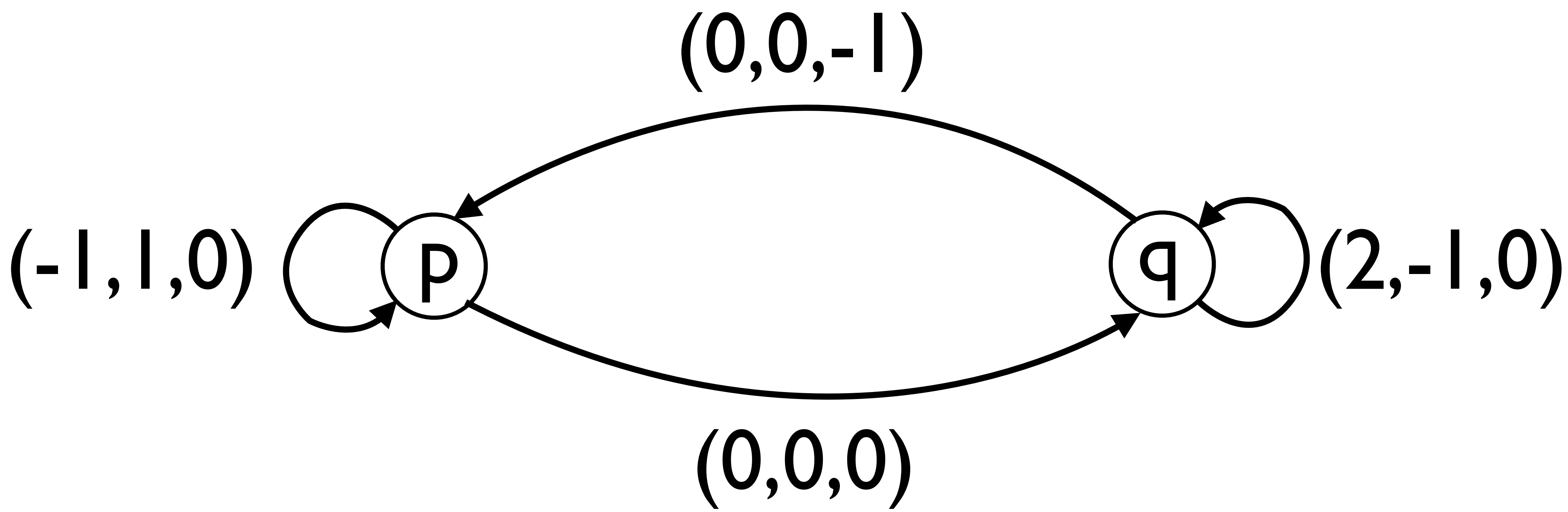


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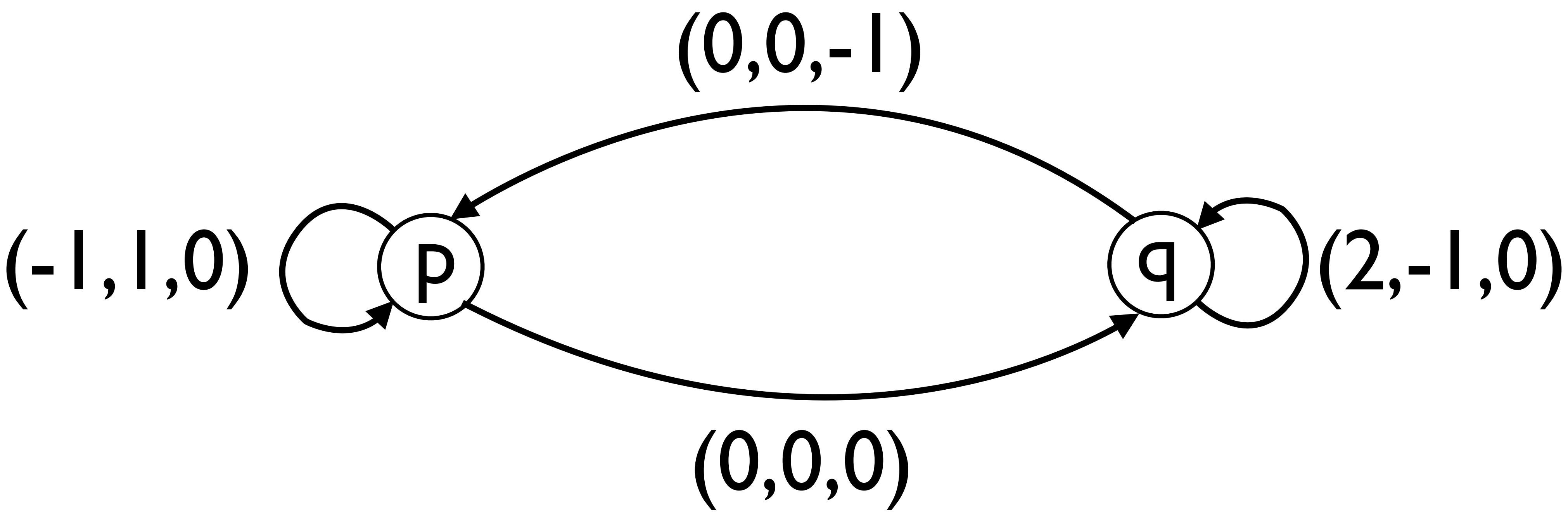
(-1, 1, 0, +1, 0, -1, 0)

# Why complicated?




For every state and loop a new dimension

(0, 0, -1, +1, -1, 0, 0) (0, 0, 0, -1, +1, 0, 0)

(0, 0, 0, -1, 0, +1, 0) (0, 0, 0, 0, -1, 0, +1)

(-1, 1, 0, +1, 0, -1, 0) (2, -1, 0, 0, +1, 0, -1)

# Hopcroft-Pansiot example

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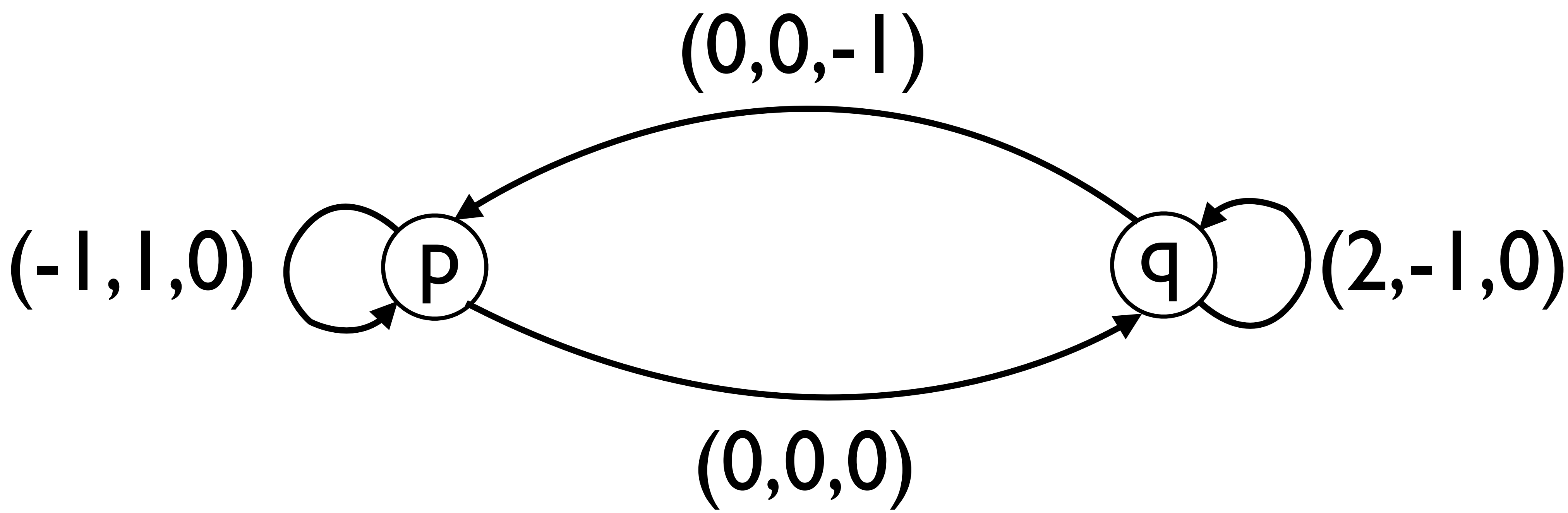

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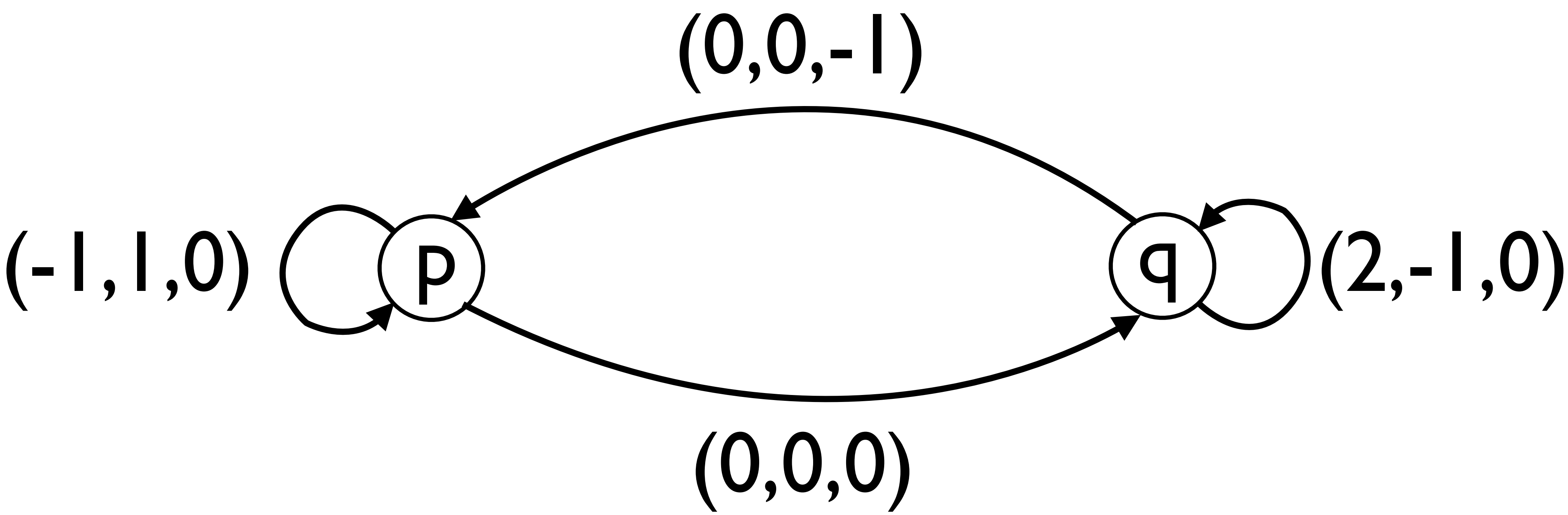
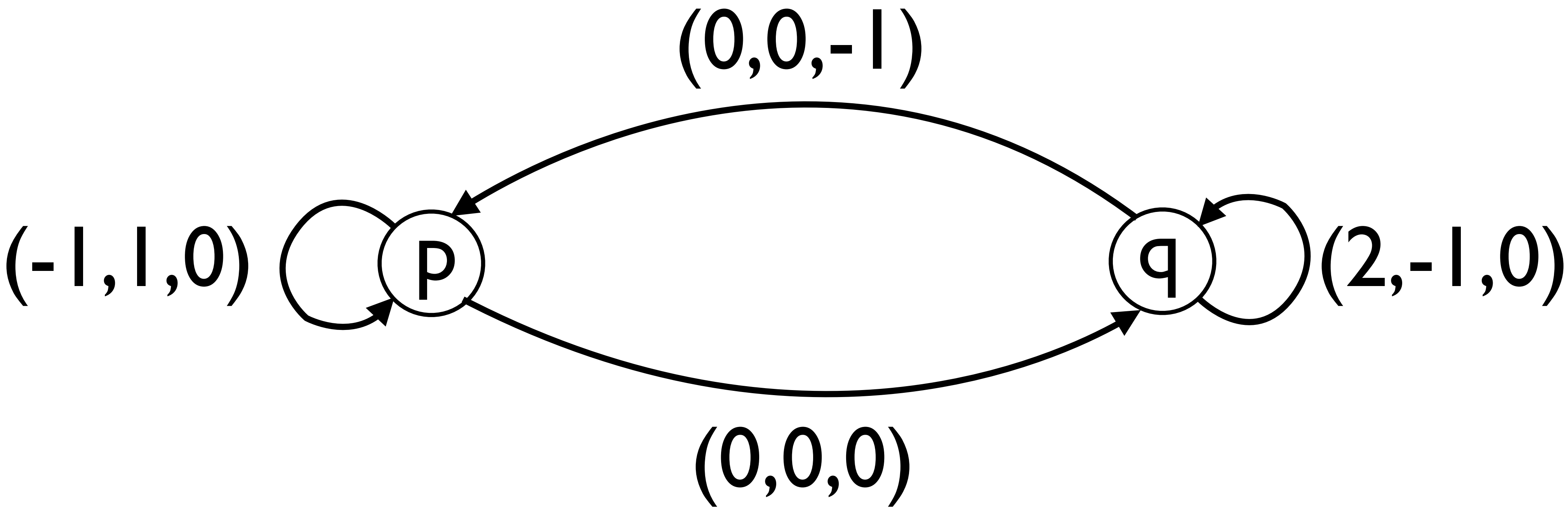


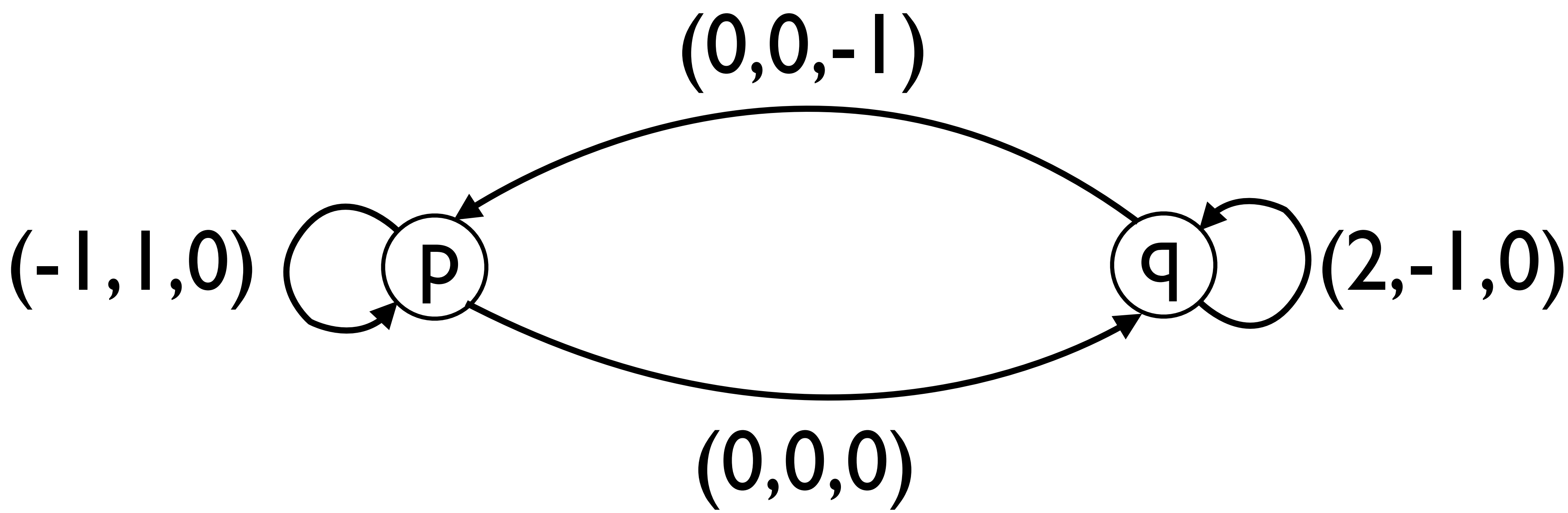

p(k,0,n)

# Hopcroft-Pansiot example



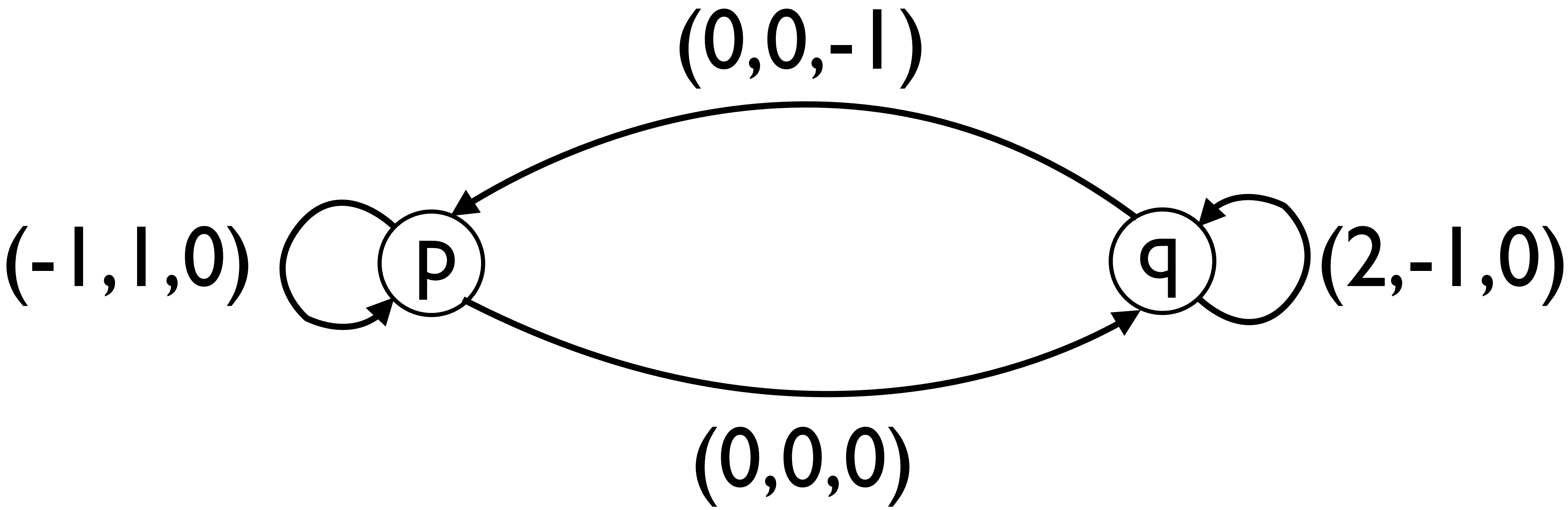

$p(k,0,n) \longrightarrow p(0,k,n)$

# Hopcroft-Pansiot example



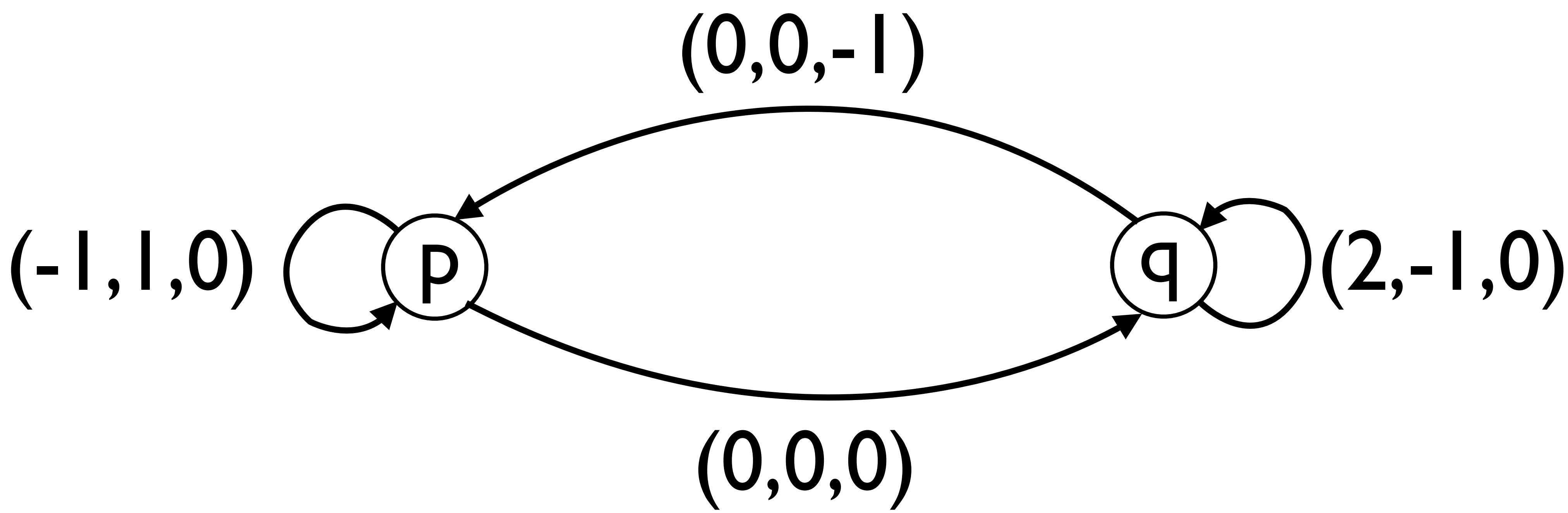

$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n)$$

# Hopcroft-Pansiot example




$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n) \longrightarrow q(2k,0,n)$$

# Hopcroft-Pansiot example




$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n) \longrightarrow q(2k,0,n) \longrightarrow p(2k,0,n-1)$$

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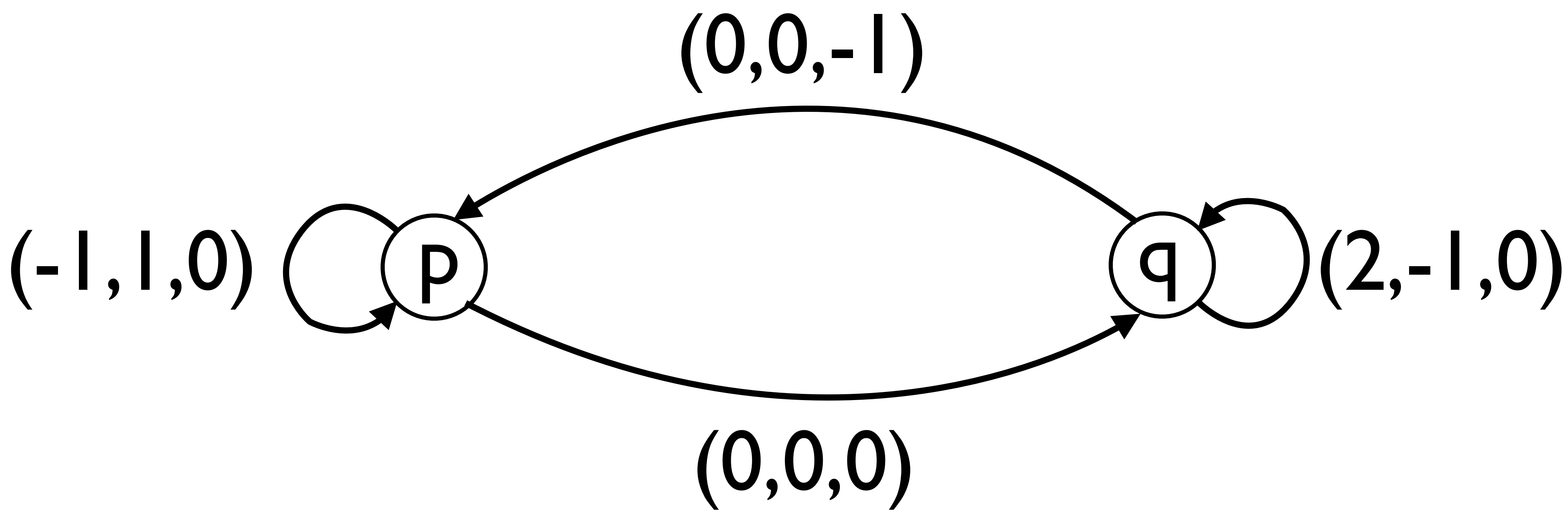


$$p(k,0,n) \longrightarrow p(0,k,n) \longrightarrow q(0,k,n) \longrightarrow q(2k,0,n) \longrightarrow p(2k,0,n-1)$$

$p(1,0,n)$

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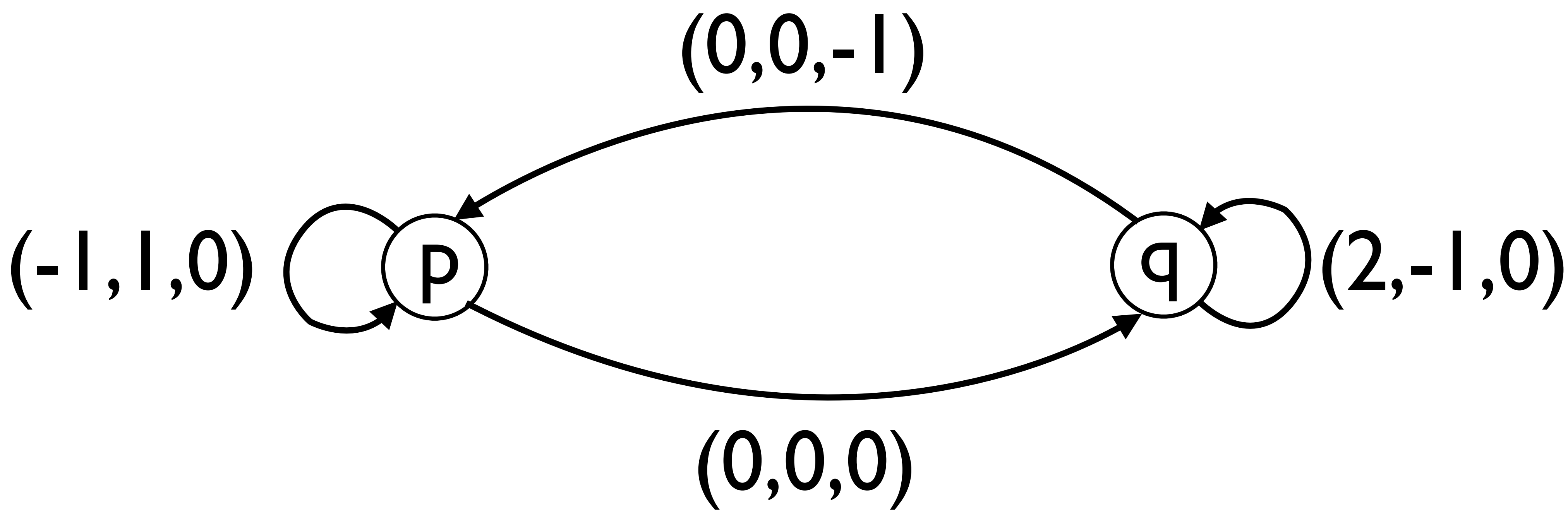


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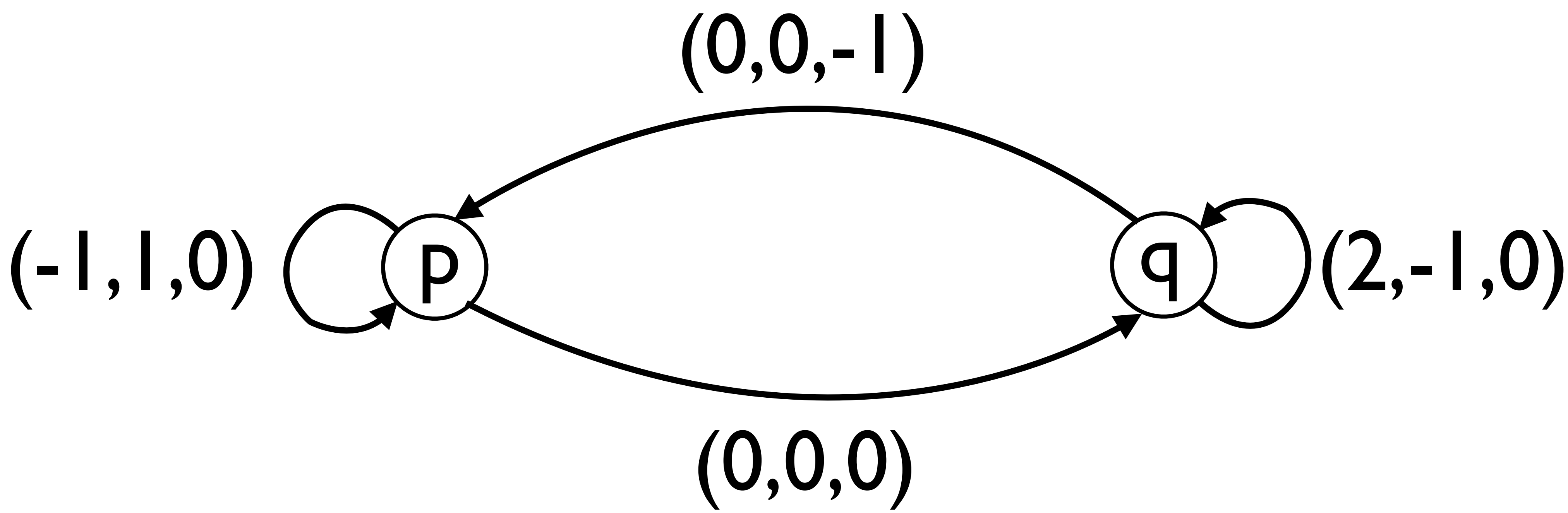


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$$p(1,0,n) \longrightarrow p(2,0,n-1) \longrightarrow$$

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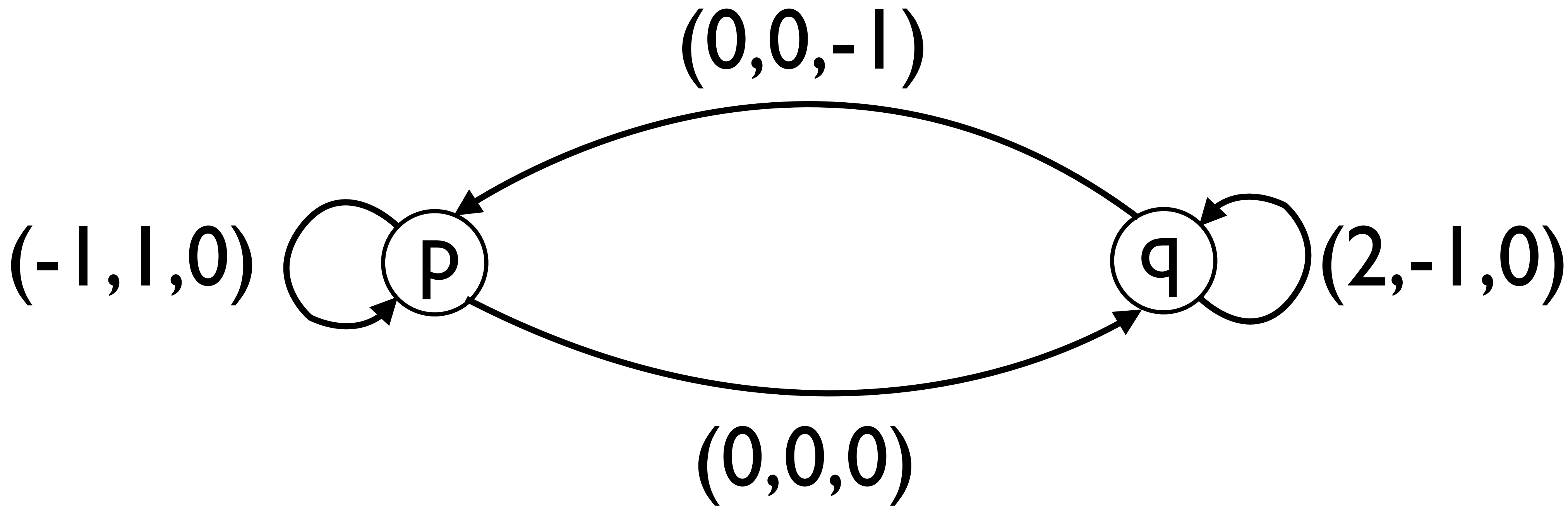


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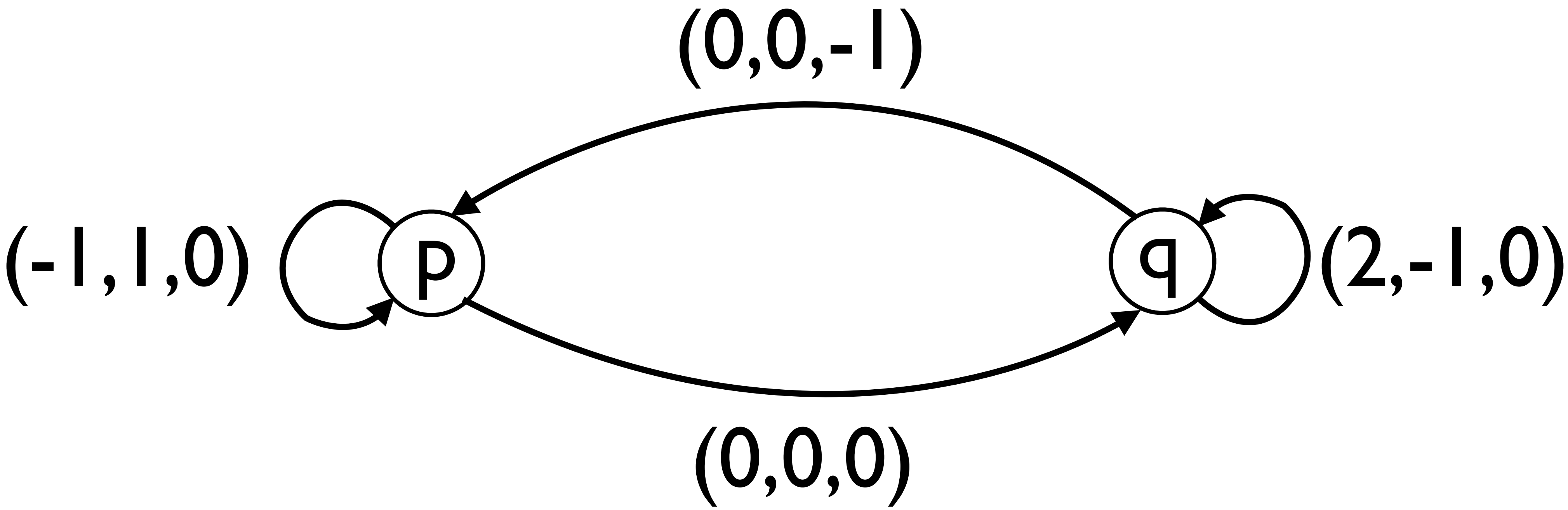


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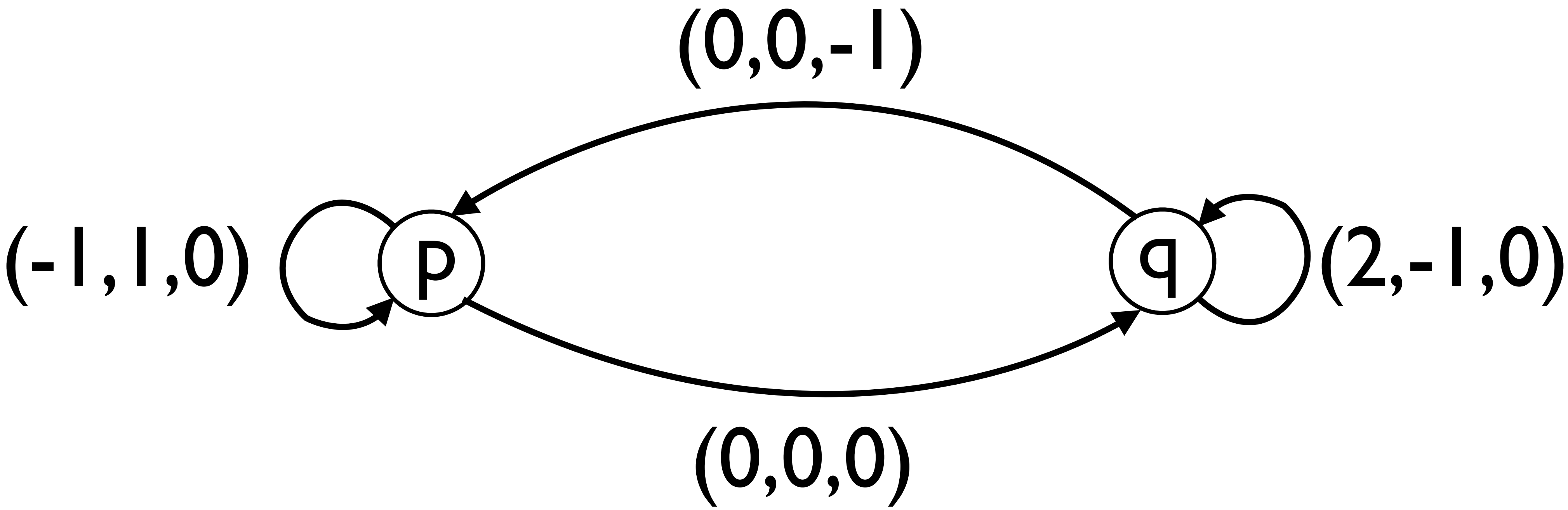


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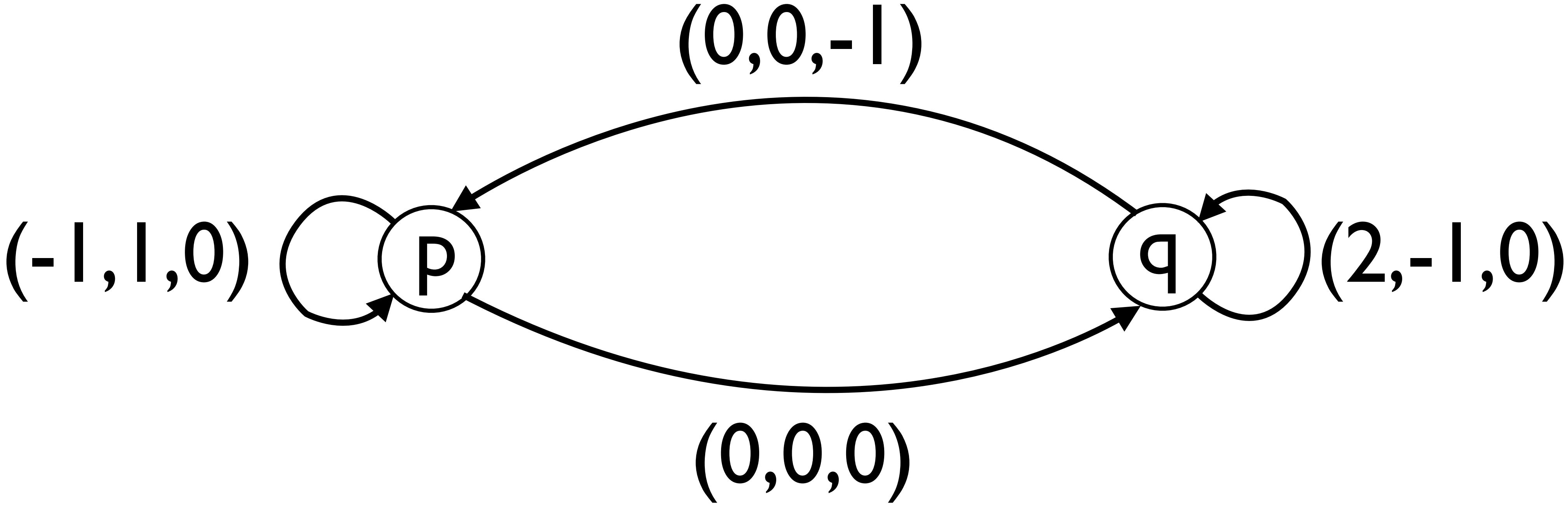


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Each $p(x, y, 0)$ for $x+y= 2^n$ is reachable

# Nesting the example

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$p(1, 0, n, 1)$

# Nesting the example

$$p(1, 0, n, 1) \longrightarrow$$

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$$p(1, 0, n, 1) \longrightarrow p(2^n, 0, 0, 1)$$

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$$p(1, 0, n, 1) \longrightarrow p(2^n, 0, 0, 1)$$

$$\downarrow$$

# Nesting the example

$$
\begin{tikzcd}
p(1, 0, n, 1) \arrow[r] & p(2^n,0,0, 1) \arrow[d] \\
& r(2^n,0,0, 1)
\end{tikzcd}
$$

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$$\begin{tikzcd} p(1, 0, n, 1) \arrow[r] & p(2^n,0,0, 1) \arrow[d] \\ {} & r(2^n,0,0, 1) \arrow[l] \end{tikzcd}$$

# Nesting the example

$$\begin{array}{ccc} p(1, 0, n, 1) & \longrightarrow & p(2^n,0,0,1) \\ & & \downarrow \\ r(0,0,0,2^{2^n}) & \longleftarrow & r(2^n,0,0,1) \end{array}$$

# Nesting the example

$$
\begin{tikzcd}
p(1, 0, n, 1) \arrow[r] & p(2^n,0,0, 1) \arrow[d] \\
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\end{tikzcd}
$$

Size of finite reachability set can 2-exp, 3-exp, ...

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$$\begin{tikzcd} p(1, 0, n, 1) \arrow[r] & p(2^n,0,0, 1) \arrow[d] \\ r(0,0,0, 2^{2^n}) & r(2^n,0,0, 1) \arrow[l] \end{tikzcd}$$

Size of finite reachability set can 2-exp, 3-exp, ...

Even ackermann size is possible

# Long paths?

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we have:

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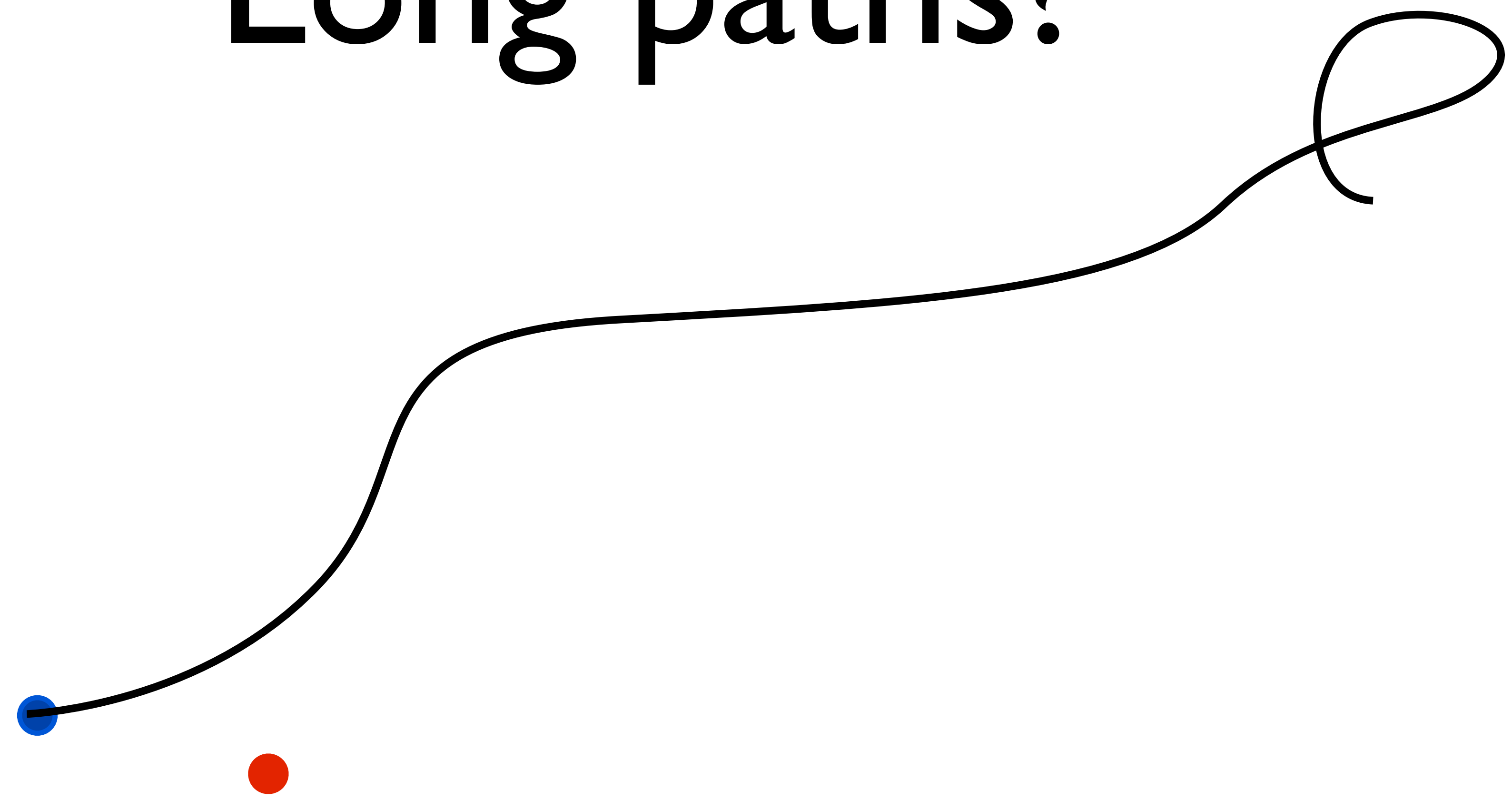

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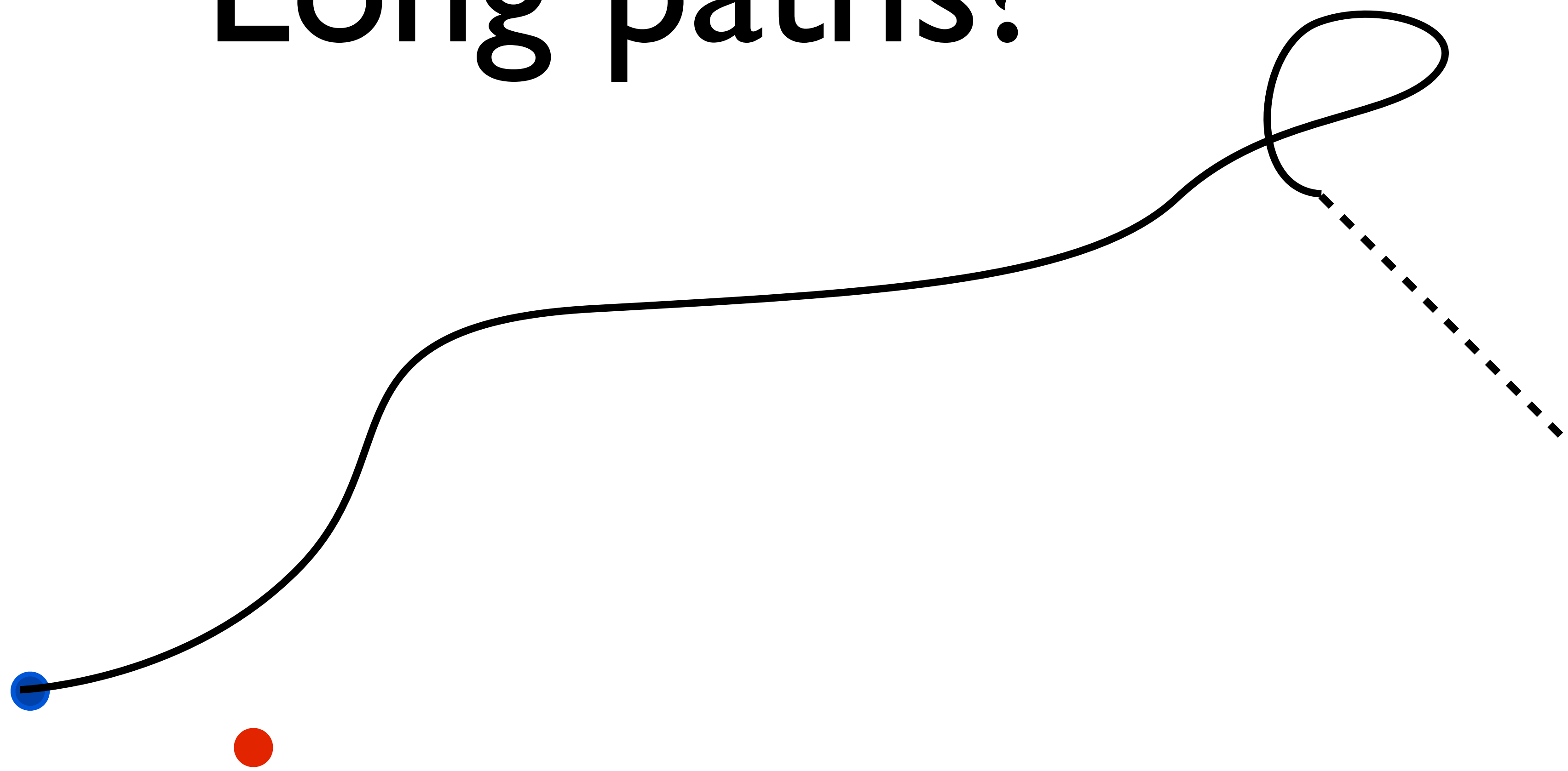

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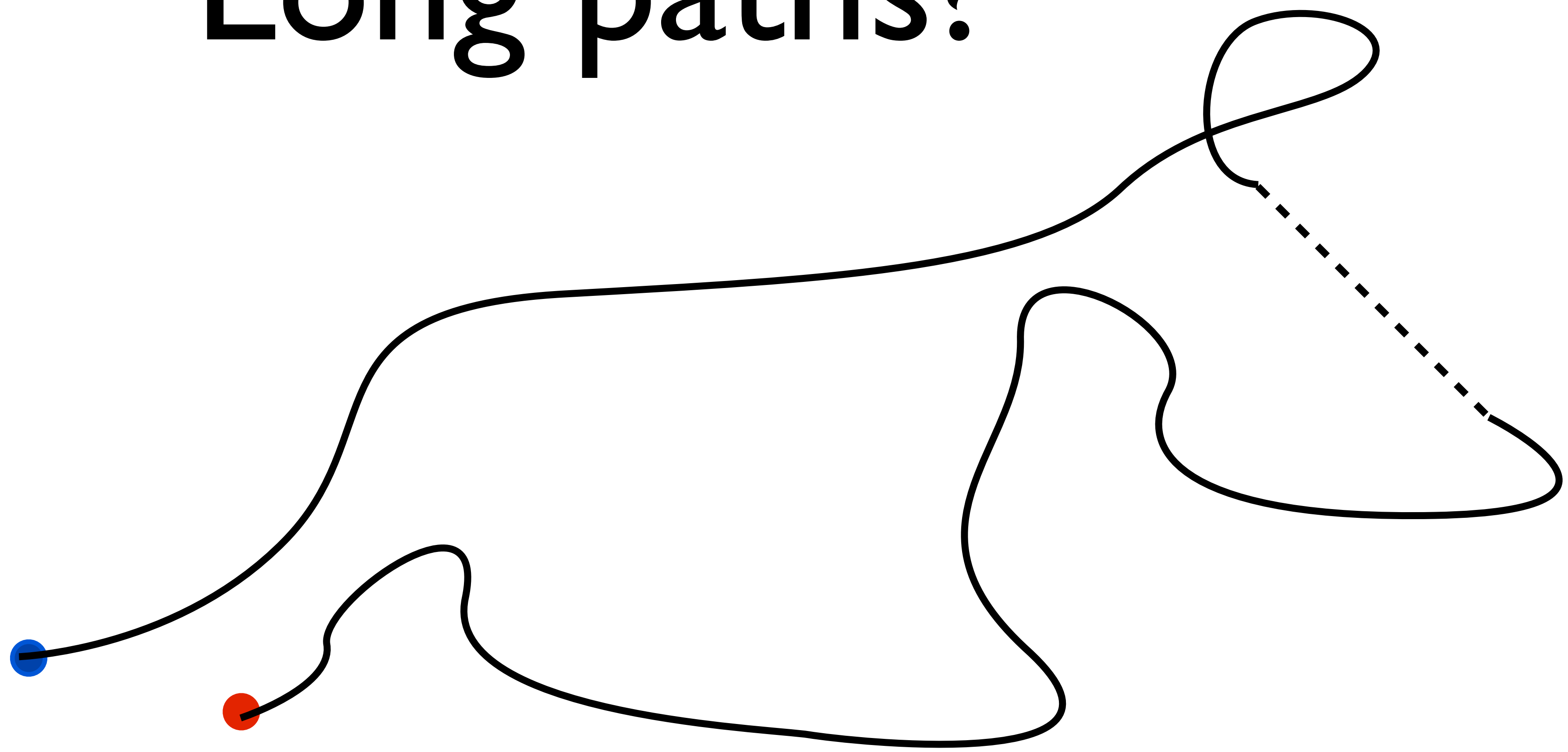

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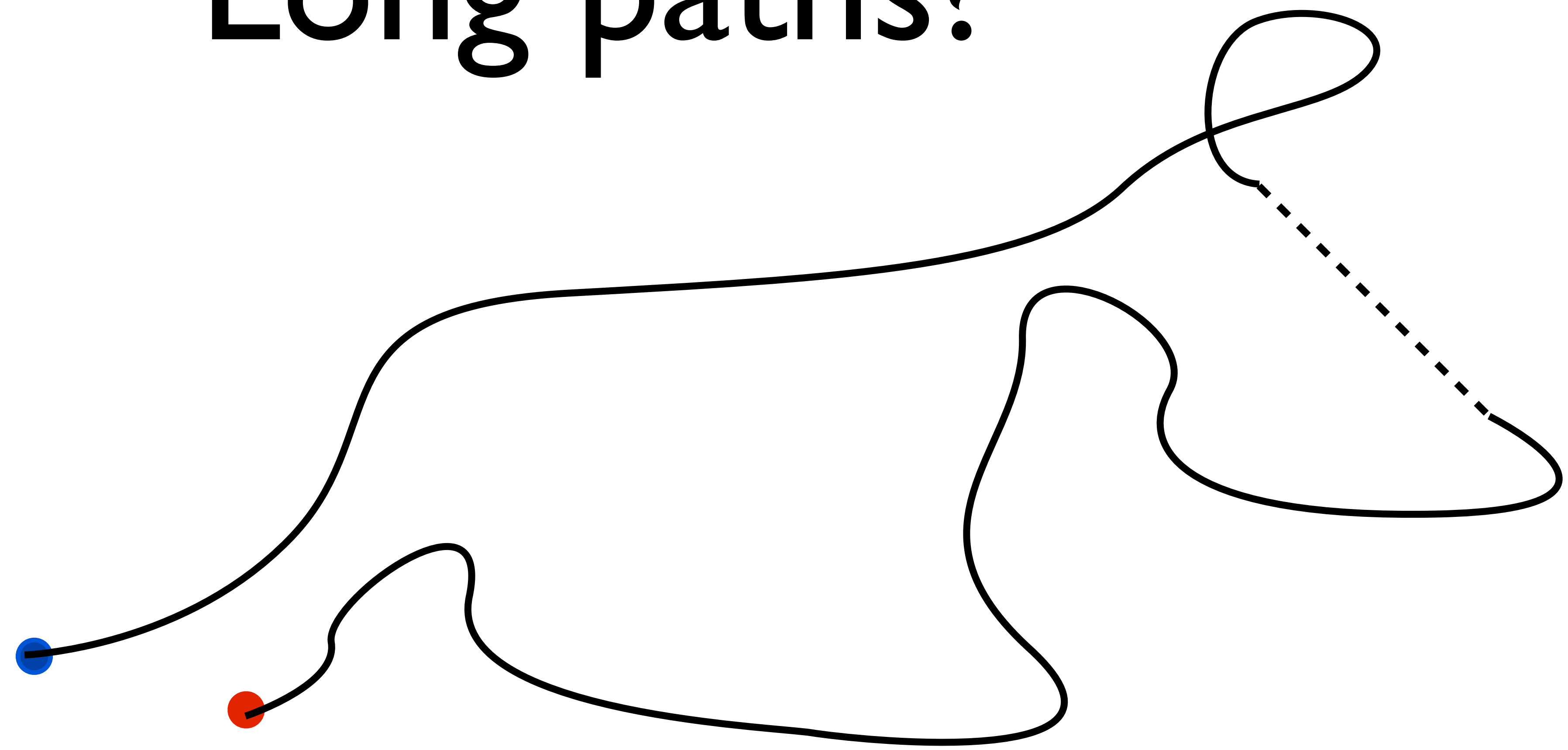

we have also:

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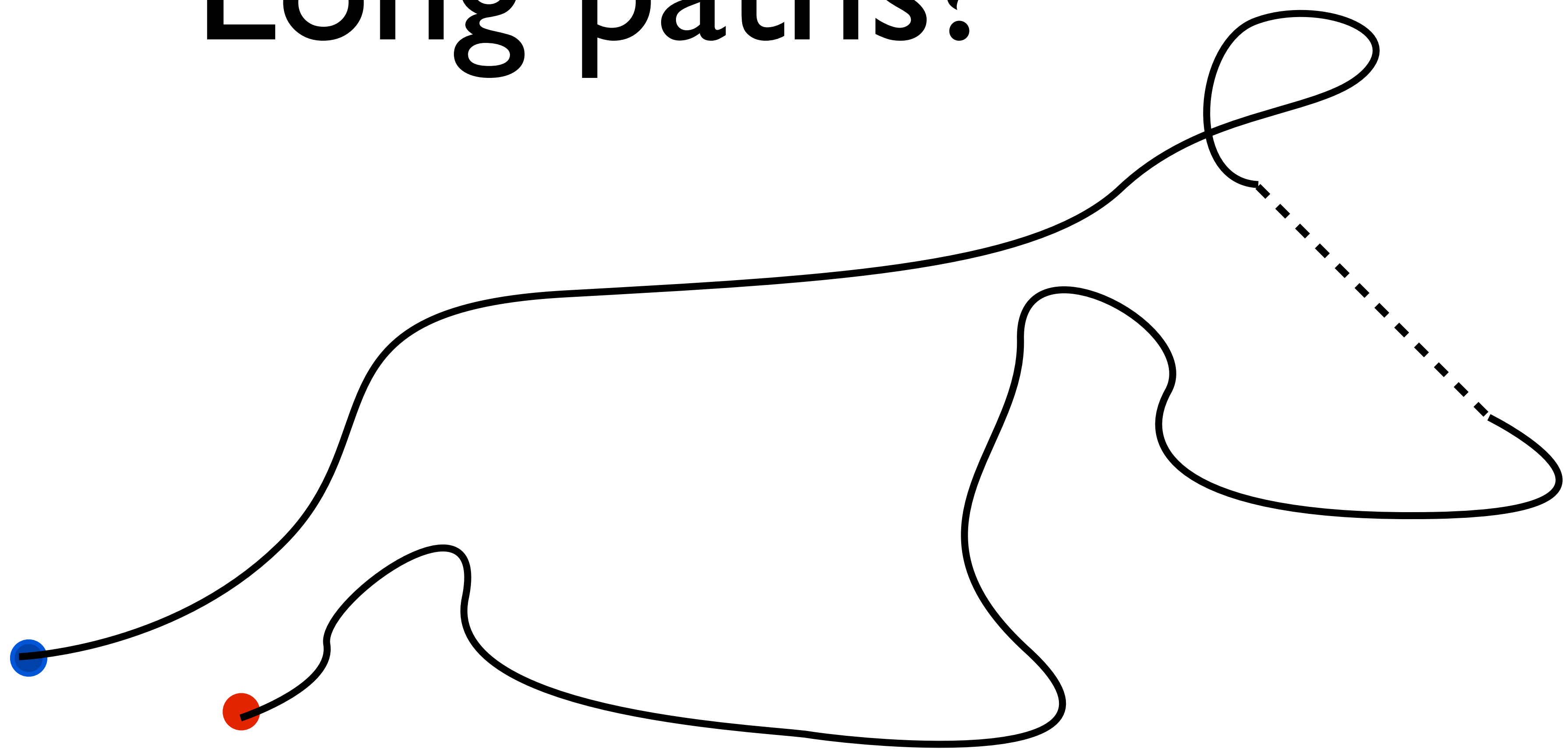

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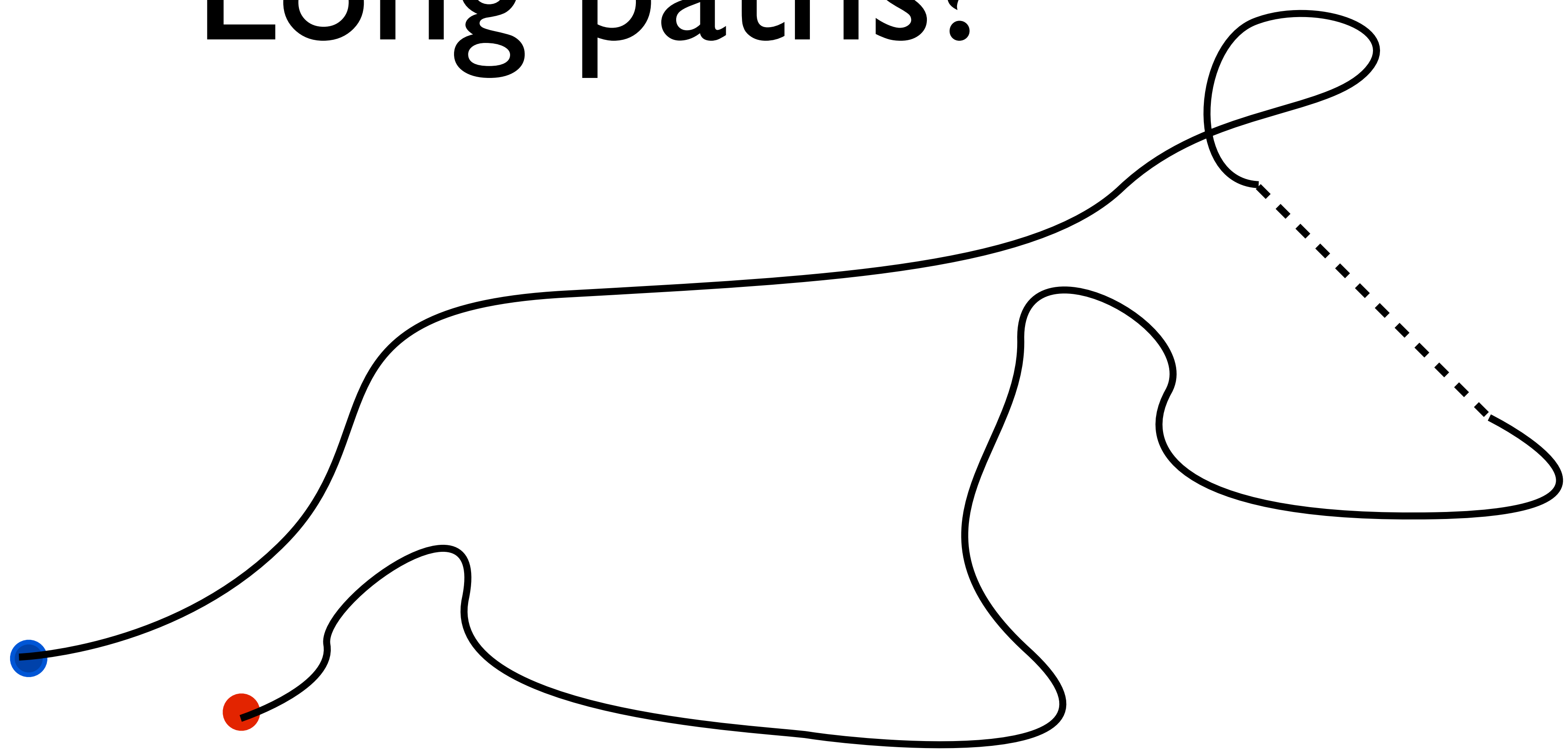

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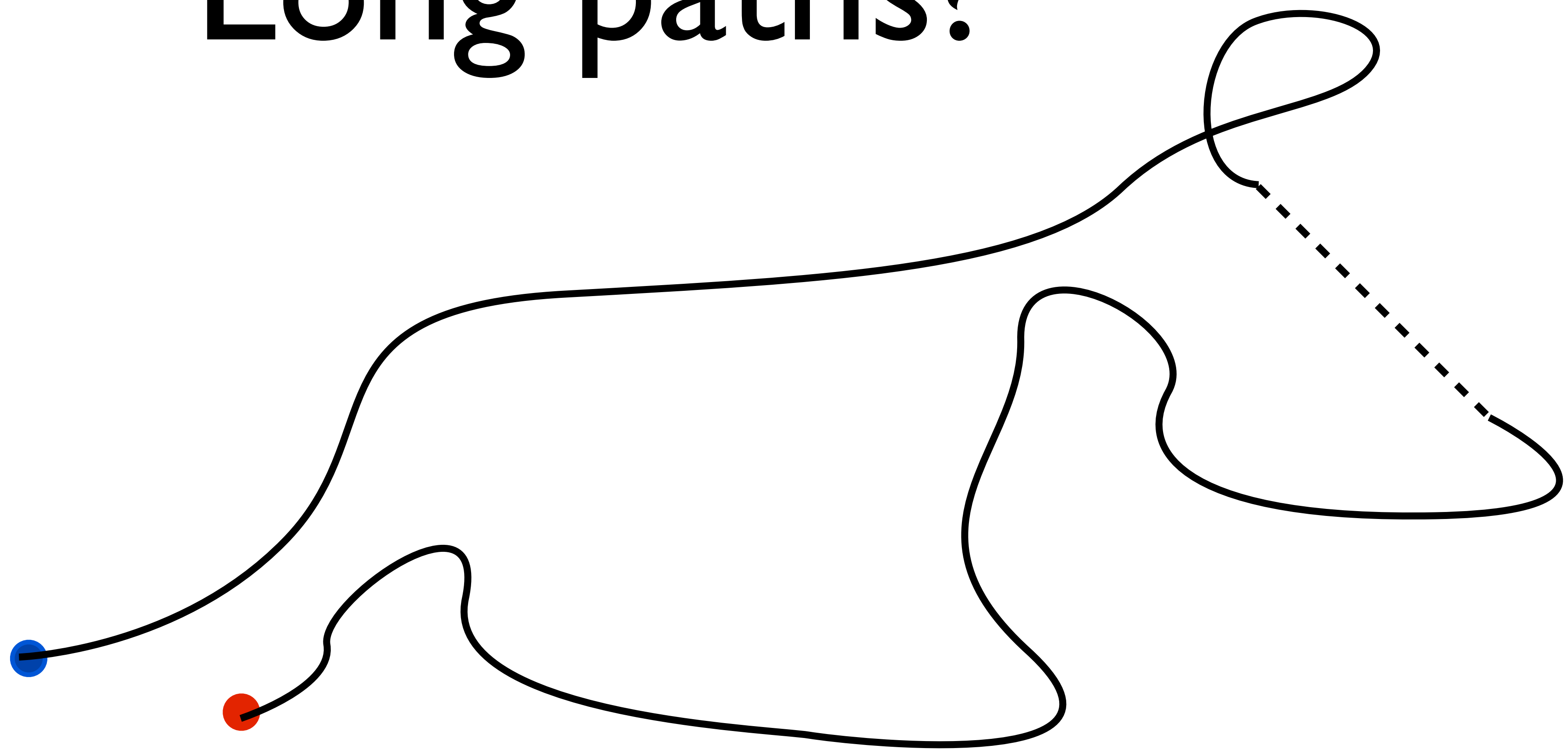

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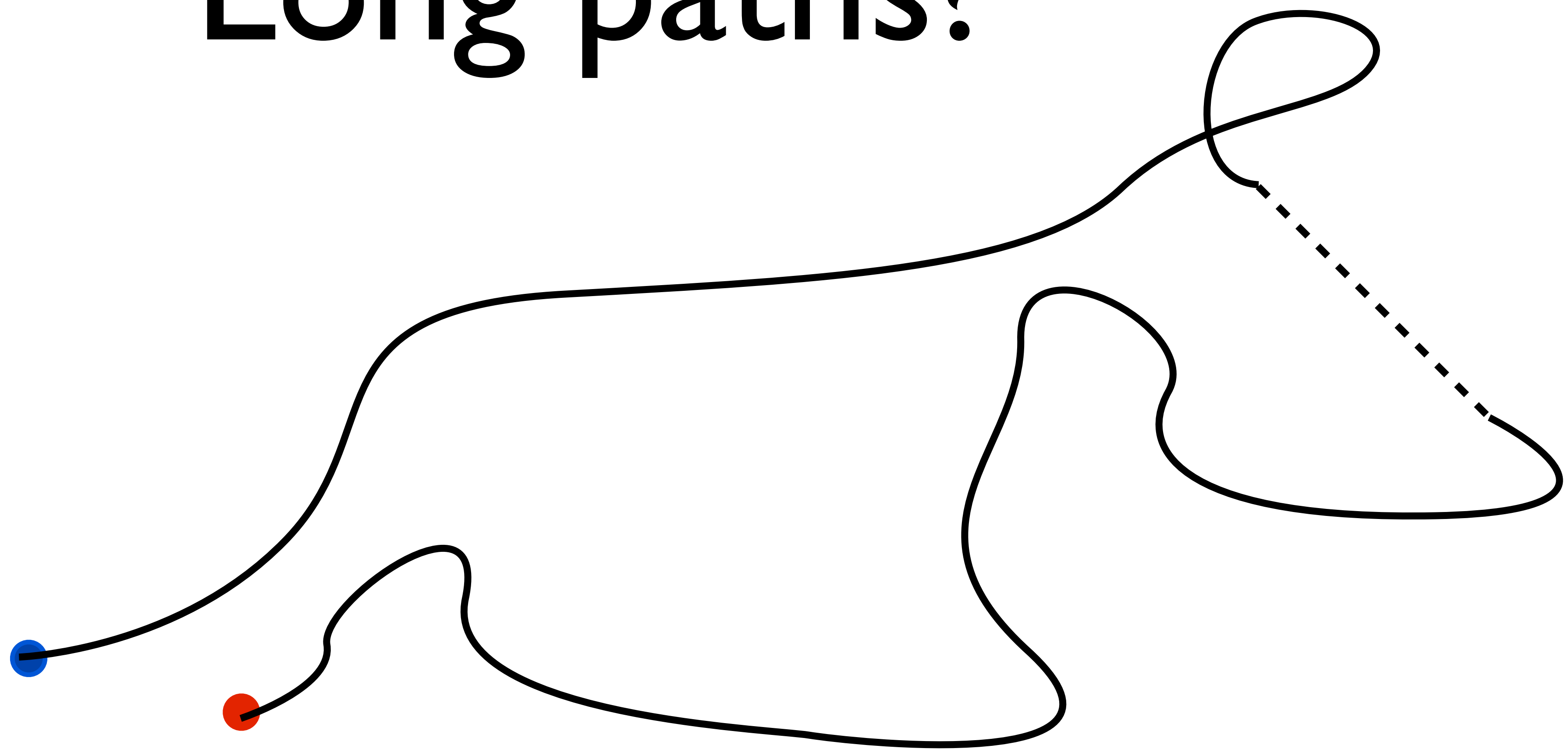

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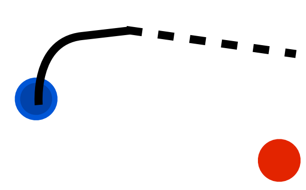

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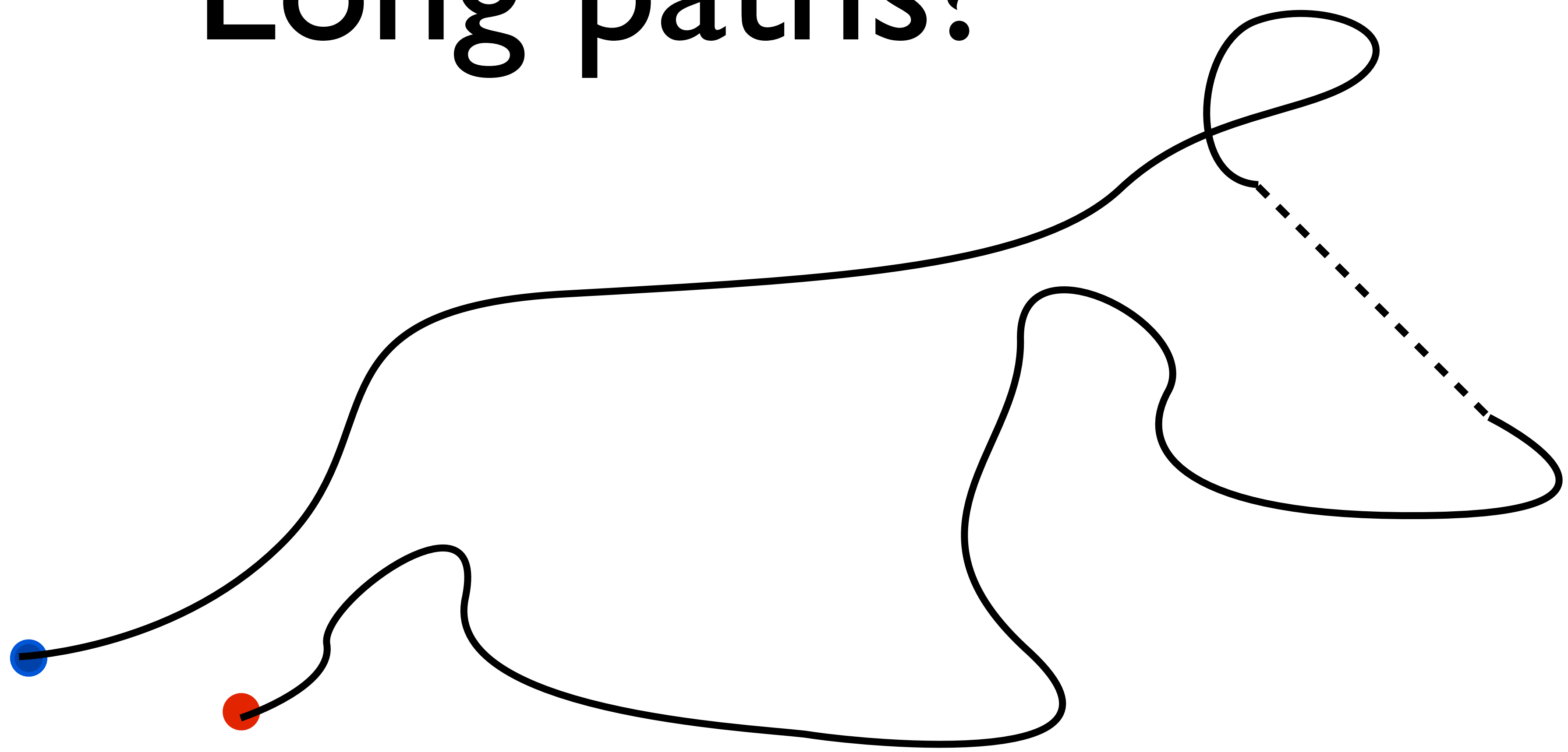

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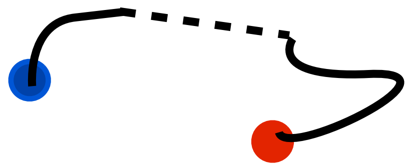

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This is not true for coverability!

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VASS (VAS with states) in dimension 4 such that
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is doubly-exponential

This is not true for coverability!

For every VASS in dimension 4
shortest path from source s above target t
is exponential (or no path)

# Fractional equation

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**Lemma**

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and all $a_i$, $b_i$, $a$ and $b$ are at most exponential in $k$.

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$$(1+2^k / 2^k)^{2^k} \approx e$$

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# VASS building blocks

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$$p(0,0,0,0) \longrightarrow p(Kb,0,0,K)$$

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$p(0, 0, 0, 0) \longrightarrow p(Kb, 0, 0, K)$ $(+b, 0, 0, +1)$ in p

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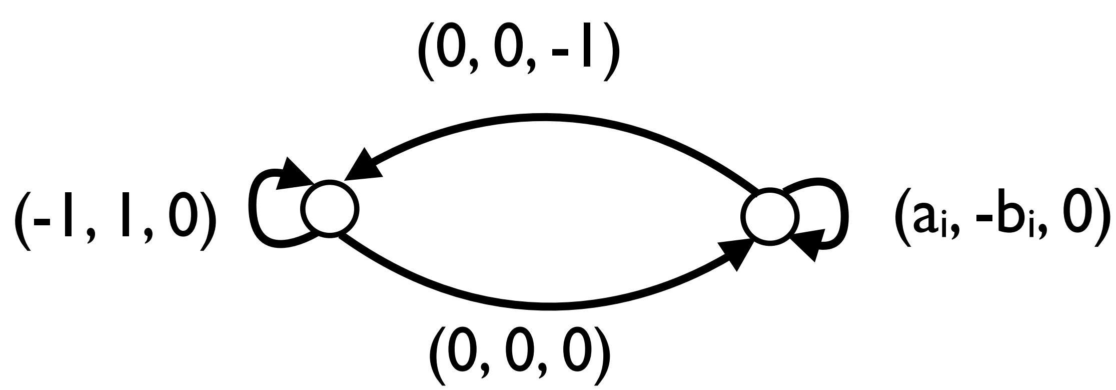

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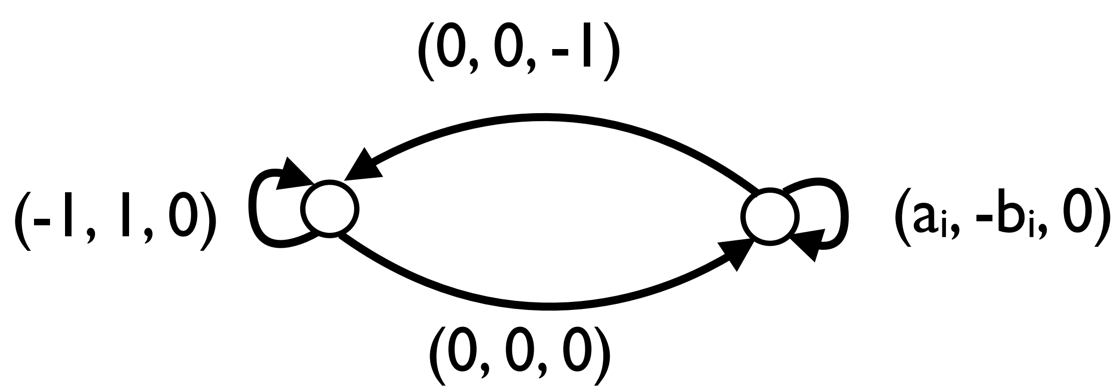


$$(Cb_i^n, 0, n) \longrightarrow (Ca_ib_i^{n-1}, 0, n-1) \longrightarrow \ldots \longrightarrow (Ca_i^n, 0, 0)$$

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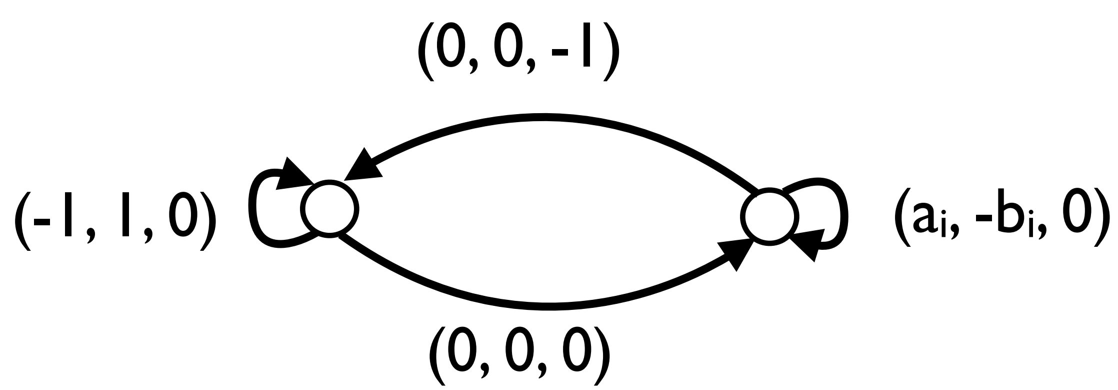


$(Cb_i^n, 0, n) \longrightarrow (Ca_ib_i^{n-1}, 0, n-1) \longrightarrow \ldots \longrightarrow (Ca_i^n, 0, 0)$

one can multiply by at most n times by at most $a_i / b_i$

# VASS implementation

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can I reach q(0,0,0,0) from p(0,0,0,0)?

# VASS implementation

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$$(a_1 \,/\, b_1)^{2^1} \cdot (a_2 \,/\, b_2)^{2^2} \cdot \ldots \cdot (a_k \,/\, b_k)^{2^k} = a \,/\, b$$

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adds to first coordinate **K**b

# VASS implementation

$$(a_1 / b_1)^{2^1} \cdot (a_2 / b_2)^{2^2} \cdot \ldots \cdot (a_k / b_k)^{2^k} = a / b$$

adds to first coordinate $\mathbf{K}b$ $\quad$ $p(\mathbf{K}b, 0, 0, \mathbf{K})$

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$q(\leq \mathbf{K}a, \geq 0, \geq 0, \mathbf{K})$

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$\mathbb{N}$

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multiplies it at most $2^1$ times by $\leq a_1 / b_1$

...

multiplies it at most $2^k$ times by $\leq a_k / b_k$

$\mathbf{N}$ $\qquad$ subtracts from it $\mathbf{K}a$ $\qquad$ $q(\leq \mathbf{K}a, \geq 0, \geq 0, \mathbf{K})$

$\mathbf{N}$ divisible by $b_k^{2^k}$

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$\mathbf{N}$

**K doubly exponential**

Thank you!