Maximal Spread of Coherent Distributions

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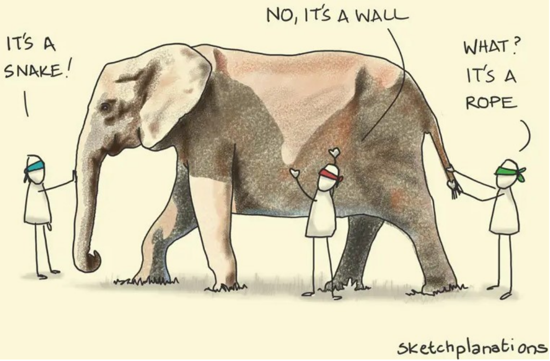


Definition (Dawid, DeGroot, Mortera, 1995)

The probability distribution π on $[0,1]^n$ is coherent: there is $(\Omega, \mathcal{F}, \mathbb{P})$ and random vector $X \sim \pi$ with $X_i = \mathbb{P}(E|\mathcal{G}_i)$ a.e. $(i = 1, \ldots, n)$ for some $E \in \mathcal{F}$ and σ-fields $\mathcal{G}_1, \ldots, \mathcal{G}_n \subseteq \mathcal{F}$.

- different expert opinions (microeconomics), probability forecasts (statistics), group beliefs (game theory).

Example. **A**lice, **B**ob, **C**indy, and **D**ave are playing cards, short 20-card deck (from 10s up) is used. Each player gets two cards, $a_i, b_i, c_i, d_i, i = 1, 2$ - face down. Rest, 12 cards remain hidden.

ALICE	**A♠**	**A♥**	*plays fair*	$X_1 \approx 0.10$
BOB	**A♦**	**K♠**	*spying on Alice*	$X_2 = 0.25$
CINDY	**J♣**	**10♣**	*plays fair*	$X_3 < 0.01$
DAVE	**A♣**	**Q♦**	*marked the aces*	$X_4 = 1.00$

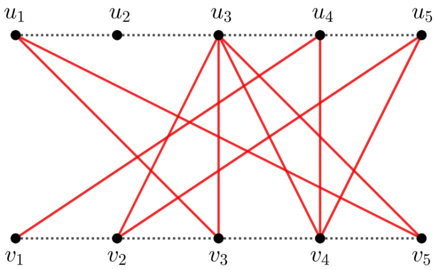
- All four players are asked to estimate the odds of

$$E = \{\{\mathbf{A♠A♥A♦A♣}\} \subseteq \{a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}\}.$$

- The knowledge of the players is represented by σ-algebras

$$\mathcal{G}_1 = \sigma(a_1, a_2), \quad \mathcal{G}_2 = \sigma(a_1, a_2, b_1, b_2), \quad \mathcal{G}_3 = \sigma(c_1, c_2),$$

  and a bit more complicated algebra $\mathcal{G}_4$ (which contains 1_E).
- We have $(X_1, X_2, X_3, X_4) \in \mathcal{C}_4$.



	v_1	v_2	v_3	v_4	v_5	X
Y	0.2	0.4	0.4	0.6	0.4	
u_1	0	0	1	0	1	0.4
u_2	0	0	0	0	0	0
u_3	0	1	1	1	1	0.8
u_4	1	0	0	1	0	0.4
u_5	0	1	0	1	0	0.4

Example. Consider the bipartite graph (U, V, E), where $E \subseteq U \times V$ denotes the set of edges.

- Let $\omega \in U \times V$ be random; $R(\omega)$, $C(\omega)$ be its row and column.
- Set $X(\omega) = \deg(u_\omega)/|V|$ and $Y(\omega) = \deg(v_\omega)/|U|$.
- We have $X = \mathbb{P}(E|R)$ and $Y = \mathbb{P}(E|C)$, so $(X, Y) \in \mathcal{C}_{\mathcal{I}}$.

Maximal Spread of Coherent Distributions?

Theorem (Arieli, Babichenko, Sandomirskiy, Tamuz, 2021)

For $\alpha \in [0, 2]$, we have

$$\sup_{(X,Y)\in\mathcal{C}_2} \mathbb{E}|X - Y|^\alpha = 2^{-\alpha}.$$

main idea – simple geometry in L^2 framework

Theorem (Burdzy, Pal, 2021)

For $\delta \in (1/2, 1]$, we have

$$\sup_{(X,Y)\in\mathcal{C}_2} \mathbb{P}(|X - Y| \geq \delta) = \frac{2(1-\delta)}{2-\delta}.$$

main idea – discretization, long step-by-step reduction

- What if $\alpha > 2$ or $(X, Y) \in \mathcal{C}_\mathcal{I}$? What about a more general $\mathbb{E}\max_{1\leq i<j\leq n}|X_i - X_j|^\alpha$ and $\mathbb{P}\left(\max_{1\leq i<j\leq n}|X_i - X_j| \geq \delta\right)$?

Theorem (Cichomski, Osękowski, **EJP**, 2021)

$$\sup_{(X_1,\ldots,X_n)\in\mathcal{C}_n} \mathbb{E} \max_{1\leq i<j\leq n} |X_i - X_j| = \begin{cases} \frac{1}{2} & \text{if } n = 2, \\ 2-\sqrt{2} & \text{if } n = 3, \\ \frac{7}{2} - 2\sqrt{2} & \text{if } n = 4, \\ \dfrac{n-2}{n-1} & \text{if } n \geq 5. \end{cases}$$

main idea – discretization, symmetrization, step-by-step reduction

Theorem (Cichomski, Osękowski, **ALEA**, 2024)

For $\delta \in (1/2, 1]$ and any $n \geq 2$, we have

$$\sup_{(X_1,\ldots,X_n)\in\mathcal{C}_n} \mathbb{P}\left(\max_{1\leq i<j\leq n} |X_i - X_j| \geq \delta\right) = \frac{n(1-\delta)}{2-\delta} \wedge 1.$$

main idea – discretization, symmetrization, Bellman function

Theorem (Cichomski, Osękowski, **JAP**, 2024)

$$\lim_{\alpha\to\infty} \alpha \cdot \left(\sup_{(X,Y)\in\mathcal{C}_2} \mathbb{E}|X-Y|^{\alpha} \right) = \frac{2}{e}.$$

main idea – discretization, convex geometry, analysis of $\mathrm{ext}_f(\mathcal{C}_2)$

Theorem (Cichomski, Petrov, **ECP**, 2023)

For $\delta \in (1/2, 1]$, *we have*

$$\sup_{(X,Y)\in\mathcal{C}_{2,\mathcal{I}}} \mathbb{P}(|X-Y| \geq \delta) = 2\delta(1-\delta).$$

main idea – prove it for bipartite $G = (U, V, E)$, $|U| = |V| = n$; *for* $n \geq k > n/2$, *we have*

$$\sum_{1\leq i,j\leq n} 1\left\{ |\mathrm{deg}(u_i) - \mathrm{deg}(v_j)| \geq k \right\} \leq 2k(n-k).$$

Theorem (Doob's martingale inequality)

Fix $p > 1$ and let $(X_i, \mathcal{F}_i)_{i=1}^n$ be a martingale with $X_i \in L^p$, $i = 1, 2, \ldots, n$. We have

$$\left(\mathbb{E} \max_{1 \le i \le n} |X_n|^p\right)^{\frac{1}{p}} \le \frac{p}{p-1} \cdot \left(\mathbb{E}|X_n|^p\right)^{\frac{1}{p}} .$$

Theorem (Cichomski, Osękowski, **PMS**, 2023)

Let random variable $\xi \in \{0, 1\}$ a.e. and fix a family of σ-fields $\left\{\mathcal{F}_i^j\right\}_{1 \le i \le n,\ 1 \le j \le k}$ with $\mathcal{F}_1^j \subseteq \cdots \subseteq \mathcal{F}_n^j$, $1 \le j \le k$. We have

$$\left(\mathbb{E} \max_{i,j} |X_i^j|^p\right)^{\frac{1}{p}} \le \left(1 + \frac{k}{p-1}\right)^{\frac{1}{p}} \cdot \mathbb{P}(\xi = 1)^{\frac{1}{p}}.$$

main idea – analysis of maximal operators on star-shaped domain