

A taste of descriptive combinatorics

Ronnie Chen

MIMUW
19 March 2026

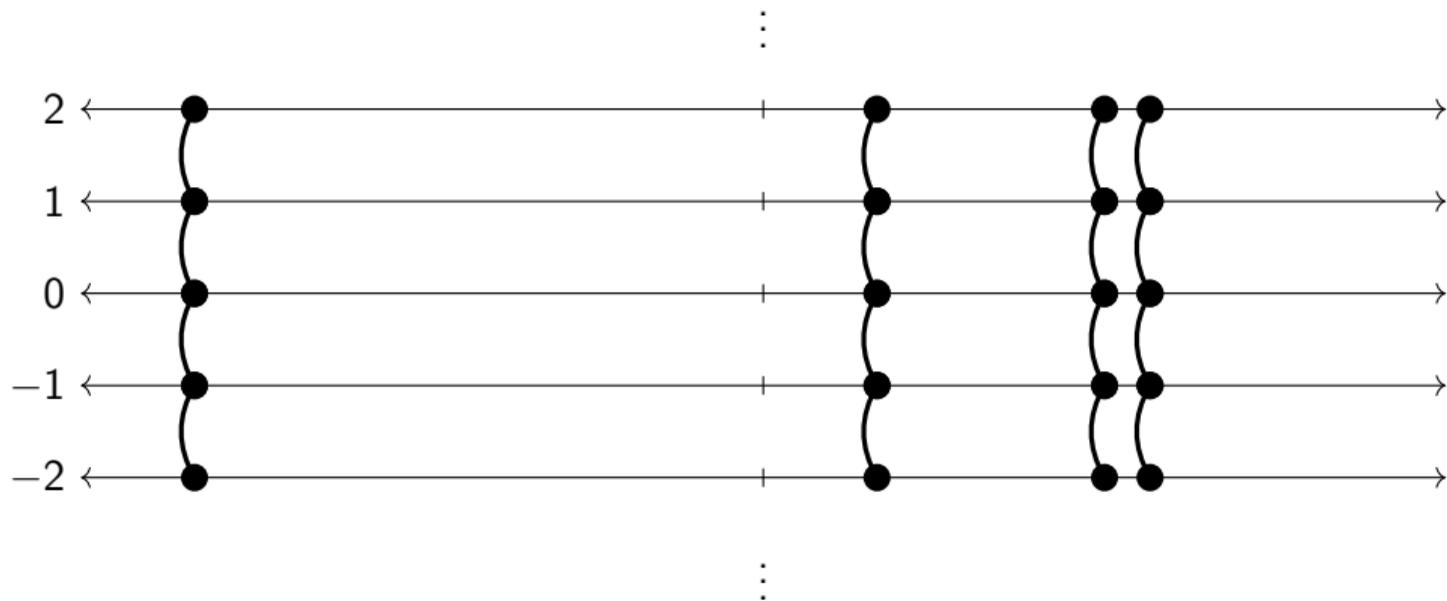
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Let $X = \mathbb{R} \times \mathbb{Z}$. Draw an edge from (x, n) to $(x, n + 1)$ for each $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.



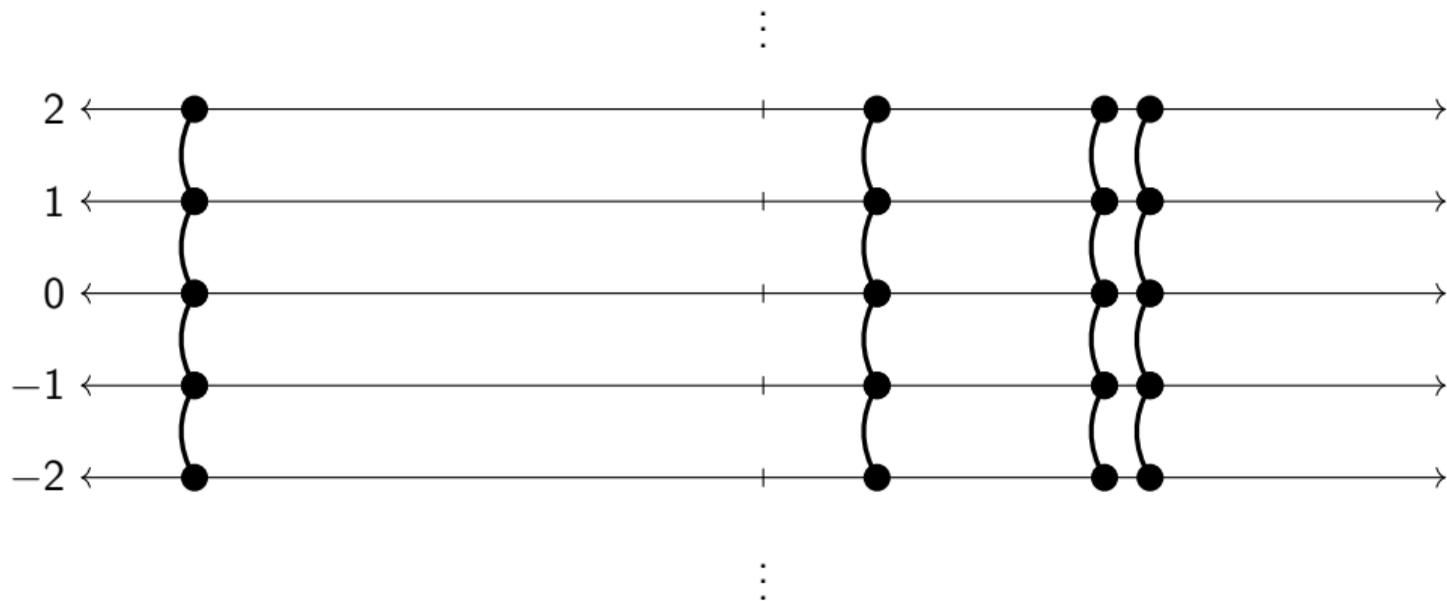
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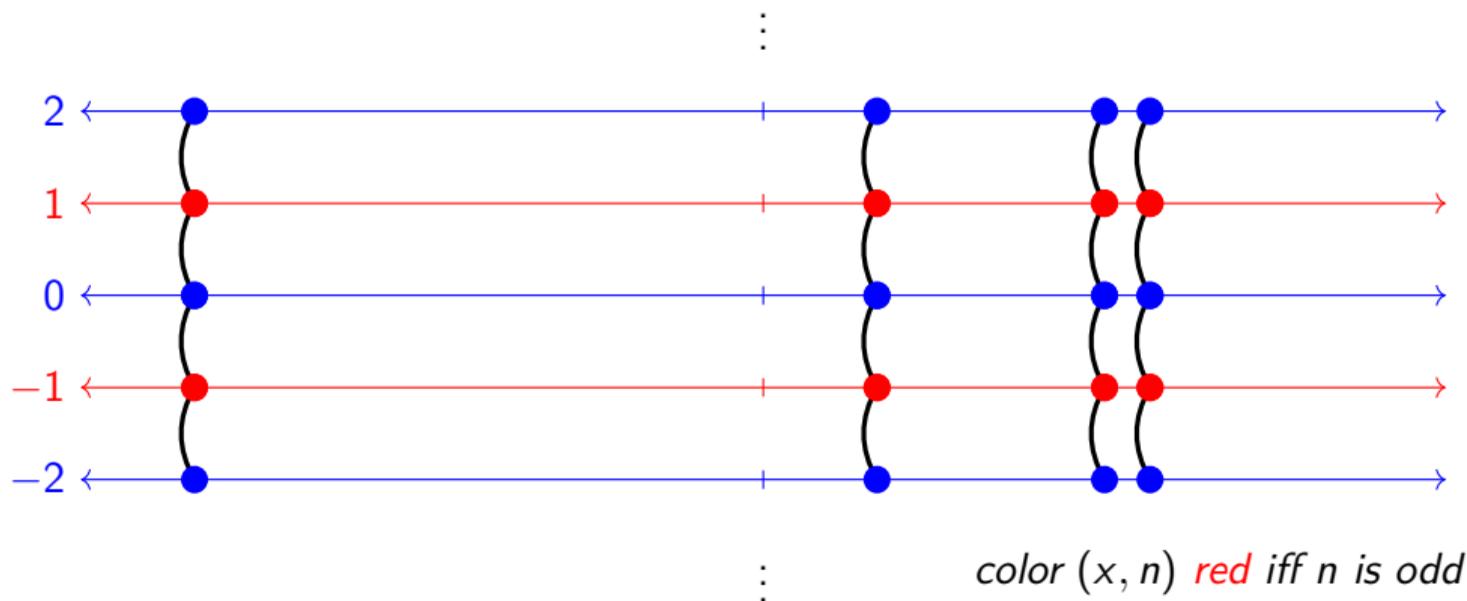
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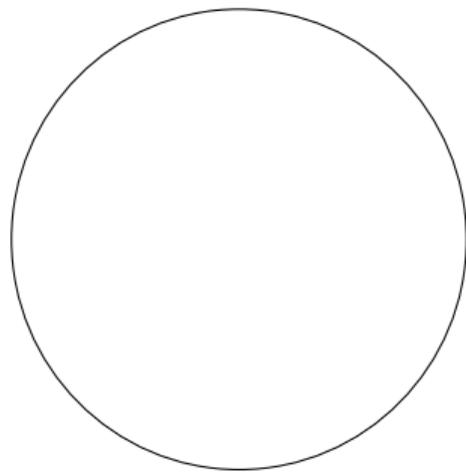


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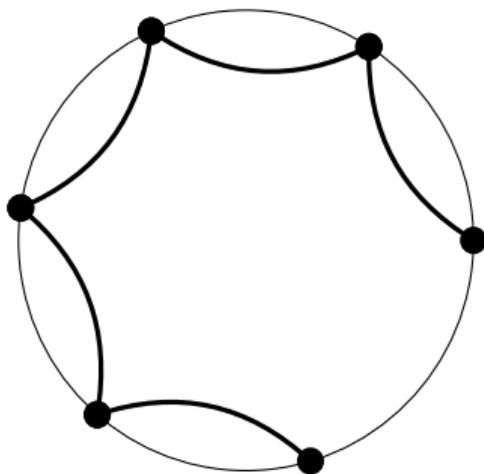
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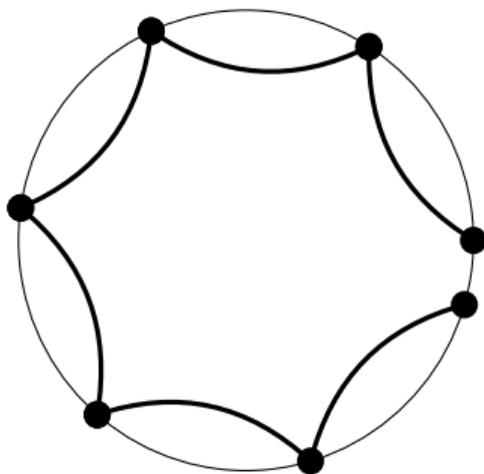
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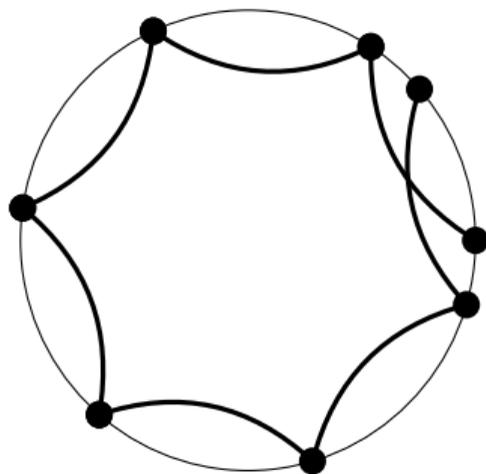
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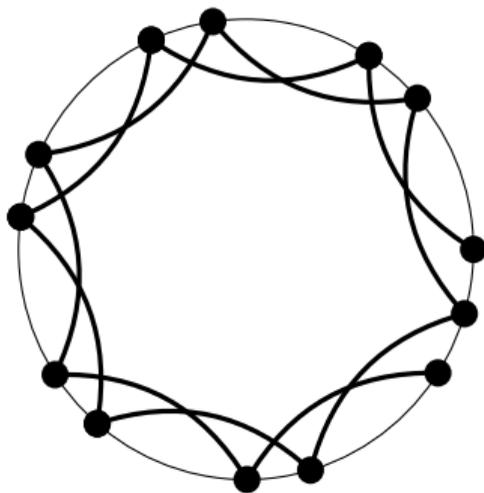
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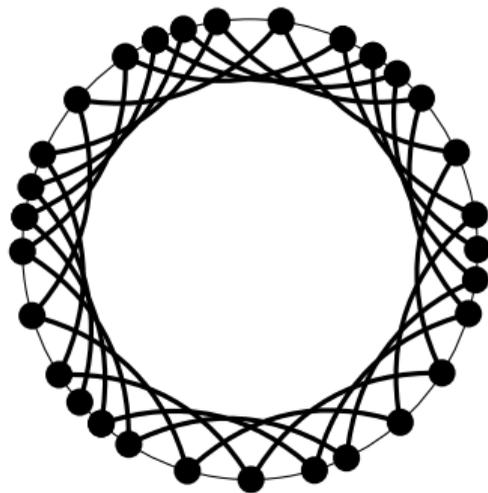
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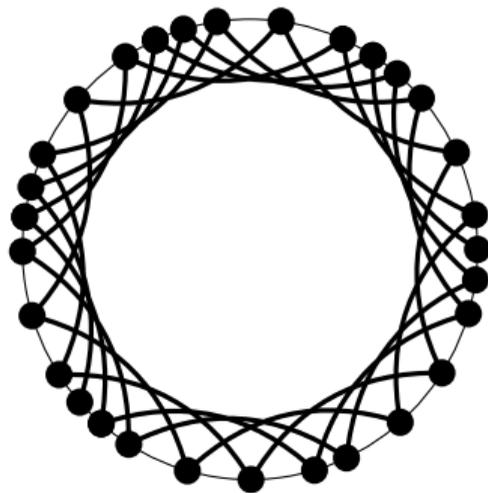
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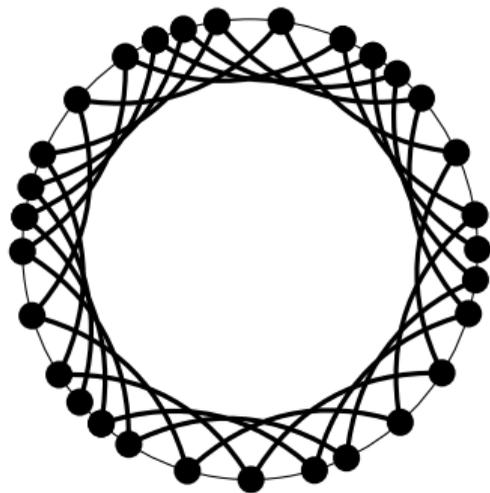


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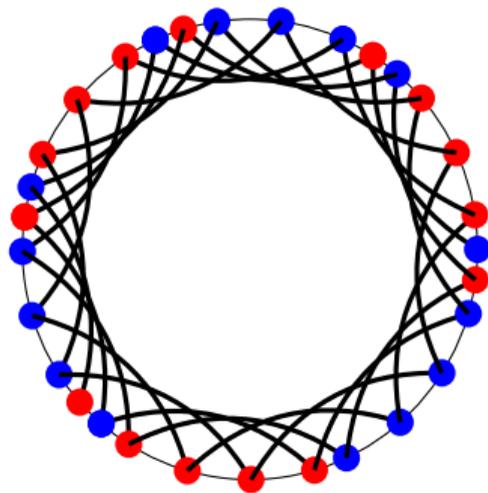
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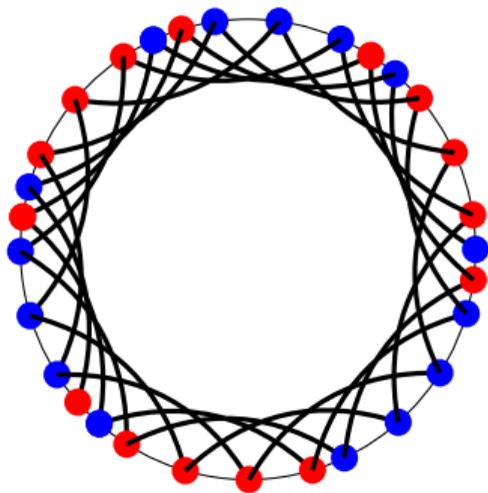
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Theorem (classical)
*It is not possible to write
down the coloring rule!*

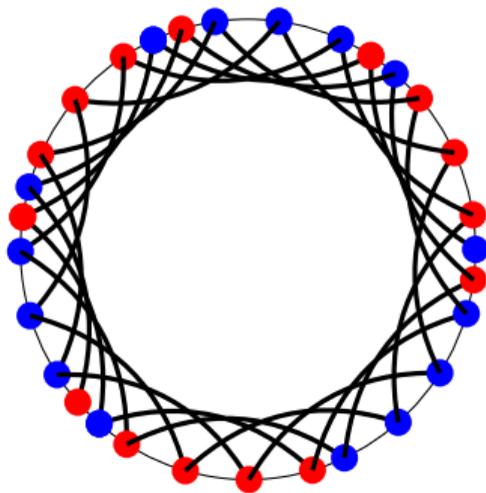
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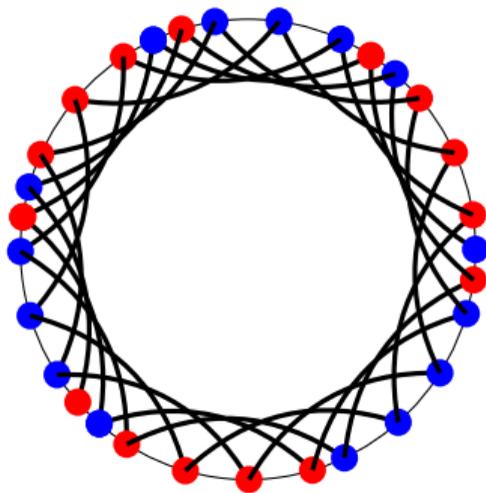


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Note that $|\mathbb{R}| = |\mathbb{R} \times \mathbb{Z}| = |\mathbb{T}| = 2^{\aleph_0}$, while each connected component is countable. So the graphs $\mathbb{R} \times \mathbb{Z} \cong \mathbb{T}$ are both 2^{\aleph_0} many \mathbb{Z} -lines.

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Descriptive set theory (DST) studies infinite but “explicitly definable” structures on $\mathbb{R}, \mathbb{T}, \mathbb{R}^3$, etc., often using tools from analysis (measures, topological arguments, etc.)

Examples of combinatorial problems in DST

- (Kechris–Solecki–Todorcevic 1999) A “definable” graph with degree $\leq d$ can be “definably” colored with $d + 1$ colors. (“definable” = Borel)
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- (classical) If every action of a group G admits a forest of lines, then G is amenable.
- (Gao–Jackson 2015) If G is abelian, then every “definable” action admits a “definable” forest of lines (called “hyperfinite”).