Suprema of stochastic processes

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Introduction/Motivation

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$$\|g\|_{2k} = \sqrt[2k]{1 \cdot 3 \cdots (2k-1)} \overset{\text{p. means}}{\geq} \frac{k}{\sum_{l=1}^k \frac{1}{\sqrt{2l-1}}} \geq \frac{k}{10 + \int_1^k \frac{1}{\sqrt{x}} dx} \geq \frac{\sqrt{k}}{22}.$$

$$\|g\|_p \geq \begin{cases} \|g\|_1 = 1 & , p \in [1,2) \\ \|g\|_{2[\frac{p}{2}]} \geq \frac{\sqrt{p}}{234} & p \geq 2. \end{cases}$$

Conclusion

There exists $c, C > 0$ numerical (independent of $p \geq 1$) s.t.

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$$\text{s.t. } \frac{f}{C} \leq g \leq Cf \ (\frac{f}{C(\alpha)} \leq g \leq C(\alpha)f).$$

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Problem (Main topic of my research)

Let f be a "complicated" probabilistic quantity. Find a "simpler" expression g such that

$$f \sim g,\ f \sim^{\alpha} g \text{ or at least } f \leq C(\alpha)g, C(\alpha) \text{"small"}.$$

Typically $f = \mathbb{E}\sup_{t\in T} X_t$, $(X_t)_{t\in T}$ a stochastic process or $f = \sqrt[p]{\mathbb{E}|W(X)|^p}$, W a d-linear form, X random vector from $\mathbb{R}^d$.

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Remark: *Bounding $\sqrt[p]{\mathbb{E}|W(X)|^p}$ is strongly related to estimating suprema of stochastic processes.*

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Remark: *We don't care about the optimality of constants.*
Exception: *dependence on the dimension of the problem.*

Some results 1

Theorem (Latala 96')

Let X_i be independent, "nice", symmetric r.v.'s and $T \subset \mathbb{R}^d$ is convex. Then for $p \geq 1$

$$\sqrt[p]{\mathbb{E} \sup_{t \in T} \left| \sum_i t_i X_i \right|^p} \sim \mathbb{E} \sup_{t \in T} \left| \sum_i t_i X_i \right| + \sup_{t \in T} \sqrt[p]{\mathbb{E} \left| \sum_i t_i X_i \right|^p}.$$

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Theorem (Bednorz-Martynek-Meller 24')

Let K be a "nice" convex set and $X = (X_1, \ldots, X_d) \sim Unif(K)$. Let $X_1', \ldots, X_d'$ be independent, $\forall_{i\leq d}\ X_i \stackrel{D}{=} X_i'$. Then for any set $T \subset \mathbb{R}^d$, symmetric w.r.t. $\{x_i = 0\}$, $i = 1\ldots, d$

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^d t_i X_i \leq C\mathbb{E}\sup_{t\in T}\sum_{i=1}^d t_i X_i'.$$

Some Results 2

Theorem (Meller 22')

(X_i) independent, with the density $C(r_i)e^{-|x|^{r_i}}$, and $1 \le r_i < \infty$ (they are not i.i.d). For any $p \ge 1$ and any $a_{ij} \in \ell_q$ s.t. $a_{ij} = a_{ji}$ (irrelevant) and $a_{ii} = 0$ (important)

$$\sqrt[p]{\mathbb{E}\|\sum_{ij} a_{ij} X_i X_j\|_{\ell_q}^p} \sim^q \sup_{x,y \in B(X,p)} \|\sum_{ij} a_{ij} x_i y_j\|_{\ell_q} + \|\sqrt{\sum_{ij} a_{ij}^2}\|_{\ell_q}$$

$$+ \sup_{x \in B(X,p)} \|\sqrt{\sum_j \left(\sum_i x_i a_{ij}\right)^2}\|_{\ell_q} + \sup_{x \in B(X,p), \varphi \in B_{q'}} \sum_i x_i \sqrt{\sum_j \varphi^2(a_{ij})}$$

where $B(X,p) = \{x : \sum_i (x_i^2 \mathbf{1}_{|x_i| \le 1} + x_i^{r_i} \mathbf{1}_{|x_i| > 1}) \le p\}$.

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TLDR: *Formula on the right is deterministic (=simpler).*