Property A and duality in linear programming

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What is … property A?

Property A

Property A was introduced in 2000 and turns out to be of great importance in many areas of mathematics [1]. Perhaps the most striking example is the following implication that follows from results in [4].

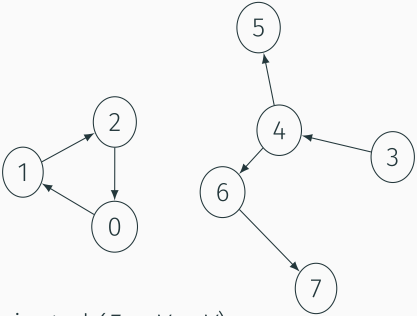
> *If group G has Property A then the Novikov conjecture is true for all closed manifolds with fundamental group G.*

The Novikov conjecture asserts homotopy invariance of higher signatures of smooth manifolds. Is is one of most important unsolved problems in topology.

[1] P. Nowak, G. Yu, ”What is … property A?”, Notices AMS (2008)

Graph as a metric space

Path-length metric on a graph.



- Graphs are oriented ($E \subset V \times V$).
- We allow infinite distance.
- If $ij \in E$ is used to denote an edge, then i is the source and j is the target vertex of ij.

Minimal variation of probability measures at scale S

Optimization Problem I. Let $G = (V, E)$ be a graph and let $S \geq 0$. Find minimal $\varepsilon = \varepsilon_{S,G}$ (variation) and a family $\{\xi_i \colon V \to \mathbb{R}\}_{i \in V}$ of functionals (probability measures) such that

1. Each ξ_i is a probability measure, i.e.

$$\|\xi_i\|_1 = 1 \text{ and } \xi_i \geq 0 \text{ for each } i \in V;$$

2. Variation on edge ij does not exceed ε, i.e.

$$\|\xi_i - \xi_j\|_1 \leq \varepsilon \text{ for each } ij \in E;$$

3. Each ξ_i is supported by $B(i, S)$, i.e.

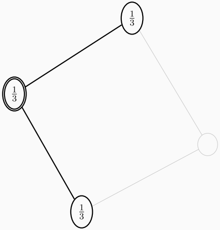
$$\operatorname{supp} \xi_i = \{j \in V \colon \xi_i(j) > 0\} \subset B(i, S) \text{ for each } i \in V,$$

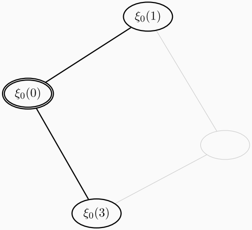
where $B(i, S)$ is ball of radius S centered at i.

Minimal example: square graph at scale 1

Optimization Problem I. Find minimal $\varepsilon_{S,G} > 0$ with functionals ξ_i on G satisfying

1. $\|\xi_i\|_1 = 1$ and $\xi_i \geq 0$ for each $i \in V$;
2. $\|\xi_i - \xi_j\|_1 \leq \varepsilon$ for each $ij \in E$;
3. $\operatorname{supp} \xi_i = \{j \in V \colon \xi_i(j) > 0\} \subset B(i, S)$ for each $i \in V$.





	$\boldsymbol{\xi_0}$	ξ_1	ξ_2	ξ_3
0	$\mathbf{\frac{1}{3}}$	$\frac{1}{3}$		$\frac{1}{3}$
1	$\mathbf{\frac{1}{3}}$	$\frac{1}{3}$	$\frac{1}{3}$	
2		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
3	$\mathbf{\frac{1}{3}}$		$\frac{1}{3}$	$\frac{1}{3}$

This is optimal solution with objective $\varepsilon = \frac{2}{3}$.

Optimality of the solution is not trivial to show.

Property A

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Property A

Let G be a graph and for each $S \geq 0$ let $\varepsilon_{S,G}$ be the minimal variation of probability measures on G at scale S, i.e. the solution of Optimization Problem I at scale S. We say that G has **property A** iff

$$\lim_{S \to \infty} \varepsilon_{S,G} = 0.$$

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- To prove property A for G it is enough to find upper bounds $\varepsilon_{S,G} \leq \hat{\varepsilon}_{S,G}$ such that
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- To prove property A for G it is enough to find upper bounds $\varepsilon_{S,G} \leq \hat{\varepsilon}_{S,G}$ such that
$$\lim_{S\to\infty} \hat{\varepsilon}_{S,G} = 0.$$
- To prove that G does not have property A we have to show that
$$\limsup_{S\to\infty} \varepsilon_{S,G} > 0.$$
so we have to consider optimal solutions $\varepsilon_{G,S}$.

Reduction to finite graphs

- If $S \geq \operatorname{diam} G$, then $\varepsilon_{S,G} = 0$, so property A is trivial for finite graphs.

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Theorem
Let G be a graph and assume that $G = \bigcup_{n \in \mathbb{N}} G_n$, with $G_1 \subset G_2 \subset G_3 \subset \cdots$ an ascending sequence of convex subgraphs of G. Let ε_{S,G_n} be the minimal variation of probability measures at scale S for graph G_n. Graph G has property A iff

$$\lim_{S \to \infty} \lim_{n \to \infty} \varepsilon_{S,G_n} = 0.$$

Reduction to finite graphs

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$$\lim_{S\to\infty} \lim_{n\to\infty} \varepsilon_{S,G_n} = 0.$$

Important case: disjoint sum of finite subgraphs.

Graphs without property A

Graph G has property A iff

$$\lim_{S\to\infty}\lim_{n\to\infty}\varepsilon_{S,G_n}=0.$$

To check that

$$\lim_{S\to\infty}\lim_{n\to\infty}\varepsilon_{S,G_n}\neq 0$$

we need a lower bound on ε_{S,G_n}.

Property A as a linear problem and the dual problem

Linear programming formulation of minimal variance problem

Optimization Problem I. Find minimal $\varepsilon_{S,G} > 0$ with functionals ξ_i on G satisfying

1. $\|\xi_i\|_1 = 1$ and $\xi_i \geq 0$ for each $i \in V$;
2. $\|\xi_i - \xi_j\|_1 \leq \varepsilon$ for each $ij \in E$;
3. $\operatorname{supp} \xi_i = \{j \in V\colon \xi_i(j) > 0\} \subset B(i, S)$ for each $i \in V$.

Primal problem.

$$\text{minimize} \quad e$$

$$\begin{aligned}
\text{subject to} \quad & \sum_{j \in V} x_{i,j} = 1 \text{ for each } i \in V, \\
& x_{i,j} = 0 \text{ for each } i \in V, j \in V \setminus B(i,S), \\
& x_{j,k} - x_{i,k} \leq e_{ij,k} \text{ for each } ij \in E, k \in V, \\
& x_{i,k} - x_{j,k} \leq e_{ij,k} \text{ for each } ij \in E, k \in V, \\
& \sum_{k \in V} e_{ij,k} \leq e \text{ for each } ij \in E, \\
& x_{i,j}, e_{ij,k}, e \geq 0
\end{aligned}$$

The dual problem

Dual problem.

$$\text{maximize} \quad \sum_{i \in V} \eta_i$$

$$\text{subject to}$$

$$\begin{aligned}
\sum_{ij \in E} \kappa_{ij} &\le 1 \ , \\
\varphi_{k,ij} &\le \kappa_{ij} \text{ for each } ij \in E, k \in V, \\
-\varphi_{k,ij} &\le \kappa_{ij} \text{ for each } ij \in E, k \in V, \\
\sum_{mi \in E, m \in V} \varphi_{k,mi} - \sum_{im \in E, m \in V} \varphi_{k,im} &\ge \eta_i \ \text{ for each } k \in V, i \in B(k,S), \\
\eta_i, \varphi_{k,ij} \in \mathbb{R}, \ \ \kappa_{ij} &\ge 0
\end{aligned}$$

Primal vs dual

Primal

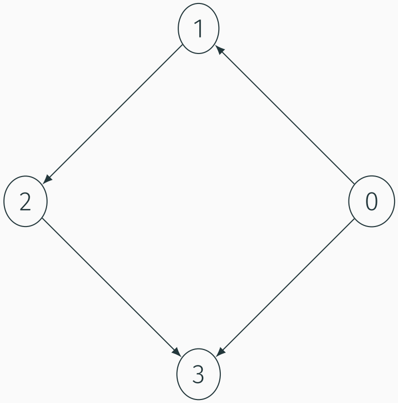
$$\begin{aligned} \min \quad & e \\ \text{s.t.} \quad & \sum_{j \in V} x_{i,j} = 1, \\ & x_{i,j} = 0, \\ & x_{j,k} - x_{i,k} \leq e_{ij,k}, \\ & x_{i,k} - x_{j,k} \leq e_{ij,k}, \\ & \sum_{k \in V} e_{ij,k} \leq e, \\ & x_{i,j}, e_{ij,k}, e \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} \max \quad & \sum_{i \in V} \eta_i \\ \text{s.t.} \quad & \\ & \sum_{ij \in E} \kappa_{ij} \leq 1, \\ & \varphi_{k,ij} \leq \kappa_{ij}, \\ & -\varphi_{k,ij} \leq \kappa_{ij}, \\ & \sum_{mi \in E, m \in V} \varphi_{k,mi} - \sum_{im \in E, m \in V} \varphi_{k,im} \geq \eta_i, \\ & \eta_i, \varphi_{k,ij} \in \mathbb{R}, \kappa_{ij} \geq 0 \end{aligned}$$

Examples: primal and dual solutions

Square graph



Square graph - proof of optimality

$$\|\xi_1-\xi_0\|_1 = |\xi_1(0)-\xi_0(0)|+|\xi_1(1)-\xi_0(1)|+|\xi_1(2)-\xi_0(2)|+|\xi_1(3)-\xi_0(3)| \leq \varepsilon$$

$$\Downarrow$$

$$-\frac{1}{12}(\xi_1(0)-\xi_0(0))+\frac{1}{12}(\xi_1(1)-\xi_0(1))+\frac{1}{4}(\xi_1(2)-0)-\frac{1}{4}(0-\xi_0(3)) \leq \frac{1}{4}\varepsilon$$

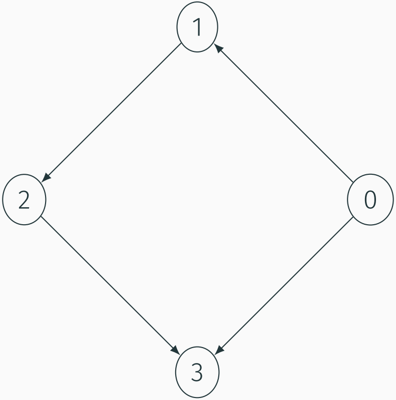
Duality

Theorem
Dual problem at scale S is dual to Primal problem at scale S. In particular, for each admissible solution of each problem, we have

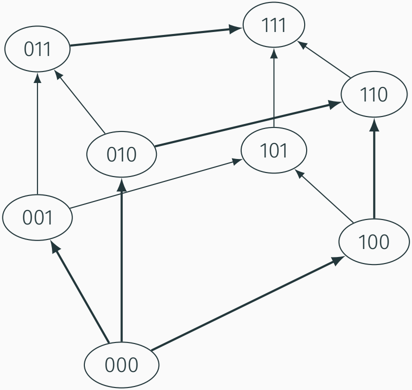
$$\sum_{i \in V} \eta_i \leq e$$

and the optimal solutions are equal.

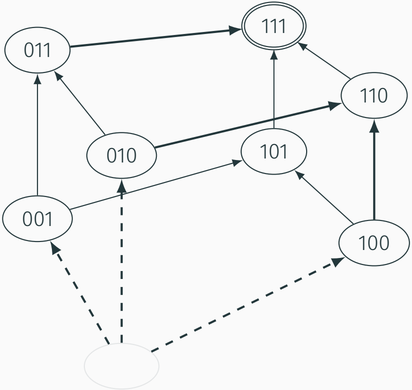
Square graph - solution by hand



Cube graph, $S = 2$, primal solution



Cube graph, $S = 2$, dual solution



Primal relaxation: Cheeger constant

Capacity and supply is implicit if we know flows

Note that the capacity κ and supply η is implicit in the solution of the dual problem as the optimal values for chosen pseudo-flows φ_i can be easily computed.

Dual Problem

$$\text{maximize} \quad \sum_{i \in V} \eta_i$$

$$\text{subject to}$$

$$\sum_{ij \in E} \kappa_{ij} \leq 1 \quad ,$$

$$\varphi_{k,ij} \leq \kappa_{ij} \text{ for each } ij \in E, k \in V,$$

$$-\varphi_{k,ij} \leq \kappa_{ij} \text{ for each } ij \in E, k \in V,$$

$$\sum_{mi \in E, m \in V} \varphi_{k,mi} - \sum_{im \in E, m \in V} \varphi_{k,im} \geq \eta_i \text{ for each } k \in V, i \in B(k,S),$$

$$\eta_i, \varphi_{k,ij} \in \mathbb{R}, \quad \kappa_{ij} \geq 0$$

Averaged solutions

Theorem

Let G be a graph. Let Γ be a group that acts on G by automorphisms.

If Γ acts transitively both on edges and on vertices of G, then there exists an optimal solution of the dual problem such that $\eta_i = \eta_j$ *for each* $i, j \in V$ *and* $\varepsilon_{ij} = \frac{1}{|E|}$ *for each* $ij \in E$.

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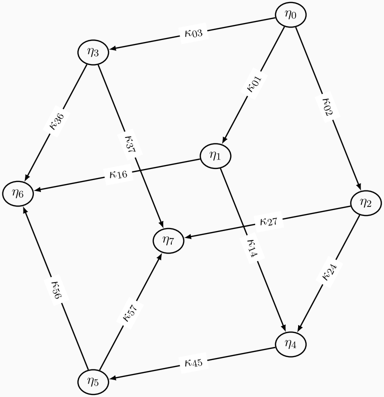
This is not always the case.

Forced equal capacities and supplies



i	$\sigma_{0,i}$	$\sigma_{1,i}$	$\sigma_{2,i}$	$\sigma_{3,i}$	$\sigma_{4,i}$	$\sigma_{5,i}$	$\sigma_{6,i}$	$\sigma_{7,i}$	$\min_k \sigma_{k,i}$
0	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$		$\frac{1}{28}$	$\frac{1}{28}$	$\eta_0 = \frac{1}{28}$
1	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$		$\eta_1 = \frac{1}{28}$
2	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$		$\frac{1}{28}$	$\eta_2 = \frac{1}{28}$
3	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$		$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\eta_3 = \frac{1}{28}$
4	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$		$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\eta_4 = \frac{1}{28}$
5		$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\eta_5 = \frac{1}{28}$
6	$\frac{1}{28}$	$\frac{1}{28}$		$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\eta_6 = \frac{1}{28}$
7	$\frac{1}{28}$		$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\frac{1}{28}$	$\eta_7 = \frac{1}{28}$
									$\sum = \frac{2}{7}$

$$\sigma_{k,i} = \sum_{mi \in E, m \in V} \varphi_{k,mi} - \sum_{im \in E, m \in V} \varphi_{k,im}$$

The dual problem with extra constraints

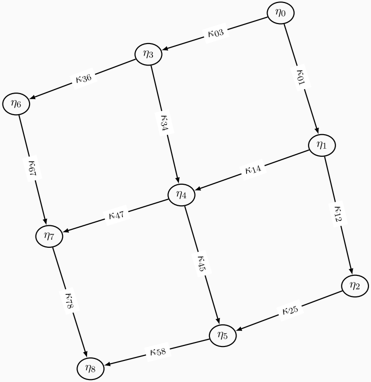
Uniform flows.

$$\text{maximize} \quad |V| \cdot \eta$$

subject to

$$\begin{aligned}
\varepsilon_{ij,k} &\le \frac{1}{|E|} \text{ for each } ij \in E, k \in V, \\
-\varepsilon_{ij,k} &\le \frac{1}{|E|} \text{ for each } ij \in E, k \in V, \\
\sum_{j \in V, ji \in E} \varepsilon_{ji,k} - \sum_{j \in V, ij \in E} \varepsilon_{ij,k} &\ge \eta \quad \text{for each } k \in V, i \in B(k,S), \\
\eta, \varepsilon_{ij,k} &\in \mathbb{R}
\end{aligned}$$

3×3 grid, Uniform Flows problem at scale $S = 1$



Minimum isoperimetric number

Let $S \subset V$. The **edge boundary of** S is $\partial S = E[S, V \setminus S]$. For $S \neq \emptyset$ we let

$$\varphi(S) = \frac{|\partial S|}{|S|}$$

be the **isoperimetric number of** S.

Let $S \geq 0$ be a scale on a graph G. We let

$$\gamma(G, \mathcal{S}) = \min_{T \subset B(i,S), i \in V, T \neq \emptyset} \varphi(T)$$

be the **Cheeger constant** of G at scale S.

Uniform Flows and Cheeger constant

Theorem

The optimal solution of Uniform Flow problem at scale S is equal to

$$\frac{|V|}{|E|}\gamma(G,S),$$

the Cheeger constant of G at scale S multiplied by $\frac{|V|}{|E|}$.

Minimal isoperimetric number over $B(k, S)$

Minimal isoperimetric number

$$\text{maximize} \quad \eta$$

$$\text{subject to}$$

$$\begin{aligned} \varepsilon_{ij} &\leq 1 \text{ for each } ij \in E, \\ -\varepsilon_{ij} &\leq 1 \text{ for each } ij \in E, \\ \sum_{j \in V, ji \in E} \varepsilon_{ji} - \sum_{j \in V, ij \in E} \varepsilon_{ij} &\geq \eta \text{ for each } i \in B(k, S), \\ \eta, \varepsilon_{ij} &\in \mathbb{R} \end{aligned}$$

Minimal isoperimetric number over $B(k, S)$

Minimal isoperimetric number - the dual

$$\begin{aligned}
&\text{minimize} \quad \sum_{ij\in E} |a_i - a_j| \\
&\text{subject to} \\
&\sum_{i\in S} a_i = 1, \\
&\quad a_i = 0 \ \text{ for each } i \in V \setminus B(k, S), \\
&\quad a_i \geq 0 \ \text{ for each } i \in B(k, S)
\end{aligned}$$

Theorem
There exists an optimal solution of the above problem with all non-zero values equal.

Minimal isoperimetric number over $B(k, S)$

Minimal isoperimetric number - the dual

$$\begin{aligned}
&\text{minimize} \quad \sum_{ij \in E} |a_i - a_j| \\
&\text{subject to} \\
&\sum_{i \in S} a_i = 1, \\
&\qquad a_i = 0 \text{ for each } i \in V \setminus B(k, S), \\
&\qquad a_i \geq 0 \text{ for each } i \in B(k, S)
\end{aligned}$$

For such solution, if we take $T = \{i \colon a_i > 0\}$, then the value of the objective function is $\frac{|\partial T|}{|T|} = \eta(T)$. But this is the isoperimetric number of T - and it is the minimal one.

Minimal isoperimetric number over $B(k, S)$

Therefore the minimal isoperimetric number dual is equivalent to:

Minimal isoperimetric number - the dual reinterpreted

Maximize η such that for each $T \subset B(k, S)$ we have

$$\eta \leq \frac{|\partial T|}{|T|}.$$

This is the Cheeger constant. Remember that we rescaled the original problem by $\frac{|E|}{|V|}$.

The dual problem to Uniform flows

Uniform flows - Cheeger constant times $\frac{|V|}{|E|}$.

$$\text{maximize} \quad |V| \cdot \eta$$

$$\text{subject to}$$

$$\varepsilon_{ij,k} \leq \frac{1}{|E|} \text{ for each } ij \in E, k \in V,$$

$$-\varepsilon_{ij,k} \leq \frac{1}{|E|} \text{ for each } ij \in E, k \in V,$$

$$\sum_{j \in V, ji \in E} \varepsilon_{ji,k} - \sum_{j \in V, ij \in E} \varepsilon_{ij,k} \geq \eta \quad \text{for each } k \in V, i \in B(k, S),$$

$$\eta, \varepsilon_{ij,k} \in \mathbb{R}$$

Mean property A - the dual problem to Uniform Flows

For each $S > 0$ find $\varepsilon_{S,G}$ and a family of functionals $\{\psi_i\}$ on G satisfying

1. $\{\psi_i\}$ has norm 1 *on average*, i.e.

$$\frac{1}{|V|}\|\sum_{i\in V}\psi_i\|_1 = 1,$$

   and $\psi_i \geq 0$ for each $i \in V$.

2. $\{\psi_i\}$ has ε-variation *on average*, i.e.

$$\frac{1}{|E|}\sum_{ij\in E}\sum_{k\in V}|\psi_i(k) - \psi_j(k)| = \frac{1}{|E|}\sum_{ij\in E}\|\psi_i - \psi_j\|_1 \leq \varepsilon.$$

3. $\operatorname{supp}\psi_i \subset B(i, S)$ for each $i \in V$.

If $\lim_{S\to\infty}\varepsilon_{S,G} = 0$, then G has **mean property A**.

Applications: hypercubes and graphs with large girth

Hypercubes

Theorem

Let Q_n be the n-dimensional hypercube graph. The minimal variation of probability measures for Q_n at scale S is

$$\varepsilon_{S,Q_n} = \frac{2\binom{n-1}{S}}{\sum_{k=0}^{S}\binom{n}{k}}.$$

Cube graph, $S = 2$

Theorem

Let Q_n be the n-dimensional hypercube graph. The minimal variation of probability measures for Q_n at scale S is

$$\varepsilon_{S,Q_n} = \frac{2\binom{n-1}{S}}{\sum_{k=0}^{S}\binom{n}{k}}.$$

For $n = 3$, $S = 2$ we have

$$\varepsilon = \frac{2 \cdot \binom{2}{2}}{\binom{3}{0} + \binom{3}{1} + \binom{3}{2}} = \frac{2}{7}.$$

(We found this number before.)

Hypercubes

Corollary (P. Nowak, [2])
The disjoint union

$$\coprod_{n\in\mathbb{N}} \{0,1\}^n$$

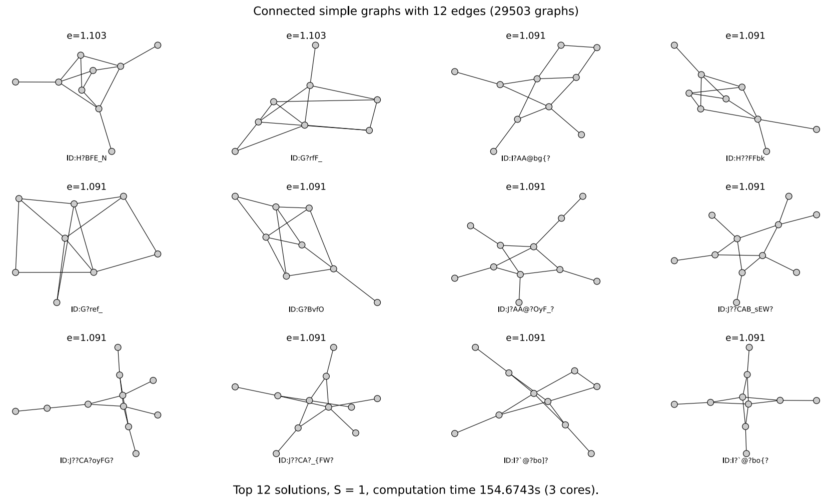
with ℓ_1 metric does not have property A.

Proof.
The proof follows from the observation that for each $S \geq 0$ we have

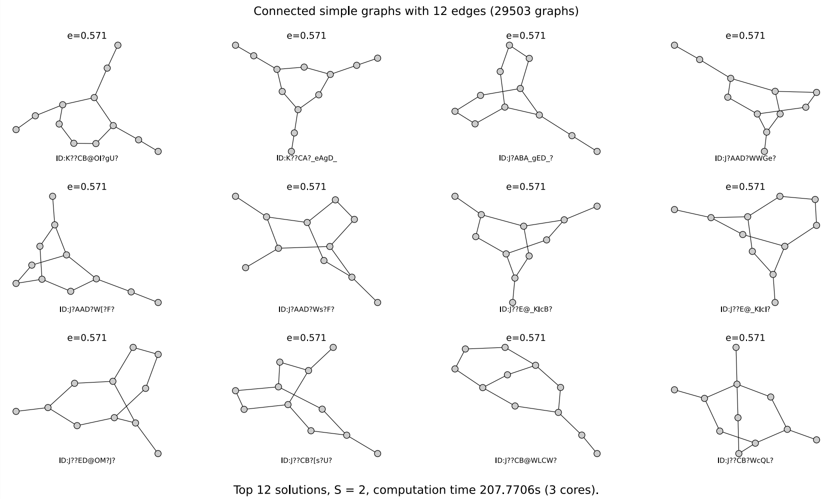
$$\lim_{n\to\infty} \frac{2\binom{n-1}{S}}{\sum_{k=0}^{S}\binom{n}{k}} = 2.$$

□

Some experimental results and a quiz



Some experimental results and a quiz



Large girth

Theorem
Let $G(d, c)$ be a d-regular graph with girth c. Let $2S + 1 < c$. The minimal variation of probability measures for $G(d, c)$ at scale S is

$$\frac{2(d-1)^S(2-d)}{2-d(d-1)^S}.$$

Corollary (R. Willett, [3])
Suppose d_i is a bounded sequence of integers with $d_i \geq 3$ and suppose c_i is a sequence of integers going to infinity. Then, the disjoint union of the graphs $G(d_i, c_i)$ fails to have property A.

Proof.
The proof follows from the observation that

$$\lim_{S\to\infty} \frac{2(d-1)^S(2-d)}{2-d(d-1)^S} = 2 - \frac{4}{d}.$$

□

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Thank you for your attention!

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