The homological algebra of strict polynomial functors

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What is a polynomial representation?

Let **k** be a field of characteristic $p > 0$. We consider the group GL_n (over **k**) as an affine variety.

Polynomial representation of GL_n

1. V – a vector space
2. $\phi : \mathrm{GL}_n \to \mathrm{GL}(V)$ – a group homomorphism and a morphism of affine varieties
3. the coordinate functions are polynomials in the coordinate functions on GL_n (they don't involve $\frac{1}{\det(-)}$).

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It is more convenient to study representations via modules over the group algebra $\mathbf{k}[\mathrm{GL_n}]$. The category of homogeneous polynomial representations of GL_n of degree d is equivalent to the category of finite-dimensional left modules over the Schur algebra $S(n, d) \simeq (\mathrm{End}(\mathbf{k}^n)^{\otimes d})^{\Sigma_d}$.

The category $\mathcal{P}_d$

$\mathcal{V}ect_{\mathbf{k}}$ – the category of finite dimensional $\mathbf{k}-$vector spaces.

Strict polynomial functor

Roughly speaking, a strict polynomial functor of degree d is a functor $\mathcal{V}ect_k \to \mathcal{V}ect_k$, which acts on morphisms via a homogeneous polynomial map.
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Examples of strict polynomial functors of degree d

The tensor power $I^{\otimes d}$, the symmetric power S^d, the exterior power Λ^d, the divided power Γ^d

Significance of $\mathcal{P}_d$

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- It turns out that homological computations are much easier in $\mathcal{P}_d$ than in the module categories because of different tools specific to functor categories, invisible at the level of modules.
- Homological algebra? Why? In the modular case there are representations, which are not direct sums of the irreducible representations. To measure the obstruction of that one uses the homological algebra.

My research

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$\mathcal{P}_p$

- The complete computations of Ext-groups in $\mathcal{P}_p$ between structural functors important from the representation-theoretic point of view
- Some structural results regarding the category $\mathcal{P}_p$, e.g., the description of the injective envelopes of simple functors, the existence of a Kazhdan-Lusztig theory, the explicit structure of some Ext-algebras corresponding to the important families of functors
- A generalization of the obtained results for $d = p$ to the case of some subcategories of $\mathcal{P}_d$ for higher degrees d (deeply connected to the combinatorics of Young diagrams). This provides the complete information about the homological algebra of $\mathcal{P}_d$ for $p < d < 2p$.

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$\mathcal{P}_{2p}$

- The injective envelopes of simple functors
- The complete computation of Ext-groups between Schur functors, important from the representation-theoretic point of view
- In progress: the computation of Ext-groups between simple and Schur functors. Goal: the existence of a Kazhdan-Lusztig theory in $\mathcal{P}_{2p}$

Thank you for your attention! ☺