Exceptional holonomy

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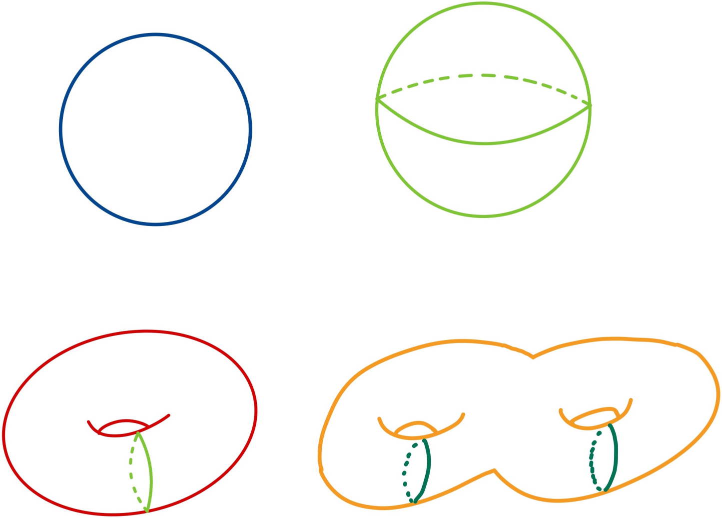
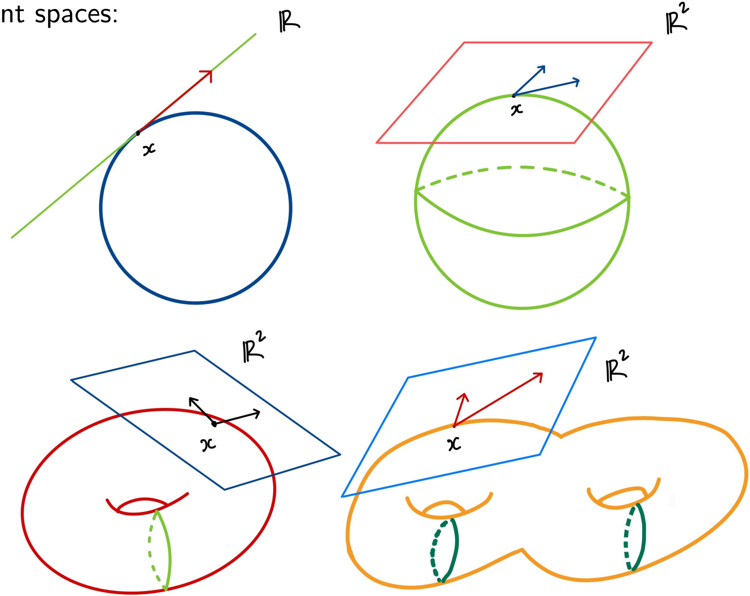
Differential geometry

Differential geometry studies smooth manifolds:



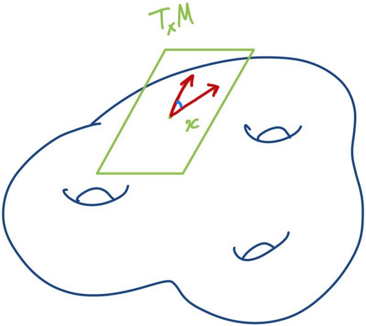
Differential geometry

Tangent spaces:



Riemannian geometry

Riemannian geometry - put an inner product $\langle \cdot, \cdot \rangle_x$ on each $T_x M$. (1850s)



Equipped with $g_x = \langle \cdot, \cdot \rangle_x$, we can measure:

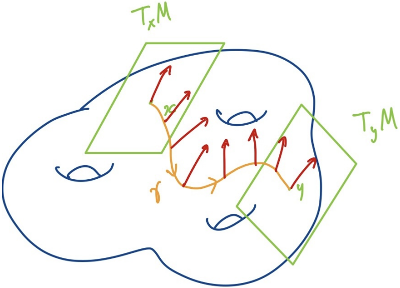
- angles between vectors,
- distance between points,
- volume of manifold,...

Riemannian geometry

There is a notion of parallel transport

$$P_\gamma : T_x M \to T_y M$$

for a curve $\gamma : [0,1] \to M$.

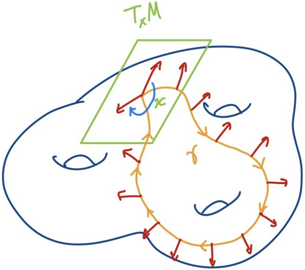


Holonomy group

This leads to the notion of <u>holonomy group</u>:

$$\mathrm{Hol}_x = \text{“group generated by } P_{\gamma,x} \text{ around closed loops”} \subseteq \mathrm{GL}(n,\mathbb{R})$$

introduced by Schouten (1918) and Cartan (1926).



This process converts ‘geometry’ into ‘linear algebra’

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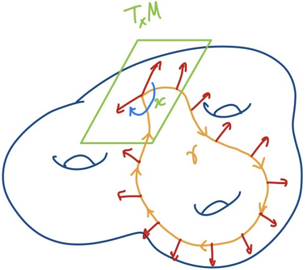

Cartan (1926): What are the possible groups for Hol_x?

Berger's classification

In 1955, Berger gave a classification for Hol_x of (M, g):

Dimension of M	Holonomy group	Name
n	$\mathrm{SO}(n)$	Riemannian
$2n$	$\mathrm{U}(n)$	Kähler
$2n$	$\mathrm{SU}(n)$	Calabi-Yau
$4n$	$\mathrm{Sp}(n)$	HyperKähler
$4n$	$\mathrm{Sp}(n)\mathrm{Sp}(1)$	Quaternion-Kähler
7	G_2	G_2
8	$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$

Remark:

- Any Riemannian manifold can be decomposed into these components.
- Berger's list only gives the the possible holonomy groups - it does NOT say that there are such examples.

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First 2 cases are trivial:

- $\mathrm{SO}(n)$ is generic e.g. S^n
- $\mathrm{U}(n)$ - algebraic varieties are Kähler e.g. $\{P(z_1, ..., z_n) = 0\} \subset \mathbb{C}^n$.

$\mathrm{SO}(n) = \{A \mid AA^T = \mathrm{Id}, \det(A) = 1\}$ $\quad$ $\mathrm{U}(n) = \{A \mid A\bar{A}^T = \mathrm{Id}\}$ (over $\mathbb{C}$)

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Over the following 20 years or so, examples are established:

- Wolf (1965) - $\mathrm{Sp}(n)\mathrm{Sp}(1)$ case
- Calabi (1957) and Yau (1978) - $\mathrm{SU}(n)$, $\mathrm{Sp}(n)$ cases

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G_2 and $\mathrm{Spin}(7)$ cases are called the exceptional holonomy cases. They were not believed to exist... until Bryant (1987) proved they did!

This opens up a new branch of research in differential geometry!

Origins: G_2 and $\mathrm{Spin}(7)$ geometry

- Local existence of G_2 and $\mathrm{Spin}(7)$ metrics - Bryant (1987)

$$\begin{aligned} \text{General } \mathrm{G}_2 \text{ metric} &\Rightarrow 49 \text{ 1st order PDEs} \\ \text{General } \mathrm{Spin}(7) \text{ metric} &\Rightarrow 56 \text{ 1st order PDEs} \end{aligned}$$

- Explicit non-compact examples - Bryant-Salamon (1989)
- Compact examples - Joyce (1996-1999)
- Witten (1995) proposes *M*-theory as generalisation of string theory: this is 11-dimensional and $11 = 4 + 7 \longleftarrow$ this comes from G_2

There are by now "hundreds of thousands" of examples - constructed by glueing PDE techniques, algebraic geometry (Fano, CY), ODE analysis,... but we don't really understand much about these manifolds

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Properties

What makes G_2 and $\mathrm{Spin}(7)$ interesting?

- Einstein manifolds - Ricci flat!
- 'Real version' of Calabi-Yau 3-folds - but no algebraic geometry
- Minimal submanifolds (think holo curves and special lagrangians)
- 'Gauge theoretic invariants' - akin to Donaldson theory for 4-manifolds

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Thank you for listening!