# Products of Hurewicz, Menger and Lindelöf spaces

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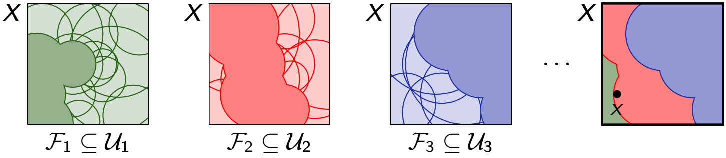
# Combinatorial covering properties

Menger: for every sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of $X$ there are finite families $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ covers $X$

Hurewicz: for every sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of $X$ there are finite families $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ such that $\{\, n : x \notin \bigcup \mathcal{F}_n \,\}$ is finite for all $x \in X$




$$\sigma\text{-compact} \to \text{Hurewicz} \to \text{Menger} \to \text{Lindelöf}$$

**Theorem (Hurewicz)**

*Assume that $X$ is Lindelöf and zero-dimensional*

- *$X$ is Menger $\leftrightarrow$ continuous image of $X$ into $[\mathbb{N}]^\infty$ is nondominating*
- *$X$ is Hurewicz $\leftrightarrow$ continuous image of $X$ into $[\mathbb{N}]^\infty$ is bounded*

# Previously

**Theorem ($\mathfrak{d} = \aleph_1$)**

*Every productively Lindelöf space is productively Menger.*

**Lemma**

*Assume that* $X = \{ x_\alpha : \alpha < \omega_1 \}$ *is a dominating scale in* $[\mathbb{N}]^\infty$*, i.e.,*

- $x_\alpha \leq^* x_\beta$ *for* $\alpha < \beta$*,*
- *for each* $a \in [\mathbb{N}]^\infty$ *there is* $\alpha$ *with* $a \leq^* x_\alpha$*.*

*A space* $X \cup \mathrm{Fin}$*, where*

- *points from* $X$ *are isolated,*
- *points from* $\mathrm{Fin}$ *have the same neighborhoods as in* $\mathrm{P}(\mathbb{N})$*,*

*is productively Menger but not productively Lindeöf.*

# In today's episode

**Theorem ($\mathfrak{d} = \aleph_1$)**

*Every productively Menger space is productively Hurewicz.*

$X$ is concentrated: there is a countable $D \subseteq X$ such that every closed $A \subseteq X \setminus D$ is countable

**Lemma (Miller, Tsaban, Zdomskyy)**

*If $X$ is concentrated and $Y$ is Hurewicz, then $X \times Y$ is Menger.*

Menger: for every sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ of $X$ there are finite families $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ covers $X$

Hurewicz: for every sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \dots$ of $X$ there are finite families $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that $\{\, n : x \notin \bigcup \mathcal{F}_n \,\}$ is finite for all $x \in X$

**Theorem (Hurewicz)**

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# In today's episode

$[\mathbb{N}]^\infty \supseteq X$ is a cFin-scale: $|X| \geq \mathfrak{b}$ and for each function $b \in [\mathbb{N}]^\infty$, there is $c \in [\mathbb{N}]^\infty$ such that

$$b \leq^\infty c \leq^* x$$

for all but less than $\mathfrak{b}$ functions $x \in X$,
e.g., unbounded $X = \{\, x_\alpha : \alpha < \mathfrak{b} \,\}$ with $x_\alpha \leq^* x_\beta$ for $\alpha < \beta$.

**Lemma ($\mathfrak{b} = \aleph_1$)**

*Let $X \subseteq [\mathbb{N}]^\infty$ be a cFin-scale. A space $X \cup \mathrm{Fin}$, where*

- *points from $X$ are isolated,*
- *points from $\mathrm{Fin}$ have the same neighborhoods as in $\mathrm{P}(\mathbb{N})$,*

*is productively Hurewicz.*

**Lemma (Miller)**

*It is consistent with CH that there is a cFin-scale that is Menger.*

**Theorem**

*It is consistent with CH that there is a productively Hurewicz space that is not productively Menger.*

# Main results

**Theorem ($\mathfrak{d} = \aleph_1$)**

- *productively Lindelöf* $\to$ *productively Menger* $\to$ *productively Hurewicz*
- *None of these implications is reversible.*

**Lemma ($\mathfrak{d} = \aleph_1$)**

*Assume that* $X \subseteq [\mathbb{N}]^\infty$ *is a dominating scale. A space* $X \cup \mathrm{Fin}$, *where*

- *points from* $X$ *are isolated,*
- *points from* $\mathrm{Fin}$ *have the same neighborhoods as in* $\mathrm{P}(\mathbb{N})$,

*is productively Menger but not productively Lindeöf.*

**Lemma ($\mathfrak{b} = \aleph_1$)**

*Let* $X \subseteq [\mathbb{N}]^\infty$ *be a* $\mathrm{cFin}$*-scale. A space* $X \cup \mathrm{Fin}$, *where*

- *points from* $X$ *are isolated,*
- *points from* $\mathrm{Fin}$ *have the same neighborhoods as in* $\mathrm{P}(\mathbb{N})$,

*is productively Hurewicz.*

**Corollary ($\mathfrak{b} = \aleph_1 < \mathfrak{d}$)**

*There is a productively Hurewicz space that is not productively Menger.*

# References

P. Szewczak, B. Tsaban, *Products of general Menger spaces*, Topology and its Applications 255 (2019), 41–55.