Probabilistic programming semantics for name generation

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A syntax for "probabilistic programming" (actually "name generation")

The following is the grammar for ν-calculus

terms

$$M = x \mid \underbrace{\lambda x . M}_{x \mapsto M} \mid \underbrace{MM}_{\text{application of}}$$

$$\mid \text{true} \mid \text{false} \mid \text{if } M \text{ then } M \text{ else } M$$

$$\mid M = M \mid \nu n\, M$$

generate "random" name n

types $\overbrace{\underset{\text{boolean}}{B} \mid \underset{\text{names}}{N}}^{\text{ground}} \mid \sigma \to \sigma$

1st order types $\quad G_1 \to (G_2 \to \dots G_n)$

G_i - ground types

2nd order $\quad (B \to B) \to N$

<u>Examples</u>

1) $\nu n \nu m \; n = m \qquad B$

2) $\nu n \; \lambda x \; x = n \qquad N \to B$

3) $\lambda x \; \text{false} \qquad N \to B$

operational semantics expresses iterated evaluation of terms $M \Downarrow C$ stands for

M evaluates to C

Examples 1) $(\lambda x\ \text{false})(n) \Downarrow \text{false}$

2) $\nu n \nu m\ n = m \Downarrow \text{false}$

Def (Observational equivalence)

If M_1 and M_2 are ν-terms, of the same type,

$M_1 \approx M_2$ if for every context $P[-]$ we have

$$P[M_1] \Downarrow b \quad \text{iff} \quad P[M_2] \Downarrow b$$

whenever $P[M_i]$ are well-formed expressions.

For 1st order types there is a formal system called logical relations that determines $\approx$

$$M_1 \approx M_2 \quad \text{iff} \quad M_1 \underline{R} M_2$$

defined inductively and resembles a proof system

Examples 1) $\nu n \nu m\ n = m \approx \text{false}$

2) $\lambda x\ \text{false} \approx \nu n \lambda x\ x = n$

(this is called the privacy equation)

Semantics

In semantics of programming languages, we often need function spaces:

to each type σ we associate a set X_σ

(e.g. $\sigma = B \quad X_\sigma = 2 = \{0, 1\}$,

$\sigma = N \qquad X_\sigma = \mathbb{R}$)

for type $\sigma \to \tau$ we want to have $X_\tau^{X_\sigma}$

In probabilistic programming we want to have
$\mathbb{R}$ as the space associated to type N
we want to treat it as a space with a σ-algebra
(measurable space)
The problem appears at the construction of function
spaces).

<u>Theorem</u> (Aumann '69) There is no σ-algebra
on $2^{\mathbb{R}}$ s.th.

$$\mathbb{R} \times 2^{\mathbb{R}} \to \mathbb{R}$$
$$(r, f) \mapsto f(r)$$

is measurable

- <u>Def</u> (Heunen-Kammar; Staton-Yang '17)

A <u>Quasi-Borel space</u> is a set X together with
a set M_X of functions $f: \mathbb{R} \to X$
satisfying

1) all constant functions are in M_X
2) if $f \in M_X$, $\alpha: \mathbb{R} \to \mathbb{R}$ is Borel
then $f\alpha \in M_X$
3) if $\mathbb{R} = \biguplus_{n \in \mathbb{N}} B_n$ $\quad B_n$ - Borel
$f_n \in M_X$, then $\bigcup_{n \in \mathbb{N}} f_n|B_n \in M_X$

This forms a category

<u>Remarks</u> 1) Every standard Borel space X is a quasi-Borel space (M_X - all Borel maps)

2) the category allows for function spaces and products and $X \mapsto \mathcal{P}(X)$

If X, Y · QBS

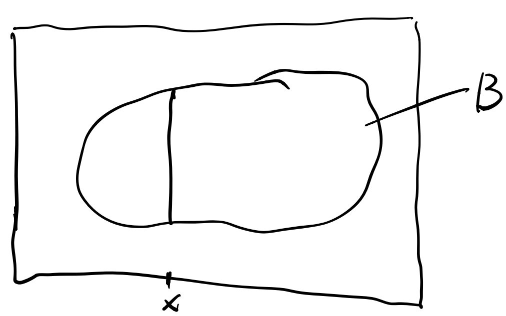
Y^X is QBS with those

$f : \mathbb{R} \to Y^X$

s.t. $f' : \mathbb{R} \times X \to Y$ is measurable

$f'(r, x) = f(r)(x)$

On $2^{\mathbb{R}}$ (this is the set of Borel subsets of $\mathbb{R}$)



$f : \mathbb{R} \to 2^{\mathbb{R}}$ is in the QBS structure if there exists $B \subseteq \mathbb{R} \times \mathbb{R}$ Borel s.t.

$f(x) = B_x$

<u>Fact</u> Every $F : \mathbb{R} \to 2^{\mathbb{R}}$ measurable in QBS is (Borel-on-Borel) - measurable

Interpretation

We can interpret ν-calculus in QBS

$$N \rightsquigarrow \mathbb{R} \qquad B \rightsquigarrow 2 = \{0,1\}$$

$$[\![\lambda \underset{x:\sigma}{x} \; M]\!] \qquad \text{“} \in X_\tau^{X_\sigma} \text{”} = \delta_f$$

$$[\![\nu n \; M]\!] = \int_{\mathbb{R}} [\![M]\!] \, d\nu$$

ν - Gaussian

Examples

1) $$[\![\lambda x \; \text{false}]\!] = \delta_\emptyset$$

2) $$[\![\nu n \; \lambda x \; x = n]\!] = \int_{\mathbb{R}} \delta_{\{n\}} \, d\nu$$

both are measures on Borel-on-Borel sets

$$\delta_\emptyset(\mathcal{A}) = \begin{cases} 1 & \text{if } \emptyset \in \mathcal{A} \\ 0 & \text{if } \emptyset \notin \mathcal{A} \end{cases}$$

$$\int_{\mathbb{R}} \delta_{\{n\}} \, d\nu = \int_{\mathbb{R}} \begin{cases} 1 & \text{if } \{n\} \in \mathcal{A} \\ 0 & \text{if } \{n\} \notin \mathcal{A} \end{cases} d\nu(n)$$

Theorem (SSSW) If M_1 and M_2 are 1st order ν-terms, then

$$M_1 \approx M_2 \quad \text{iff} \quad [\![M_1]\!] = [\![M_2]\!]$$

we will look at the special case of

$M_1 = \lambda x$ false

$M_2 = \nu n\ \lambda x\ x = n$

<u>Lemma</u> If $\mathcal{F} \in$ Borel on Borel, then

$\emptyset \in \tilde{\mathcal{F}}$ iff $\{x \in \mathbb{R} : \{x\} \in \tilde{\mathcal{F}}\}$ is co-countable

<u>Pf sketch</u>

spse $\emptyset \in \tilde{\mathcal{F}}$ but $\{x \in \mathbb{R} : \{x\} \notin \tilde{\mathcal{F}}\}$ is unctble

$$\text{WLOG} \quad \underset{\mathbb{R}}{\parallel}$$

Recall Becker's theorem

WF, UB are Borel inseparable

i.e. there exists a Borel set $B \subseteq \mathbb{R} \times \mathbb{R}$

s.t. $WF := \{x \in \mathbb{R} : |B_x| = 0\}$

$UB := \{x \in \mathbb{R} : |B_x| = 1\}$ are Borel inseparable

If $\{x : B_x \in \tilde{\mathcal{F}}\}$ is Borel b/c $\mathcal{F}$ was Borel-on-Borel

but it separates WF from UB. $\square$

This Lemma implies that

$$\delta_\emptyset(\mathcal{F}) = \int_{\mathbb{R}} \delta_{\{n\}}(\mathcal{F})\, d\nu(n)$$

for every $\mathcal{F}$ Borel-on-Borel.

In general, we introduce and use a "normal form"

$$M \mapsto \langle M \rangle$$

s.th. $M_1 \approx M_2$ iff $\langle M_1 \rangle = \langle M_2 \rangle$