Graph Width Parameters. Dependencies, Algorithms and Decompositions.

Wojciech Nadara

University of Warsaw, Poland

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Plan

Discerning the title

Graph width parameters

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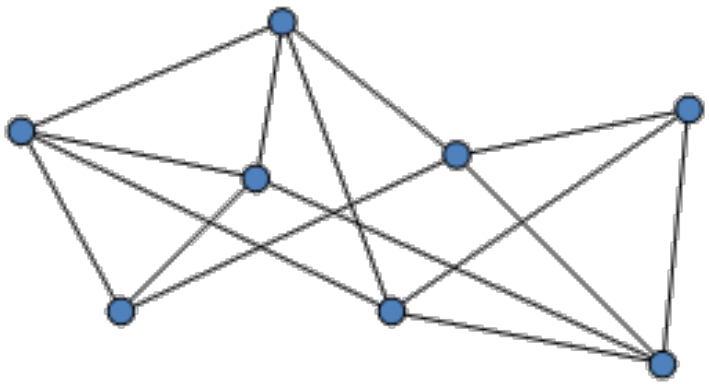
Graph width parameters

What are graphs?

Graphs are...

What are graphs?



Graphs are... dots and segments

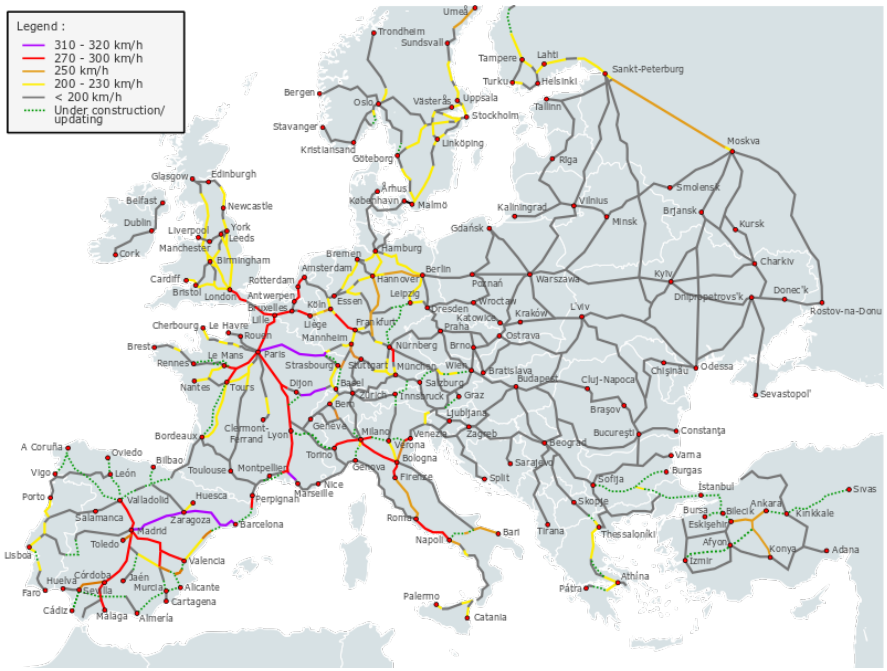
What are graphs for?

Modelling:

What are graphs for?

Modelling:

- road networks



What are graphs for?

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What are graphs for?

Modelling:

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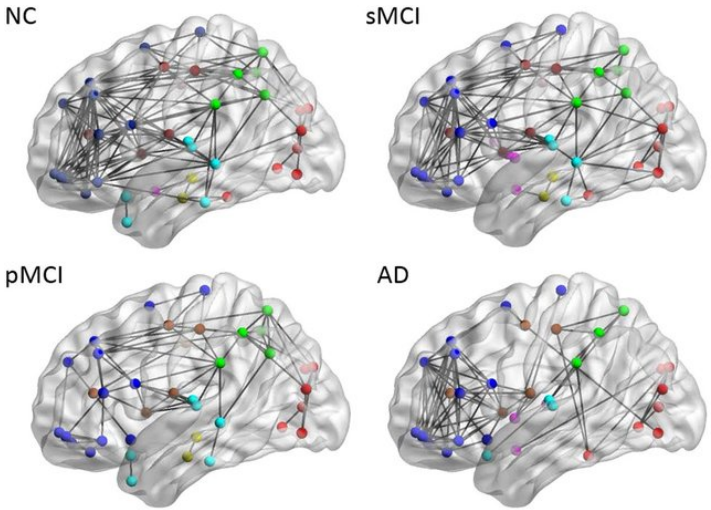
- neural connections



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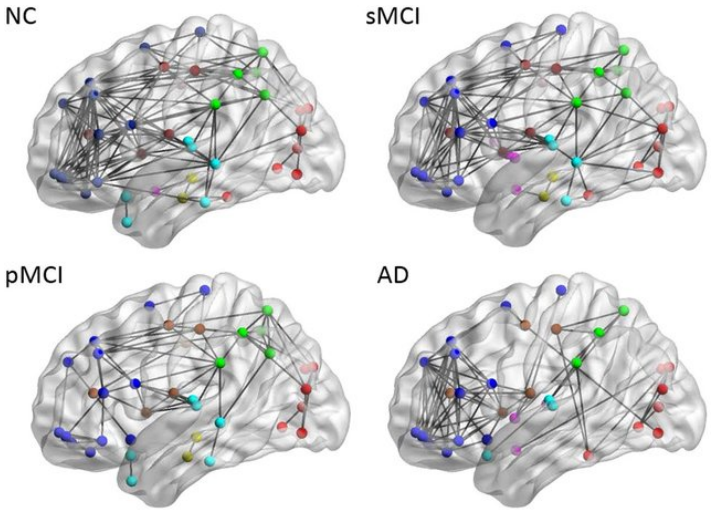


- and many many more...

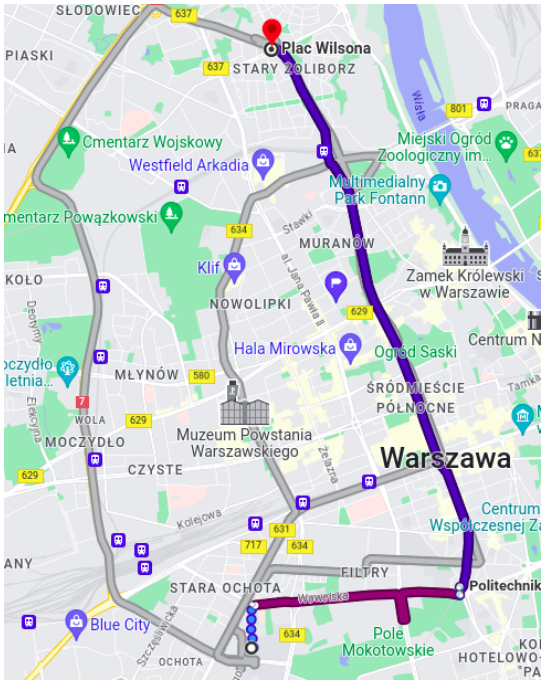
Graph problems examples

Problems examples:

Graph problems examples

Problems examples:

- Getting from point A to point B



Graph problems examples

Problems examples:

- Getting from point A to point B
- Maximum flow problem

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- Getting from point A to point B
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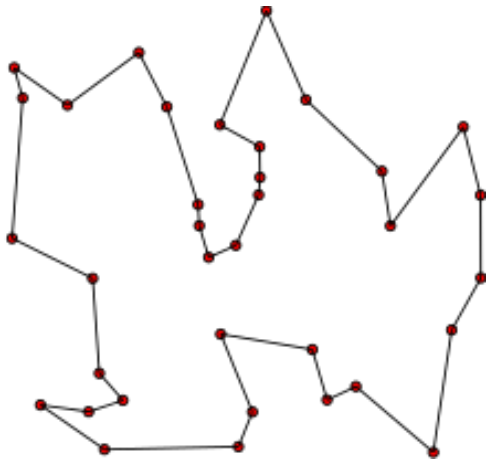
- Travelling salesman problem



Graph problems examples

Problems examples:

- Getting from point A to point B
- Maximum flow problem
- Travelling salesman problem
- Minimum balanced cut

Graph problems examples

EASY

- Getting from point A to point B
- Maximum flow problem

HARD

- Travelling salesman problem
- Minimum balanced cut

Dealing with hardness

How do we deal with hard problems?

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A few directions:

Dealing with hardness

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A few directions:

- Approximation

Dealing with hardness

How do we deal with hard problems?

A few directions:

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- Heuristics

Dealing with hardness

How do we deal with hard problems?

A few directions:

- Approximation
- Heuristics
- **Exact solutions on *easy* instances**

Discerning the title

Graph **width parameters**
Dependencies, Algorithms
and **Decompositions**.

Width parameters

Decompositions $\Leftrightarrow$ ways of capturing the structure of a graph

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Low-width decompositions $\Leftrightarrow$ *easy* graphs

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Various graph width parameters:

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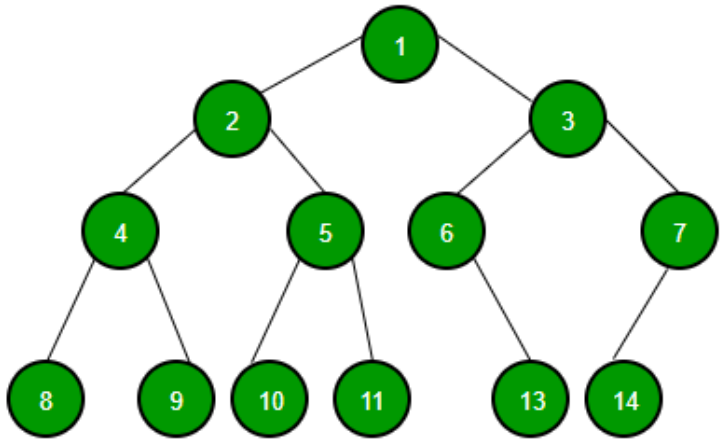
- ...

Trees

Trees for normal people:

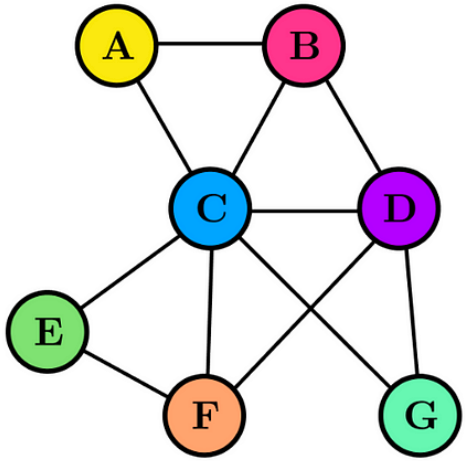
Trees for computer scientists:

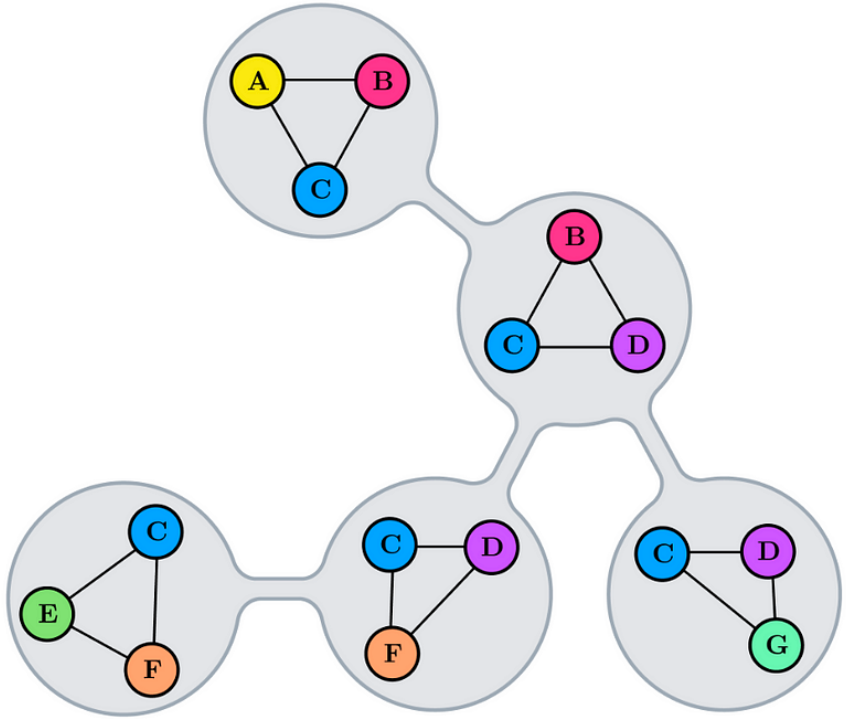


Trees are easy to process!

Beyond trees - treewidth

Some graphs can be viewed as "fat trees"





The thinner the decomposition - the better

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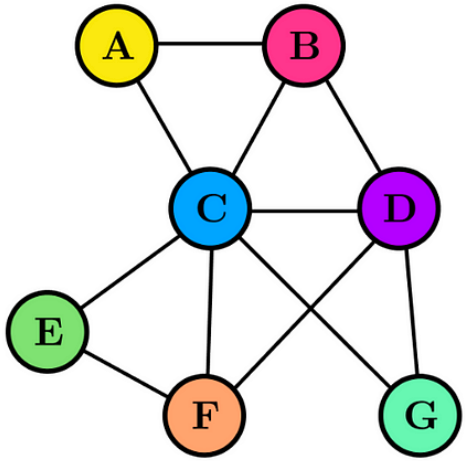

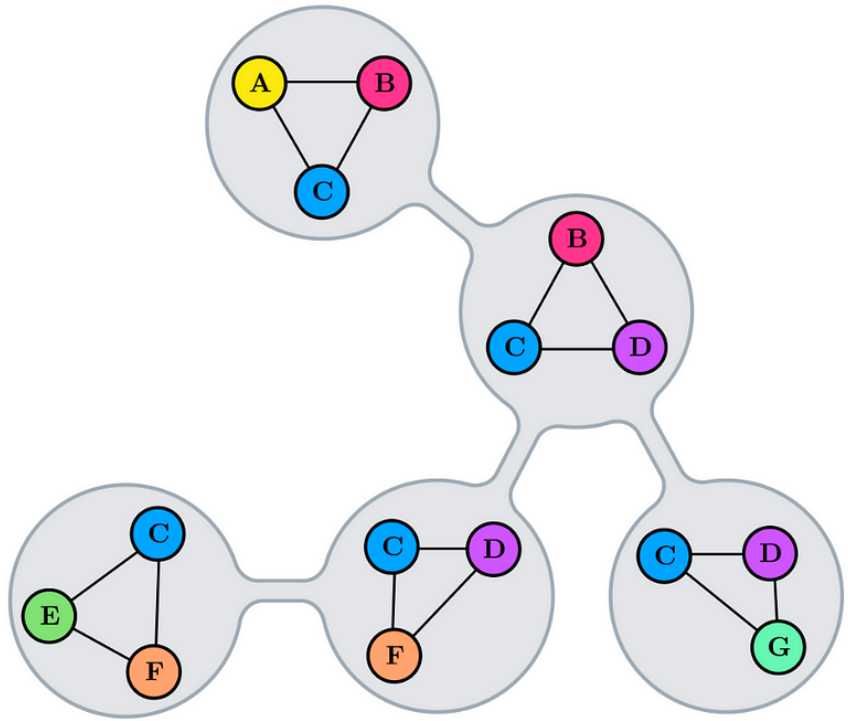

The thinner the decomposition - the better
Pathwidth is a similar concept, where we view a graph as a fat path instead of a fat tree.

Plan

Structural treedepth results

Excluded-minor characterization of treedepth

If the treedepth of a graph G is big, then either:

- treewidth of G is big
- G "contains" a high complete binary tree or
- G contains a long path

The dependencies are polynomial.

Structural treedepth results

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Treedepth obstructions

Every graph of a *small* treedepth contains a *small* subgraph of the same treedepth

Decomposition results

Approximate treedepth decomposition

Polynomial time approximate treedepth decomposition off by a factor of roughly $\mathrm{tw}(G)$

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Optimum treedepth decomposition

Polynomial **space** optimum treedepth decomposition with the time matching the best one of exact algorithms for that problem

Pathwidth results

Excluded-minor characterization of pathwidth

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Polynomial time approximate pathwidth decomposition off by a factor of roughly $\mathrm{tw}(G)$.

Thank you!