When sets take cOVERs

set theory & topology

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Institute of Mathematics

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Rothberger

$\mathcal{O}$pen covers of $X \subseteq \mathbb{R} \setminus \mathbb{Q}$

$\mathsf{S}_1(\mathcal{O}, \mathcal{O})$: $\forall\, \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O}\ \exists\ U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots\ \{\, U_n : n \in \mathbb{N} \,\} \in \mathcal{O}$

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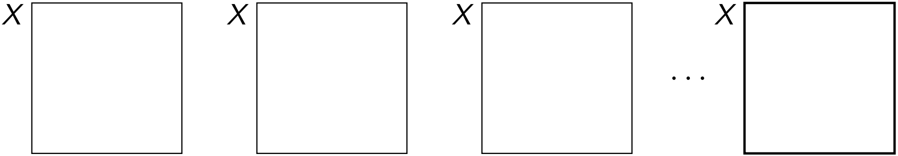

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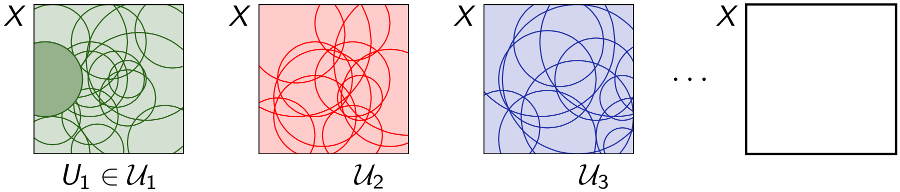

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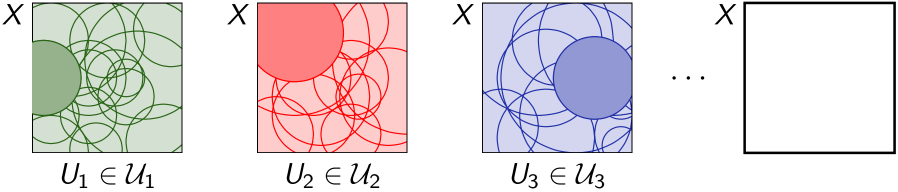

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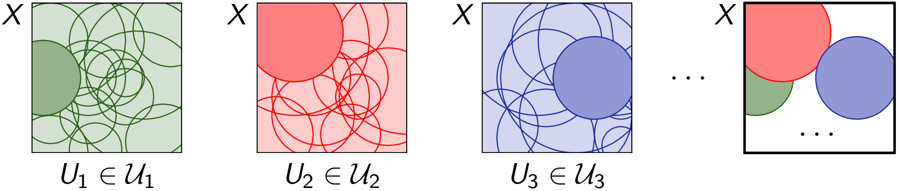

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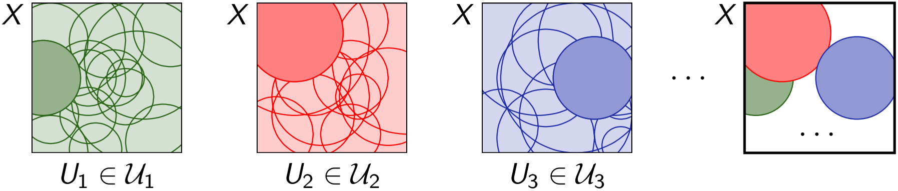

countable $\longrightarrow$ $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$

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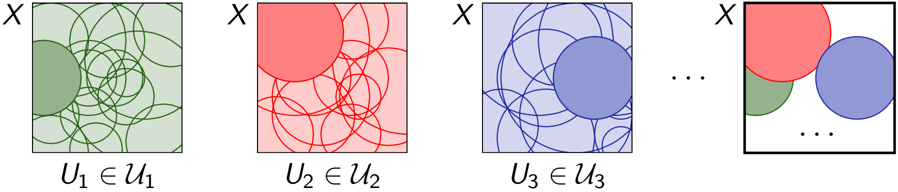

countable $\longrightarrow$ $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ $\longrightarrow$ SMZ

strong measure zero:

$\forall\ \epsilon_1, \epsilon_2, \ldots > 0\ \exists$ intervals $|I_1| \leq \epsilon_1, |I_2| \leq \epsilon_2, \ldots\ \{\, I_n : n \in \mathbb{N} \,\} \in \mathcal{O}$

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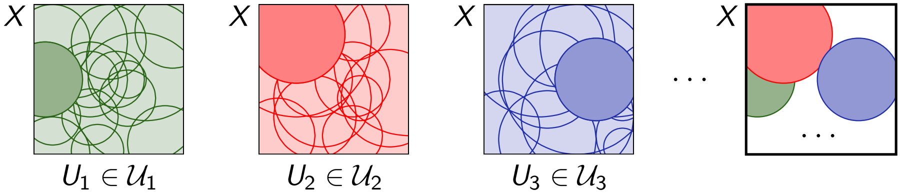

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Borel's Conjecture 1919

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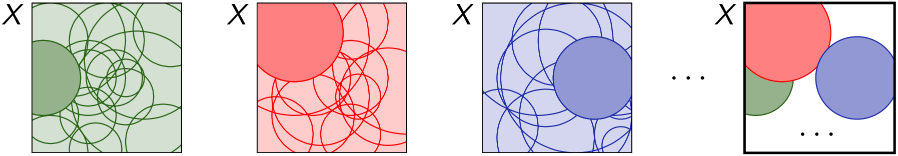

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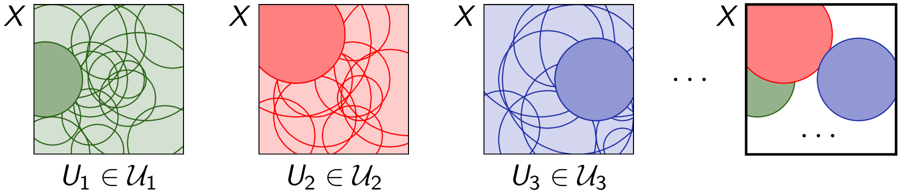

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Laver 1976: Borel's Conjecture is consistent with ZFC

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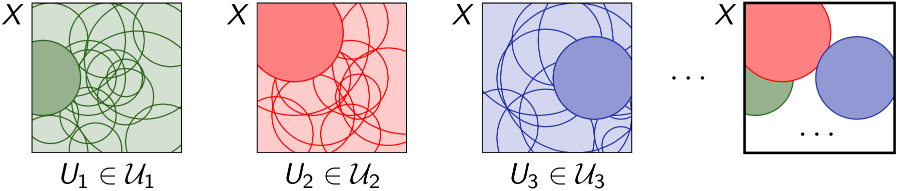

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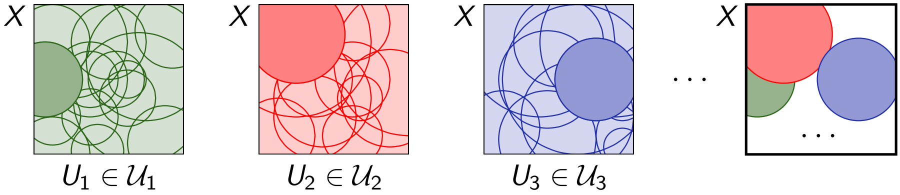

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Theorem (Recław 1994)

X is $\mathsf{S}_1(\mathcal{O},\mathcal{O}) \Leftrightarrow$ every continuous image of X into $\mathbb{N}^{\mathbb{N}}$ is guessable

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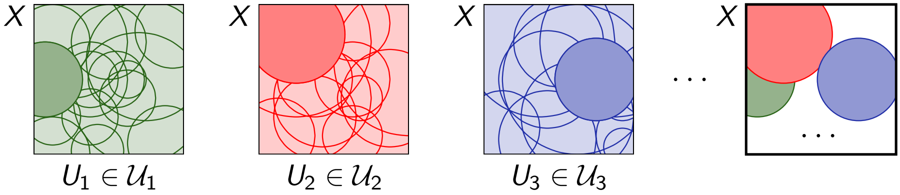

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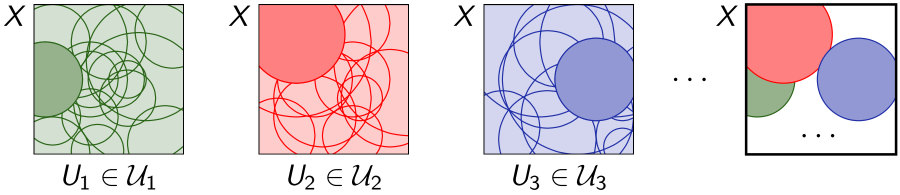

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- $|X| < \mathrm{cov}(\mathcal{M}) \Rightarrow X$ is $\mathsf{S}_1(\mathcal{O},\mathcal{O})$

Hurewicz and Menger

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γ-cover: every point is covered by almost all sets

$\mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma)$: $\forall\, \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O}$ $\exists$ fin $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots$ $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \Gamma$

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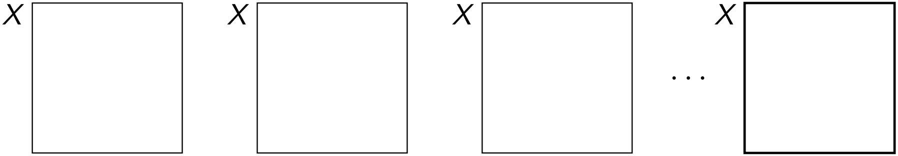

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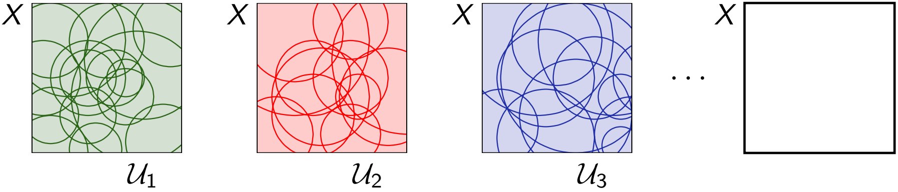

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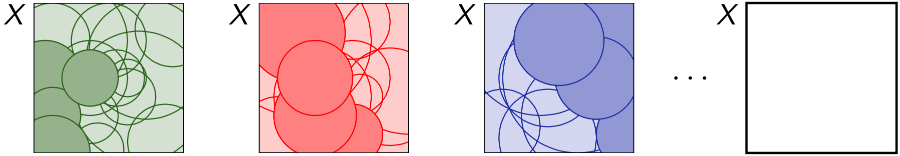

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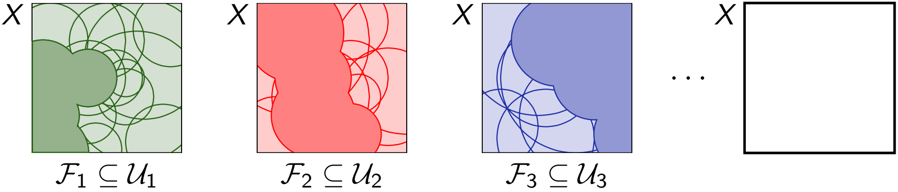

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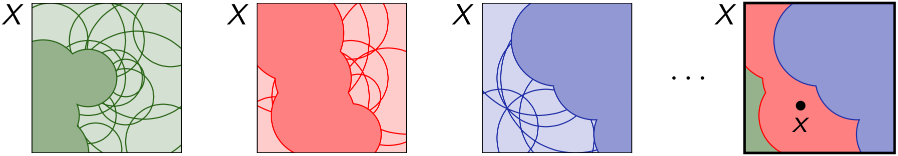

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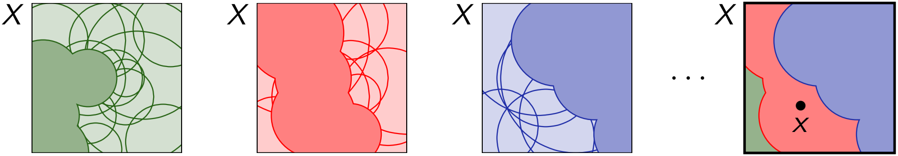

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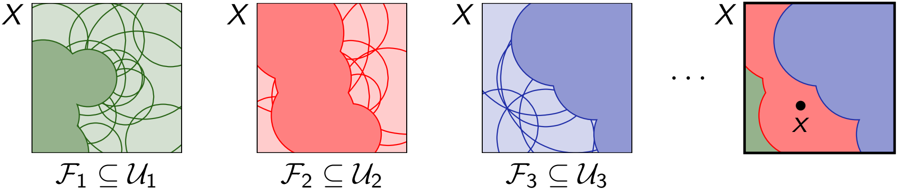

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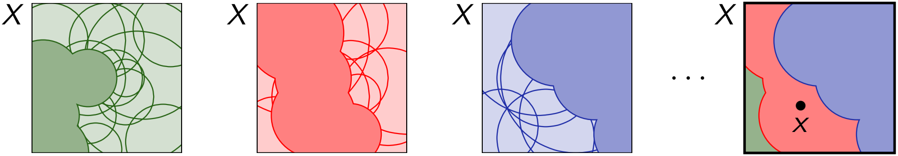

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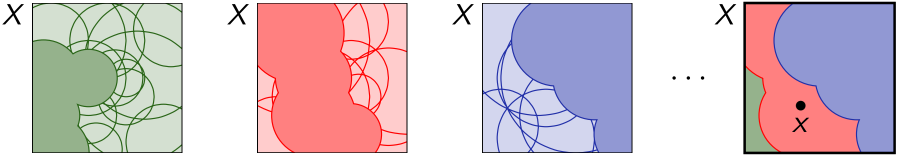

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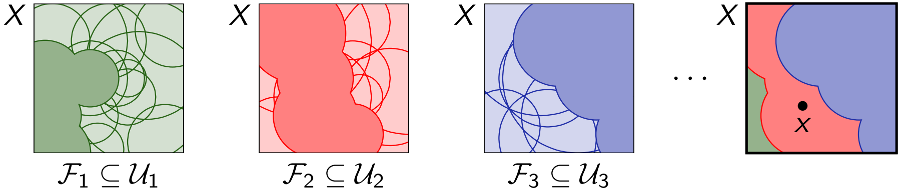

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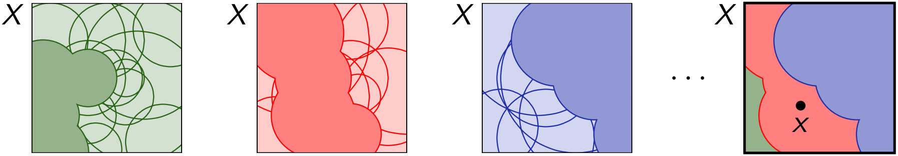

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$\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$: $\forall\, \mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathcal{O}\ \exists$ fin $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \ldots\ \bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathcal{O}$

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\begin{tikzcd}
\sigma\text{-compact} \arrow[r] & \mathsf{U}_{\mathrm{fin}}(\mathcal{O},\Gamma) \arrow[l, bend right, "\text{Hurewicz's Conjecture}"'] \arrow[l, bend left, "\times"] \arrow[r] & \mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) \arrow[l, bend right, "\text{Hurewicz's Problem}"'] \\
\aleph_1 & \mathfrak{b} & \mathfrak{d}
\end{tikzcd}
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Hurewicz and Menger

γ-cover: every point is covered by almost all sets

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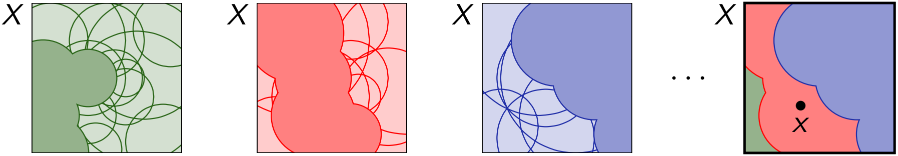

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Topological selections

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Haberl–Sz–Zdomskyy 2026: It is consistent with ZFC that

$$\text{Hurewicz + totally imperfect} \quad = \quad \text{Rothberger}$$

MIM's contribution

$$
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If $\mathsf{C}_\mathsf{p}(X)$ and $\mathsf{C}_\mathsf{p}(Y)$ are homeomorphic and X is Menger, then Y is Menger.

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Marciszewski–Pol–Zakrzewski 2026:
Assuming CH, for each Sierpiński set S there is a Hurewicz set H such that $S \times H$ is not Menger

Colorings

- $a_1, a_2, \ldots \in \mathbb{N}$, $F = \{i_1, \ldots, i_n\}$ with an increasing enumeration

$$a_F := a_{i_1} + \cdots + a_{i_n}$$

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$$a_2 + a_4 = a_F \qquad \neq \qquad a_G = a_6 + a_8 + a_{10}$$

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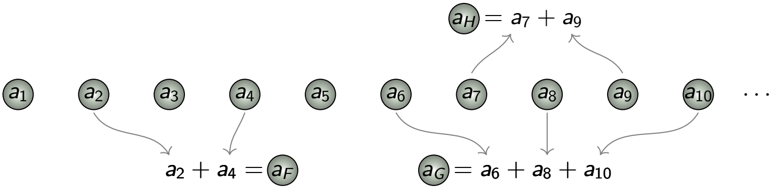

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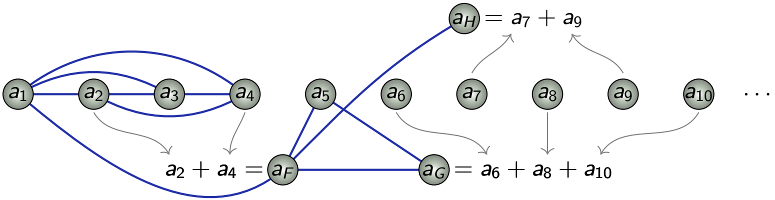

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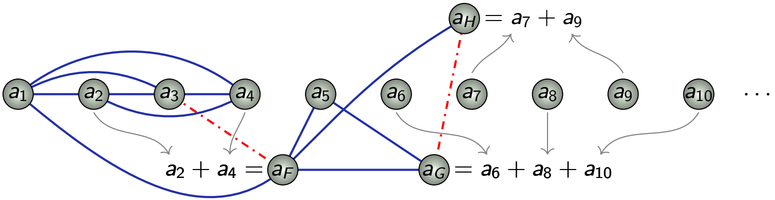

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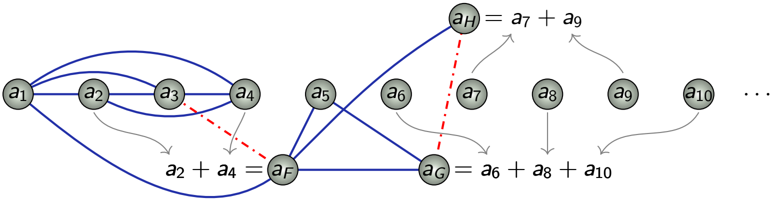

Theorem (Milliken 1975, Taylor 1976) $(\mathbb{N}, +)$

For each coloring of $[\mathbb{N}]^2$, there is a proper sequence $a_1, a_2, \ldots \in \mathbb{N}$ whose sumgraph is monochromatic.

Colorings

Theorem (Tsaban 2018; Sz 2023) $(\tau, \cup)$

If X is $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$, then for every $\mathcal{U} \in \mathcal{O}$ closed under finite unions, with no finite subcover and coloring of $[\tau]^2$, there are sets $V_1, V_2, \ldots \in \mathcal{U}$ such that

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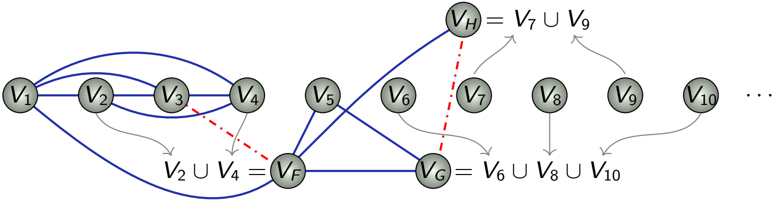

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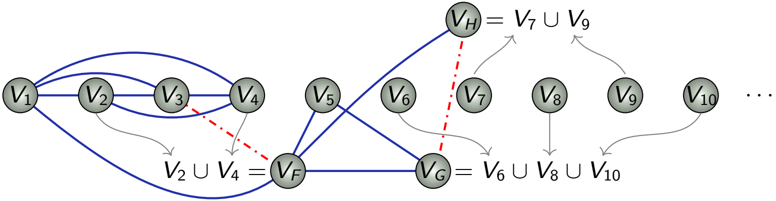

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